CISC 2210 TR11 – Introduction to Discrete Structures

Midterm 2 Exam

Apr 21, 2026

Id: ....

Problem	Maximum Points	Your Points
Induction	100	
Recursion	100	
Counting	100	
Combinatorics	100	
Probability	100	

Structure, problem selection, and credit:

- You have 2 hours to complete the exam.
- There are 5 problems. Each problem is a "mini-exam" by itself with a 5% weight in the final grade for the class. However, the grade of each individual problem counts only if it is higher than the final exam grade.

  Strategy: It is better to try first answering the questions relating to topics you have mastered. Note that since there is no cumulative grade, one fully correct answer is better than two or more partially correct answers.
- You will get only partial credit if you fail to justify or prove your answers. You will get 20% of the credit for any problem or part of a problem if you leave the allocated space for the answer empty. You will get zero credit for wrong answers.

Honor code: Students are expected to do this exam **by themselves** without any external help from other people, the Internet, books, notes, or calculators. Cheaters will be punished severely. At minimum, they will fail the exam, but they may fail the whole class. In addition, students who cheat risk disciplinary measures by Brooklyn College and CUNY.

Problem 1: Induction (100 credits)

Prove **by induction** that $n! > 5n$ for $n \geq 4$.

Remark: Full credit will be given **only** to correct proofs that include all the stages of "proof by induction".

Problem 2: Recursion (100 credits)

Define the following recursive formula for all positive integers $n \geq 1$:

$$T(n) = \begin{cases} 2 & \text{for } n = 1 \\ 4 & \text{for } n = 2 \\ 6 & \text{for } n = 3 \\ 3T(n-2) - T(n-1) - T(n-3) + 4 & \text{for } n \geq 4 \end{cases}$$

Problem 2a (30 credits)

Compute $T(n)$ for $n = 4$, $n = 5$, and $n = 6$.

Problem 2b (20 credits)

Express $T(n)$ for $n \geq 5$ as a function of $T(n-2)$, $T(n-3)$, and $T(n-4)$ by eliminating the $T(n-1)$ term through a top-down evaluation.

Problem 2c (10 credits)

Guess a closed-form expression for $T(n)$. Justify your guess.

Problem 2d (40 credits)

Prove that your guess is correct.

Problem 3: Counting (100 credits)

Definition: For $n \geq 2$, a permutation $\pi = (\pi(1), \ldots . \pi(n))$ of the numbers $1, 2, \ldots, n$ is an almost-identity permutation (AID-permutation) if exactly 2 numbers are displaced (i.e., have $\pi(i) \neq i$).

Example: For $n = 7$, $(1, 2, 6, 4, 5, 3, 7)$ is an AID-permutation. Only 3 and 6 are displaced.

Problem 3a (20 credits)

List all the AID-permutations for $n = 2$.

Problem 3b (20 credits)

List all the AID-permutations for $n = 3$.

Problem 3c (20 credits)

List all the AID-permutations for $n = 4$.

Problem 3d (30 credits)

For $n \geq 2$, derive a formula for the total number of AID-permutations as a function of n. Justify your answer.

Problem 3e (10 credits)

For $n \geq 2$, determine the number of AID-permutations (as a function of n) in which the two displaced numbers are at adjacent positions. Justify your answer.

Problem 4: Combinatorics (100 credits)

This question asks you to prove four combinatorial identities involving binomial coefficients. You may apply combinatorial (counting) arguments in your proofs. For algebraic proofs recall that:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Remark/Hint: One possible way to prove the four identities is the following. Start by proving the fourth identity. Next, explain why the third identity is a special case of the fourth one. Next, explain why the second identity is a special case of the third one. Finally, explain why the first identity is a special case of the second one.

Problem 4a (40 credits)

Prove the following identity:

$$\binom{8}{4}\binom{4}{2} = \binom{8}{2}\binom{6}{2}$$

Problem 4b (20 credits)

Prove the following identity for $t \geq 4$:

$$\binom{t}{4}\binom{4}{2} = \binom{t}{2}\binom{t-2}{2}$$

Problem 4c (20 credits)

Prove the following identity for $t \geq s \geq 2$:

$$\binom{t}{s}\binom{s}{2} = \binom{t}{2}\binom{t-2}{s-2}$$

Problem 4d (20 credits)

Prove the following identity for $t \geq s \geq r \geq 0$:

$$\binom{t}{s}\binom{s}{r} = \binom{t}{r}\binom{t-r}{s-r}$$

Problem 5: Probability (100 credits)

Coins X and Y, both biased, are flipped simultaneously.

- Coin X: The probability for Heads is 3/4 and the probability for Tails is 1/4.
- Coin Y: The probability for Heads is 1/3 and the probability for Tails is 2/3.

Justify your answers to the following five questions.

Problem 5a (20 credits)

What is the probability that both coins X and Y show Tails?

Problem 5b (20 credits)

What is the probability that both coins X and Y show Heads?

Problem 5c (20 credits)

What is the probability that one coin shows Heads while the other coin shows Tails?

Problem 5d (20 credits)

What is the probability that both coins X and Y show Heads given that at least one of them shows Heads?

Problem 5e (20 credits)

What is the probability that both coins X and Y show Tails given that at least one of them shows Tails?