CISC 2210 TR2 – Introduction to Discrete Structures

Midterm 2 Exam

Apr 21, 2026

Id: ....

Problem	Maximum Points	Your Points
Induction	100	
Recursion	100	
Counting	100	
Combinatorics	100	
Probability	100	

Structure, problem selection, and credit:

- You have 2 hours to complete the exam.
- There are 5 problems. Each problem is a “mini-exam” by itself with a 5% weight in the final grade for the class. However, the grade of each individual problem counts only if it is higher than the final exam grade.

  Strategy: It is better to try first answering the questions relating to topics you have mastered. Note that since there is no cumulative grade, one fully correct answer is better than two or more partially correct answers.
- You will get only partial credit if you fail to justify or prove your answers. You will get 20% of the credit for any problem or part of a problem if you leave the allocated space for the answer empty. You will get zero credit for wrong answers.

Honor code: Students are expected to do this exam **by themselves** without any external help from other people, the Internet, books, notes, or calculators. Cheaters will be punished severely. At minimum, they will fail the exam, but they may fail the whole class. In addition, students who cheat risk disciplinary measures by Brooklyn College and CUNY.

Problem 1: Induction (100 credits)

Prove **by induction** that $3^{n-1} > 2^n$ for $n \geq 3$.

Remark: Full credit will be given **only** to correct proofs that include all the stages of "proof by induction".

Problem 2: Recursion (100 credits)

Define the following recursive formula for all positive integers $n \geq 1$:

$$T(n) = \begin{cases} 3 & \text{for } n = 1 \\ 6 & \text{for } n = 2 \\ 9 & \text{for } n = 3 \\ 3T(n-2) - T(n-1) - T(n-3) + 6 & \text{for } n \geq 4 \end{cases}$$

Problem 2a (30 credits)

Compute $T(n)$ for $n = 4$, $n = 5$, and $n = 6$.

Problem 2b (20 credits)

Express $T(n)$ for $n \geq 5$ as a function of $T(n-2)$, $T(n-3)$, and $T(n-4)$ by eliminating the $T(n-1)$ term through a top-down evaluation.

Problem 2c (10 credits)

Guess a closed-form expression for $T(n)$. Justify your guess.

Problem 2d (40 credits)

Prove that your guess is correct.

Problem 3: Counting (100 credits)

Definitions: Let $S_n = (1, 2, \ldots, n)$ denote the identity sequence, where the value at each position corresponds to its index. An Err_n sequence is a sequence $(a_1, a_2, \ldots, a_n)$ of length n that deviates from S_n at exactly one position (one error), where all numbers drawn from the set $\{1, 2, \ldots, n\}$. More formally, there exists a unique index $i \in \{1, 2, \ldots, n\}$ and a value $j \in \{1, 2, \ldots, n\}$ such that $a_i = j$ (where $j \neq i$), while $a_k = k$ for all $k \neq i$.

Examples: $(1, 2, 2, 4, 5)$ is an Err_5 sequence because it differs from S_5 only at the third position, where the value is 2 instead of 3 and $(1, 2, 3, 4, 5, 1)$ is an Err_6 sequence because it differs from S_6 only at the last position, where the value is 1 instead of 6.

Problem 3a (20 credits)

List all the Err_2 sequences for $n = 2$.

Problem 3b (20 credits)

List all the Err_3 sequences for $n = 3$.

Problem 3c (20 credits)

List all the Err_4 sequences for $n = 4$.

Problem 3d (30 credits)

For $n \geq 2$, derive a formula for the total number of Err_n sequences as a function of n. Justify your answer.

Problem 3e (10 credits)

For $n \geq 2$, determine the number of Err_n sequences (as a function of n) in which the condition $a_i \neq i$ implies $a_i = i - 1$ (for $i > 1$) or $a_i = i + 1$ (for $i < n$). Justify your answer.

Problem 4: Combinatorics (100 credits)

This question asks you to prove four combinatorial identities involving binomial coefficients. You may apply combinatorial (counting) arguments in your proofs. For algebraic proofs recall that:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Remark/Hint: One possible way to prove the four identities is the following. Start by proving the fourth identity. Next, explain why the third identity is a special case of the fourth one. Next, explain why the second identity is a special case of the third one. Finally, explain why the first identity is a special case of the second one.

Problem 4a (40 credits)

Prove the following identity

$$\binom{8}{2}\binom{6}{2} = \binom{8}{4}\binom{4}{2}$$

Problem 4b (20 credits)

Prove the following identity for $x \geq 4$:

$$\binom{x}{2}\binom{x-2}{2} = \binom{x}{4}\binom{4}{2}$$

Problem 4c (20 credits)

Prove the following identity for $x \geq y \geq 2$:

$$\binom{x}{2}\binom{x-2}{y-2} = \binom{x}{y}\binom{y}{2}$$

Problem 4d (20 credits)

Prove the following identity for $x \geq y \geq z \geq 0$:

$$\binom{x}{z}\binom{x-z}{y-z} = \binom{x}{y}\binom{y}{z}$$

Problem 5: Probability (100 credits)

Two **fair** dice are thrown: one is a **5-sided** die labeled with the numbers $\{1, 2, 3, 4, 5\}$ and the other is a **3-sided** die labeled with the numbers $\{1, 2, 3\}$.

Justify your answers to the following five questions.

Problem 5a (20 credits)

What is the probability that both dice show an even number?

Problem 5b (20 credits)

What is the probability that both dice show an odd number?

Problem 5c (20 credits)

What is the probability that one die shows an even number while the other die shows an odd number?

Problem 5d (20 credits)

What is the probability that both dice show an even number, given that at least one of them shows an even number?

Problem 5e (20 credits)

What is the probability that both dice show an odd number, given that at least one of them shows an odd number?