CISC 2210 TR2 – Introduction to Discrete Structures

Midterm 2 Exam – Solutions

Nov 7, 2023

1. Prove the correctness of the following identity for any integer $n \geq 1$.

$$\sum_{i=1}^{n} i + \sum_{i=n+1}^{2n} i \;=\; n(2n+1)$$

Core identity:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

A direct proof I. First combine the two terms in the left side of the identity as follows.

$$\sum_{i=1}^{n} i + \sum_{i=n+1}^{2n} i = (1+2+\cdots+n) + ((n+1)+(n+2)+\cdots+2n) = \sum_{i=1}^{2n} i$$

By replacing the upper limit n with $2n$, the core identity becomes

$$\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} = n(2n+1)$$

A direct proof II. Evaluate the left side of the identity as follows and apply the core identity

$$\begin{aligned}
\sum_{i=1}^{n} i + \sum_{i=n+1}^{2n} i &= \frac{n(n+1)}{2} + ((n+1)+(n+2)+\cdots+2n) \\
&= \frac{n(n+1)}{2} + ((n+1)+(n+2)+\cdots+(n+n)) \\
&= \frac{n(n+1)}{2} + \sum_{i=1}^{n}(n+i) \\
&= \frac{n(n+1)}{2} + \sum_{i=1}^{n} n + \sum_{i=1}^{n} i \\
&= \frac{n(n+1)}{2} + n^2 + \frac{n(n+1)}{2} \\
&= n(n+1) + n^2 \\
&= n^2 + n + n^2 \\
&= 2n^2 + n \\
&= n(2n+1)
\end{aligned}$$

Proof by induction:

- *Notations.*

$$\begin{aligned} L(n) &= \sum_{i=1}^{n} i + \sum_{i=n+1}^{2n} i \\ R(n) &= n(2n+1) \end{aligned}$$

- *Induction base.* Prove that $L(1) = R(1)$:

$$L(1) = 1 + 2 = 3 = 1 \cdot (2 \cdot 1 + 1) = R(1)$$

- *Induction hypothesis.* Assume that $L(k) = R(k)$ for $k \geq 1$:

$$\sum_{i=1}^{k} i + \sum_{i=k+1}^{2k} i = k(2k+1)$$

- *Inductive step.* Prove that $L(k+1) = R(k+1)$ for $k \geq 1$:

$$\begin{aligned} L(k+1) &= \sum_{i=1}^{k+1} i + \sum_{i=(k+1)+1}^{2(k+1)} i \\ &= \sum_{i=1}^{k+1} i + \sum_{i=k+2}^{2k+2} i \\ &= \left(\sum_{i=1}^{k} i + (k+1) \right) + \left(\sum_{i=k+1}^{2k} i + ((2k+1) + (2k+2) - (k+1)) \right) \\ &= \sum_{i=1}^{k} i + \sum_{i=k+1}^{2k} i + ((k+1) + (2k+1) + (2k+2) - (k+1)) \\ &= L(k) + (4k+3) \\ &= R(k) + (4k+3) \\ &= k(2k+1) + (4k+3) \\ &= 2k^2 + k + 4k + 3 \\ &= 2k^2 + 5k + 3 \\ &= (k+1)(2k+3) \\ &= (k+1)(2(k+1)+1) \\ &= R(k+1) \end{aligned}$$

2. Match each one of the following 5 recursive formulas with one of the possible 5 solutions:

$$-5n \quad ; \quad 5n \quad ; \quad 5n^2 \quad ; \quad 5^n \quad ; \quad 5n!$$

(a) $T(1) = 5$; $T(n) = 5T(n-1)$ for $n > 1$

Solution: $T(n) = 5^n$.

Proof:

- *Induction base.* $T(1) = 5 = 5^1$ for $n = 1$.
- *Induction hypothesis.* Assume that $T(n-1) = 5^{n-1}$ for $n > 1$.
- *Inductive step.* Prove that $T(n) = 5^n$ for $n > 1$:

$$\begin{aligned} T(n) &= 5T(n-1) \\ &= 5 \cdot 5^{n-1} \\ &= 5^n \end{aligned}$$

(b) $T(1) = -5$; $T(n) = T(n-1) - 5$ for $n > 1$

Solution: $T(n) = -5n$.

Proof:

- *Induction base.* $T(1) = -5 = -5 \cdot 1$ for $n = 1$.
- *Induction hypothesis.* Assume that $T(n-1) = -5(n-1)$ for $n > 1$.
- *Inductive step.* Prove that $T(n) = -5n$ for $n > 1$:

$$\begin{aligned} T(n) &= T(n-1) - 5 \\ &= -5(n-1) - 5 = -5n + 5 - 5 \\ &= -5n \end{aligned}$$

(c) $T(1) = 5$; $T(n) = T(n-1) + (10n - 5)$ for $n > 1$

Solution: $T(n) = 5n^2$.

Proof:

- *Induction base.* $T(1) = 5 = 5 \cdot 1^2$ for $n = 1$.
- *Induction hypothesis.* Assume that $T(n-1) = 5(n-1)^2$ for $n > 1$.
- *Inductive step.* Prove that $T(n) = 5n^2$ for $n > 1$:

$$\begin{aligned} T(n) &= T(n-1) + (10n - 5) \\ &= 5(n-1)^2 + (10n - 5) = 5n^2 - 10n + 5 + 10n - 5 \\ &= 5n^2 \end{aligned}$$

(d) $T(1) = 5$; $T(n) = nT(n-1)$ for $n > 1$

Solution: $T(n) = 5n!$.

Proof:

- *Induction base.* $T(1) = 5 = 5 \cdot 1!$ for $n = 1$.
- *Induction hypothesis.* Assume that $T(n-1) = 5(n-1)!$ for $n > 1$.
- *Inductive step.* Prove that $T(n) = 5n!$ for $n > 1$:

$$\begin{aligned} T(n) &= n \cdot 5T(n-1) \\ &= n \cdot 5(n-1)! = 5 \cdot n(n-1)! \\ &= 5n! \end{aligned}$$

(e) $T(1) = 5$; $T(n) = T(n-1) + 5$ for $n > 1$

Solution: $T(n) = 5n$.

Proof:

- *Induction base.* $T(1) = 5 = 5 \cdot 1$ for $n = 1$.
- *Induction hypothesis.* Assume that $T(n-1) = 5(n-1)$ for $n > 1$.
- *Inductive step.* Prove that $T(n) = 5n$ for $n > 1$:

$$\begin{aligned} T(n) &= T(n-1) + 5 \\ &= 5(n-1) + 5 = 5n - 5 + 5 \\ &= 5n \end{aligned}$$

3. A legal password of length 5 must contain exactly one of the three letters X, Y, Z while the other four positions must be filled with one of the ten digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

(a) How many legal passwords of length 5 are there in which the letter must appear in the first position?

Answer: $\mathbf{30000 = 3 \cdot 10 \cdot 10 \cdot 10 \cdot 10}$

Proof: There are 3 options for the first position and 10 options for each one of the other four positions. In total there are $3 \times 10^4 = 30000$ different legal passwords of length 5 in which the letter must appear in the first position.

(b) How many legal passwords of length 5 are there in which the letter may appear in the first position or in the last position but cannot appear in the middle?

Answer: $\mathbf{60000 = (3 \cdot 10 \cdot 10 \cdot 10 \cdot 10) + (10 \cdot 10 \cdot 10 \cdot 10 \cdot 3)}$

Proof: Similarly to the proof of part (a), there are $3 \times 10^4 = 30000$ different legal passwords of length 5 in which the letter must appear in the last position. In total there are $2 \times 30000 = 60000$ different legal passwords of length 5 in which the letter may appear in the first position or in the last position but cannot appear in the middle.

(c) How many legal passwords of length 5 are there in which the letter may appear in any one of the five positions?

Answer: $\mathbf{150000 = (3 \cdot 10 \cdot 10 \cdot 10 \cdot 10) + (10 \cdot 3 \cdot 10 \cdot 10 \cdot 10) + (10 \cdot 10 \cdot 3 \cdot 10 \cdot 10) +}$
$\mathbf{(10 \cdot 10 \cdot 10 \cdot 3 \cdot 10) + (10 \cdot 10 \cdot 10 \cdot 10 \cdot 3)}$

Proof: Similarly to the proof of part (a), there are $3 \times 10^4 = 30000$ different legal passwords of length 5 in which the letter must appear in a specific position. Since there are five possible positions for the letter, it follows that in total there are $5 \times 30000 = 150000$ different legal passwords of length 5 in which the letter may appear in any one of the five positions.

(d) How many legal passwords of length 5 are there in which all the four digits must be the same and the letter may appear in any of the five positions?

Answer: $\mathbf{150 = (3 \cdot 10 \cdot 1 \cdot 1 \cdot 1) + (10 \cdot 3 \cdot 1 \cdot 1 \cdot 1) + (10 \cdot 1 \cdot 3 \cdot 1 \cdot 1) +}$
$\mathbf{(10 \cdot 1 \cdot 1 \cdot 3 \cdot 1) + (10 \cdot 1 \cdot 1 \cdot 1 \cdot 3)}$

Proof: The single letter may appear in any one of the 5 positions and there are 3 options for the letter. Since the four digits must be the same, it follows that there are only 10 options for the other four positions. In total there are $5 \times 3 \times 10 = 150$ different legal passwords of length 5 in which all the four digits must be the same and the letter may appear in any of the five positions.

(e) How many legal passwords of length 5 are there in which all the four digits must be different and the letter must appear in the first position?

Answer: $\mathbf{15120 = 3 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$

Proof: There are 3 options for the first position. In the other four positions the digits cannot repeat. Therefore, there are 10 options for the second position, 9 options for the third position, 8 options for the third position, and 7 options for the last position. In total there are $3 \times 10 \times 9 \times 8 \times 7 = 15120$ different legal passwords of length 5 in which the letter must appear in the first position and the four digits must be different.

(f) How many legal passwords of length 5 are there in which two of the digits must be even while the other two digits must be odd and the letter must appear in the last position?

Answer:

$$\begin{aligned}\mathbf{11250 = 6 \cdot 5^4 \cdot 3} \quad = \quad & (5_{even} \cdot 5_{even} \cdot 5_{odd} \cdot 5_{odd} \cdot 3) \; + \\ & (5_{even} \cdot 5_{odd} \cdot 5_{even} \cdot 5_{odd} \cdot 3) \; + \\ & (5_{even} \cdot 5_{odd} \cdot 5_{odd} \cdot 5_{even} \cdot 3) \; + \\ & (5_{odd} \cdot 5_{even} \cdot 5_{even} \cdot 5_{odd} \cdot 3) \; + \\ & (5_{odd} \cdot 5_{even} \cdot 5_{odd} \cdot 5_{even} \cdot 3) \; + \\ & (5_{odd} \cdot 5_{odd} \cdot 5_{even} \cdot 5_{even} \cdot 3)\end{aligned}$$

Proof: Assume first that the two even digits must appear in the first two positions. Since there are 5 even digits $(0, 2, 4, 6, 8,)$ and 5 odd digits $(1, 3, 5, 7, 9)$ and the digits may repeat, it follows that with this additional constraint there are 5 options for the first position, 5 options for the second position, 5 options for the third position, 5 options for the fourth position, and 3 options for the letter in the last position. In total there are $5^4 \times 3 = 1875$ different legal passwords of length 5 in which two of the digits must be even and must appear in the first two positions, two of the digits must be odd and must appear in the third and fourth positions, and the letter must appear in the last position.

Similar arguments show that there are $5^4 \times 3 = 1875$ different legal passwords of length 5 in which the two even digits must appear in two specific positions out of the first four positions, the two odd digits must appear in the other two positions out of the first four positions, and the letter must appear in the last position.

Since there are $6 = \binom{4}{2}$ ways to select the two positions in which the two even digits appear, it follows that in total there are $6 \times 5^4 \times 3 = 11250$ legal passwords of length 5 are there in which two of the digits must be even while the other two digits must be odd and the letter must appear in the last position.

4. Prove the following identity for or any two integers k and n such that $1 \leq k \leq n$:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

Proof: By definition,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It follows that,

$$\begin{aligned}\binom{n-1}{k-1} &= \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\ &= \frac{(n-1)!}{(k-1)!(n-k)!}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{n}{k}\binom{n-1}{k-1} &= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{n(n-1)!}{k(k-1)!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}\end{aligned}$$

5. You have three 3-sided fair dice. The faces of these dice are labeled with the numbers $1, 2, 3$. You throw the three dice together.

Preprocessing: The tables below list the 27 possible sums and products of the three shown numbers represented by the triplets (x, y, z) for $1 \leq x, y, z \leq 3$ in which the first dice shows x, the second dice shows y, and the third dice shows z.

triplets	**sum**	**product**
$(1,1,1)$	3	1
$(1,1,2)$	4	2
$(1,1,3)$	5	3
$(1,2,1)$	4	2
$(1,2,2)$	5	4
$(1,2,3)$	6	6
$(1,3,1)$	5	3
$(1,3,2)$	6	6
$(1,3,3)$	7	9

triplets	**sum**	**product**
$(2,1,1)$	4	2
$(2,1,2)$	5	4
$(2,1,3)$	6	6
$(2,2,1)$	5	4
$(2,2,2)$	6	8
$(2,2,3)$	7	12
$(2,3,1)$	6	6
$(2,3,2)$	7	12
$(2,3,3)$	8	18

triplets	**sum**	**product**
$(3,1,1)$	5	3
$(3,1,2)$	6	6
$(3,1,3)$	7	9
$(3,2,1)$	6	6
$(3,2,2)$	7	12
$(3,2,3)$	8	18
$(3,3,1)$	7	9
$(3,3,2)$	8	18
$(3,3,3)$	9	27

(a) What is the probability that the sum of the three shown numbers is 3?

Answer: $\mathbf{\frac{1}{27} \approx 0.037}$ $(\approx \mathbf{3.7}\%)$

Proof: Out of the 27 possible triplets of shown numbers, only in $(1, 1, 1)$ the sum of the three shown numbers is 3. Hence, the probability that the sum of the three shown numbers is 3 is $1/27$.

(b) What is the probability that the product of the three shown numbers is 4?

Answer: $\mathbf{\frac{3}{27} = \frac{1}{9} \approx 0.111}$ $(\approx \mathbf{11.11}\%)$

Proof: Out of the 27 possible triplets of shown numbers, only in $(1, 2, 2)$, $(2, 1, 2)$, and $(2, 2, 1)$ the product of the three shown numbers is 4. Hence, the probability that the product of the three shown numbers is 4 is $3/27 = 1/9$.

(c) What is the probability that the product of the three shown numbers is odd?

Answer: $\mathbf{\frac{8}{27} \approx 0.296}$ $(\approx \mathbf{29.63}\%)$

Proof: Out of the 27 possible triplets of shown numbers, only in the following 8 the product of the three shown numbers is odd:

$$(1,1,1) \quad (1,1,3) \quad (1,3,1) \quad (1,3,3) \quad (3,1,1) \quad (3,1,3) \quad (3,3,1) \quad (3,3,3)$$

Hence, the probability that the product of the three shown numbers is odd is $8/27$.

(d) What is the probability that the sum of the three shown numbers is even?

Answer: $\mathbf{\frac{13}{27} \approx 0.481}$ $(\approx \mathbf{48.15}\%)$

Proof: Out of the 27 possible triplets of shown numbers, only in the following 13 the sum of the three shown numbers is even:

$$(1,1,2) \quad (1,2,1) \quad (1,2,3) \quad (1,3,2)$$
$$(2,1,1) \quad (2,1,3) \quad (2,2,2) \quad (2,3,1) \quad (2,3,3)$$
$$(3,1,2) \quad (3,2,1) \quad (3,2,3) \quad (3,3,2)$$

Hence, the probability that the sum of the three shown numbers is even is $13/27$.

(e) Given that the product of the three shown numbers is larger than 7, what is the probability that their sum is exactly 7?

Answer: $\mathbf{\frac{6}{11} \approx 0.545}$ ($\approx$ **54.55**%)

Proof: Out of the 27 possible triplets of shown numbers, only in the following 11 triplets the product of the three numbers is larger than 7:

$$(1,3,3)\ (2,2,2)\ (2,2,3)\ (2,3,2)\ (2,3,3)\ (3,1,3)\ (3,2,2)\ (3,2,3)\ (3,3,1)\ (3,3,2)\ (3,3,3)$$

Out of these 11 triplets, only in the following 6 triplets the sum of the three numbers is exactly 7:

$$(1,3,3)\quad (2,2,3)\quad (2,3,2)\quad (3,1,3)\quad (3,2,2)\quad (3,3,1)$$

Hence, the probability that the product of the three shown numbers is exactly 7 given that their product is larger than 7 is $6/11$.

(f) Given that the sum of the three shown numbers is 4 or 5, what is the probability that their product is exactly 4?

Answer: $\mathbf{\frac{3}{9} = \frac{1}{3} \approx 0.333}$ ($\approx$ **33.33**%)

Proof: Out of the 27 possible triplets of shown numbers, only in the following 9 triplets the sum of the three numbers is 4 or 5:

$$(1,1,2)\quad (1,1,3)\quad (1,2,1)\quad (1,2,2)\quad (1,3,1)\quad (2,1,1)\quad (2,1,2)\quad (2,2,1)\quad (3,1,1)$$

Out of these 9 triplets, only in the following 3 triplets the product of the three numbers is exactly 4:

$$(1,2,2)\quad (2,1,2)\quad (2,2,1)$$

Hence, the probability that the product of the three shown numbers is exactly 4 given that their sum is 4 or 5 is $3/9 = 1/3$.

Remark: The answers can be found using probability arguments.

For example, consider part (c). The only way the product of three numbers is odd is when each one of these numbers is odd. For each dice this happens with probability $2/3$. Since the outcomes of the three dice are independent, it follows that with probability $(2/3)^3 = 8/27$ the three shown numbers are odd and as a result their product is odd. As for parts (e) and (f), the answers can be computed using Bayes' theorem for conditional probabilities.

However, with this particular problem, it is easier to establish counting arguments to solve the six problems. Mainly because the sample space which has only 27 outcomes is relatively small.