CISC 2210 TR11 – Introduction to Discrete Structures

Midterm 2 Exam – Problems and Solutions

Apr 21, 2026

Problem 1

Prove by induction that $n! > 5n$ for $n \geq 4$.

The inequality is false for $1 \leq n \leq 3$ and true for $4 \leq n \leq 5$:

$$\begin{array}{lcl} 1! = 1 & < & 5 \cdot 1 = 5 \\ 2! = 2 & < & 5 \cdot 2 = 10 \\ 3! = 6 & < & 5 \cdot 3 = 15 \\ 4! = 24 & > & 5 \cdot 4 = 20 \\ 5! = 120 & > & 5 \cdot 5 = 25 \end{array}$$

Proof by induction:

- *Induction base.* $4! = 24 > 20 = 5 \cdot 4$ for $n = 4$.
- *Induction hypothesis.* Assume that $k! > 5k$ for $k \geq 4$.
- *Inductive step.* Prove that $(k+1)! > 5(k+1)$ for $k \geq 4$.

$$\begin{array}{rcll} (k+1)! & = & (k+1)k! & (* \text{ definition of } n! *) \\ & > & (k+1) \cdot 5k & (* \text{ induction hypothesis } *) \\ & = & 5(k+1) \cdot k & (* \text{ algebra } *) \\ & > & 5(k+1) & (* \text{ because } k \geq 4 > 1 *) \end{array}$$

Another proof: There is no need for induction. The claim follows immediately because $n \geq 4$.

$$n! = n(n-1)(n-2) \cdot \dots \cdot 4 \cdot 3 \cdot 2 \geq n \cdot 3 \cdot 2 = 6n > 5n$$

Problem 2

Define the following recursive formula for all positive integers $n \geq 1$:

$$T(n) = \begin{cases} 2 & \text{for } n = 1 \\ 4 & \text{for } n = 2 \\ 6 & \text{for } n = 3 \\ 3T(n-2) - T(n-1) - T(n-3) + 4 & \text{for } n \geq 4 \end{cases}$$

(2a) Compute $T(n)$ for $n = 4$, $n = 5$, and $n = 6$.

- $T(4) = 3T(2) - T(3) - T(1) + 4 = 3 \cdot 4 - 6 - 2 + 4 = 8$.
- $T(5) = 3T(3) - T(4) - T(2) + 4 = 3 \cdot 6 - 8 - 4 + 4 = 10$.
- $T(6) = 3T(4) - T(5) - T(3) + 4 = 3 \cdot 8 - 10 - 6 + 4 = 12$.

(2b) Express $T(n)$ for $n \geq 5$ as a function of $T(n-2)$, $T(n-3)$, and $T(n-4)$ by eliminating the $T(n-1)$ term through a top-down evaluation.

$$\begin{aligned} T(n) &= 3T(n-2) - T(n-1) - T(n-3) + 4 \\ &= 3T(n-2) - (3T(n-3) - T(n-2) - T(n-4) + 4) - T(n-3) + 4 \\ &= 3T(n-2) - 3T(n-3) + T(n-2) + T(n-4) - 4 - T(n-3) + 4 \\ &= 4T(n-2) - 4T(n-3) + T(n-4) \end{aligned}$$

(2c) Guess a closed-form expression for $T(n)$.

Guess: Based on the definitions of $T(1)$, $T(2)$, and $T(3)$ and the evaluations of $T(4)$, $T(5)$, and $T(6)$, the plausible guess is $T(n) = 2n$.

(2d) Prove that your guess is correct.

Proposition: $T(n) = 2n$ For $n \geq 1$.

Proof: By induction on $n \geq 1$.

- *Induction base.*
  - $T(1) = 2 = 2 \cdot 1$ for $n = 1$.
  - $T(2) = 4 = 2 \cdot 2$ for $n = 2$.
  - $T(3) = 6 = 2 \cdot 3$ for $n = 3$.
- *Induction hypothesis.* For $n \geq 4$, assume that:
  - $T(n-1) = 2(n-1) = 2n - 2$.
  - $T(n-2) = 2(n-2) = 2n - 4$.
  - $T(n-3) = 2(n-3) = 2n - 6$.
- *Inductive step.* Prove that $T(n) = 2n$ for $n \geq 4$:

$$\begin{aligned} T(n) &= 3T(n-2) - T(n-1) - T(n-3) + 4 \\ &= 3(2n-4) - (2n-2) - (2n-6) + 4 \\ &= 6n - 12 - 2n + 2 - 2n + 6 + 4 \\ &= (6n - 2n - 2n) + (-12 + 2 + 6 + 4) \\ &= 2n \end{aligned}$$

Problem 3

Definition: For $n \geq 2$, a permutation $\pi = (\pi(1), \ldots . \pi(n))$ of the numbers $1, 2, \ldots, n$ is an almost-identity permutation (AID-permutation) if exactly 2 numbers are displaced (i.e., have $\pi(i) \neq i$).

(3a) List all the AID-permutations for $n = 2$.

Answer: Out of the two length-2 permutations $(1, 2)$ and $(2, 1)$, only $(2, 1)$ is an AID-permutation because both 1 and 2 are displaced while in the identity permutation $(1, 2)$ no number is displaced.

(3b) List all the AID-permutations for $n = 3$.

Answer: Out of the six length-3 permutations the three permutations $(1, 3, 2)$, $(3, 2, 1)$, and $(2, 1, 3)$ are AID-permutations because in each one of them exactly two numbers are displaced. In the two permutations $(2, 3, 1)$ and $(3, 1, 2)$ all three numbers are displaced. The remaining permutation $(1, 2, 3)$ is the identity permutation in which no number is displaced.

(3c) List all the AID-permutations for $n = 4$.

Answer: Out of the twenty-four length-4 permutations the following six permutations

$$(1, 2, 4, 3) \; (1, 3, 2, 4) \; (1, 4, 3, 2) \; (2, 1, 3, 4) \; (3, 2, 1, 4) \; (4, 2, 3, 1)$$

are the only AID-permutations. Each one of these permutations corresponds to one of the $6 = \binom{4}{2}$ ways to select exactly two different numbers to be displaced.

(3d) For $n \geq 2$, derive a formula for the total number of AID-permutations as a function of n.

Answer: $\binom{n}{2}$.

Explanation: There are $\binom{n}{2}$ ways to select a pair of numbers to be displaced. Once these numbers are selected, the only option is to swap their locations.

(3e) For $n \geq 2$, determine the number of AID-permutations (as a function of n) in which the two displaced numbers are at adjacent positions.

Answer: $n - 1$.

Explanation: Since an AID-permutation requires exactly two numbers to be displaced, the only way for them to be adjacent is to swap a pair of consecutive indices $(i, i + 1)$. There are $n - 1$ pairs such that $1 \leq i < i + 1 \leq n$.

Problem 4

(4a) Prove the following identity.

$$\binom{8}{4}\binom{4}{2} = \binom{8}{2}\binom{6}{2}$$

Algebraic proof:

$$\begin{aligned}
\binom{8}{4}\binom{4}{2} &= \frac{8!}{4!4!} \cdot \frac{4!}{2!2!} \\
&= \frac{4!}{4!} \cdot \frac{8!}{2!} \cdot \frac{1}{2!4!} \\
&= \frac{6!}{6!} \cdot \frac{8!}{2!} \cdot \frac{1}{2!4!} \\
&= \frac{8!}{2!6!} \cdot \frac{6!}{2!4!} \\
&= \binom{8}{2}\binom{6}{2}
\end{aligned}$$

(4b) Prove the following identity for $t \geq 4$.

$$\binom{t}{4}\binom{4}{2} = \binom{t}{2}\binom{t-2}{2}$$

Algebraic proof:

$$\begin{aligned}
\binom{t}{4}\binom{4}{2} &= \frac{t!}{4!(t-4)!} \cdot \frac{4!}{2!2!} \\
&= \frac{4!}{4!} \cdot \frac{t!}{2!} \cdot \frac{1}{2!(t-4)!} \\
&= \frac{(t-2)!}{(t-2)!} \cdot \frac{t!}{2!} \cdot \frac{1}{2!(t-4)!} \\
&= \frac{t!}{2!(t-2)!} \cdot \frac{(t-2)!}{2!(t-4)!} \\
&= \binom{t}{2}\binom{t-2}{2}
\end{aligned}$$

(4c) Prove the following identity for $t \geq s \geq 2$.

$$\binom{t}{s}\binom{s}{2} = \binom{t}{2}\binom{t-2}{s-2}$$

Algebraic proof:

$$\begin{aligned}
\binom{t}{s}\binom{s}{2} &= \frac{t!}{s!(t-s)!} \cdot \frac{s!}{2!(s-2)!} \\
&= \frac{s!}{s!} \cdot \frac{t!}{2!} \cdot \frac{1}{(s-2)!(t-s)!} \\
&= \frac{(t-2)!}{(t-2)!} \cdot \frac{t!}{2!} \cdot \frac{1}{(s-2)!(t-s)!} \\
&= \frac{t!}{2!(t-2)!} \cdot \frac{(t-2)!}{(s-2)!(t-s)!} \\
&= \binom{t}{2}\binom{t-2}{s-2}
\end{aligned}$$

(4d) Prove the following identity for $t \geq s \geq r \geq 0$.

$$\binom{t}{s}\binom{s}{r} = \binom{t}{r}\binom{t-r}{s-r}$$

Algebraic proof:

$$\begin{aligned}
\binom{t}{s}\binom{s}{r} &= \frac{t!}{s!(t-s)!} \cdot \frac{s!}{r!(s-r)!} \\
&= \frac{s!}{s!} \cdot \frac{t!}{r!} \cdot \frac{1}{(s-r)!(t-s)!} \\
&= \frac{(t-r)!}{(t-r)!} \cdot \frac{t!}{r!} \cdot \frac{1}{(s-r)!(t-s)!} \\
&= \frac{t!}{r!(t-r)!} \cdot \frac{(t-r)!}{(s-r)!(t-s)!} \\
&= \binom{t}{r}\binom{t-r}{s-r}
\end{aligned}$$

Proving the (4c), (4b), and (4a) based on (4d): To establish the correctness of all four identities, it is sufficient to only prove the most general identity, **(4d)**. The other three identities are then shown to be special cases through successive substitutions as follows:

- Proving (4d) is the initial step.
- Setting $r = 2$ in **(4d)** yields **(4c)**.
- Setting $s = 4$ in **(4c)** yields **(4b)**.
- .Setting $t = 8$ in **(4b)** yields **(4a)**.

Combinatorial (counting) proof for the general identity (4d): For three nonnegative integers $t \geq s \geq r \geq 0$, the objective is to determine the total number of ways a club of t members can form a committee of s members, and subsequently, a subcommittee of r members chosen from within that formed committee. The following are two methods to determine this number:

Method 1: Committee then subcommittee. First, select the s members for the main committee from the t club members. There are $\binom{t}{s}$ ways. Next, select the r members for the subcommittee from the s members of the already-chosen committee. There are $\binom{s}{r}$ ways. Therefore, the total number of selections using this method is $\binom{t}{s}\binom{s}{r}$.

Method 2: Subcommittee then Committee. First, select the r members for the subcommittee directly from the t club members. There are $\binom{t}{r}$ ways. Next, select the remaining $s - r$ members needed to complete the main committee from the $t - r$ members who have not yet been chosen. There are $\binom{t-r}{s-r}$ ways. Therefore, the total number of selections using this method is $\binom{t}{r}\binom{t-r}{s-r}$.

Since both methods accurately count the same set of outcomes (the selection of the nested committee and subcommittee), the resulting counts must be equal, thus establishing the identity:

$$\binom{t}{s}\binom{s}{r} = \binom{t}{r}\binom{t-r}{s-r}$$

Problem 5

Coins X and Y, both biased, are flipped simultaneously.

- Coin X: The probability for Heads is 3/4 and the probability for Tails is 1/4.
- Coin Y: The probability for Heads is 1/3 and the probability for Tails is 2/3.

(5a) What is the probability that both coins X and Y show Tails?

Answer: $\mathbf{(1/6) = (1/4) \times (2/3)}$.

Explanation: With probability (1/4) coin X shows Tails and with probability (2/3) coin Y shows Tails. Therefore, as the events are independent, with probability (1/6) (calculated as $(1/4) \times (2/3)$) both coins show Tails.

(5b) What is the probability that both coins X and Y show Heads?

Answer: $\mathbf{(1/4) = (3/4) \times (1/3)}$.

Explanation: With probability (3/4) coin X shows Heads and with probability (1/3) coin Y shows Heads. Therefore, as the events are independent, with probability (1/4) (calculated as $(3/4) \times (1/3)$) both coins show Heads.

(5c) What is the probability that one coin shows Heads while the other coin shows Tails?

Answer: $\mathbf{(7/12) = (3/4) \times (2/3) + (1/3) \times (1/4) = 1 - (1/4) - (1/6)}$.

Explanation I: First, with probability (3/4) coin X shows Heads and with probability (2/3) coin Y shows Tails. Therefore, as the events are independent, with probability (1/2) (calculated as $(3/4) \times (2/3)$) coin X shows Heads while coin Y shows Tails. Second, with probability (1/4) coin X shows Tails and with probability 1/3 coin Y shows Heads. Therefore, as the events are independent, with probability (1/12) (calculated as $(1/4) \times (1/3)$) coin X shows Tails while coin Y shows Heads. Summing these two probabilities together, it follows that with probability (7/12) (calculated as $(1/2) + (1/12)$) one coin shows Heads while the other coin shows Tails.

Explanation II: The probability that one coin shows Heads while the other coin shows Tails is the complement probability to the event that both coins show Heads or both coins show Tails. From the previous two calculations, it follows that this probability is (7/12) (calculated as $1 - (1/4) - (1/6)$).

(5d) What is the probability that both coins X and Y show Heads given that at least one of them shows Heads?

Answer: $\mathbf{(3/10) = \frac{(1/4)}{(1/4)+(7/12)}}$.

Explanation: The probability that at least one of them shows Heads is the sum of the probability that both of them show Heads and the probability that one of them show Heads while the other shows Tails. Based on previous computations this probability is (5/6) (calculated as $(1/4) + (7/12)$). Therefore, the conditional probability that both coins X and Y show Heads (calculated previously as (1/4)) given that at least one of them shows Heads is the ratio between these two probabilities, which is (3/10) (calculated as $\frac{(1/4)}{(5/6)}$).

(5e) What is the probability that both coins X and Y show Tails given that at least one of them shows Tails?

Answer: $\mathbf{(2/9) = \frac{1/6}{(1/6)+(7/12)}}$.

Explanation: The probability that at least one of them shows Tails is the sum of the probability that both of them show Tailss and the probability that one of them show Heads while the other shows Tails. Based on previous computations this probability is (3/4) (calculated as $(1/6) + (7/12)$). Therefore, the conditional probability that both coins X and Y show Tails (calculated previously as (1/6)) given that at least one of them shows Tails is the ratio between these two probabilities, which is (2/9) (calculated as $\frac{(1/6)}{(3/4)}$).