CISC 2210 TR2 – Introduction to Discrete Structures

Midterm 2 Exam – Problems and Solutions

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Problem 1

Prove by induction that $3^{n-1} > 2^n$ for $n \geq 3$.

The inequality is false for $1 \leq n \leq 2$ and true for $3 \leq n \leq 4$:

$$\begin{array}{lcl} 3^0 = 1 & < & 2^1 = 2 \\ 3^1 = 3 & < & 2^2 = 4 \\ 3^2 = 9 & > & 2^3 = 8 \\ 3^3 = 27 & > & 2^4 = 16 \end{array}$$

Proof by induction:

- *Induction base.* $3^2 = 9 > 2^3 = 8$ for $n = 3$.
- *Induction hypothesis.* Assume that $3^{k-1} > 2^k$ for $k \geq 3$.
- *Inductive step.* Prove that $3^k > 2^{k+1}$ for $k \geq 3$.

$$\begin{array}{rcll} 3^k & = & 3 \times 3^{k-1} & (* \text{ algebra } *) \\ & > & 3 \times 2^k & (* \text{ induction hypothesis } *) \\ & > & 2 \times 2^k & (* \text{ because } 3 > 2 *) \\ & = & 2^{k+1} & (* \text{ algebra } *) \end{array}$$

Another proof: The claim follows because $3^2 = 9 > 2^3 = 8$ and $3^i \geq 2^i$ for $i \geq 0$.

$$\begin{array}{rcl} 3^{n-1} & = & 3^2 \times 3^{n-3} \\ & = & 9 \times 3^{n-3} \\ & > & 8 \times 2^{n-3} \\ & = & 2^3 \times 2^{n-3} \\ & = & 2^n \end{array}$$

Problem 2

Define the following recursive formula for all positive integers $n \geq 1$:

$$T(n) = \begin{cases} 3 & \text{for } n = 1 \\ 6 & \text{for } n = 2 \\ 9 & \text{for } n = 3 \\ 3T(n-2) - T(n-1) - T(n-3) + 6 & \text{for } n \geq 4 \end{cases}$$

(2a) Compute $T(n)$ for $n = 4$, $n = 5$, and $n = 6$.

- $T(4) = 3T(2) - T(3) - T(1) + 6 = 3 \cdot 6 - 9 - 3 + 6 = 12$.
- $T(5) = 3T(3) - T(4) - T(2) + 6 = 3 \cdot 9 - 12 - 6 + 6 = 15$.
- $T(6) = 3T(4) - T(5) - T(3) + 6 = 3 \cdot 12 - 15 - 9 + 6 = 18$.

(2b) Express $T(n)$ for $n \geq 5$ as a function of $T(n-2)$, $T(n-3)$, and $T(n-4)$ by eliminating the $T(n-1)$ term through a top-down evaluation.

$$\begin{aligned} T(n) &= 3T(n-2) - T(n-1) - T(n-3) + 6 \\ &= 3T(n-2) - (3T(n-3) - T(n-2) - T(n-4) + 6) - T(n-3) + 6 \\ &= 3T(n-2) - 3T(n-3) + T(n-2) + T(n-4) - 6 - T(n-3) + 6 \\ &= 4T(n-2) - 4T(n-3) + T(n-4) \end{aligned}$$

(2c) Guess a closed-form expression for $T(n)$.

Guess: Based on the definitions of $T(1)$, $T(2)$, and $T(3)$ and the evaluations of $T(4)$, $T(5)$, and $T(6)$, the plausible guess is $T(n) = 3n$.

(2d) Prove that your guess is correct.

Proposition: $T(n) = 3n$ For $n \geq 1$.

Proof: By induction on $n \geq 1$.

- *Induction base.*
  - $T(1) = 3 = 3 \cdot 1$ for $n = 1$.
  - $T(2) = 6 = 3 \cdot 2$ for $n = 2$.
  - $T(3) = 9 = 3 \cdot 3$ for $n = 3$.
- *Induction hypothesis.* For $n \geq 4$, assume that:
  - $T(n-1) = 3(n-1) = 3n - 3$.
  - $T(n-2) = 3(n-2) = 3n - 6$.
  - $T(n-3) = 3(n-3) = 3n - 9$.
- *Inductive step.* Prove that $T(n) = 3n$ for $n \geq 4$:

$$\begin{aligned} T(n) &= 3T(n-2) - T(n-1) - T(n-3) + 6 \\ &= 3(3n-6) - (3n-3) - (3n-9) + 6 \\ &= 9n - 18 - 3n + 3 - 3n + 9 + 6 \\ &= (9n - 3n - 3n) + (-18 + 3 + 9 + 6) \\ &= 3n \end{aligned}$$

Problem 3

Definitions: Let $S_n = (1, 2, \ldots, n)$ denote the identity sequence, where the value at each position corresponds to its index. An Err_n sequence is a sequence $(a_1, a_2, \ldots, a_n)$ of length n that deviates from S_n at exactly one position (one error), where all numbers drawn from the set $\{1, 2, \ldots, n\}$. More formally, there exists a unique index $i \in \{1, 2, \ldots, n\}$ and a value $j \in \{1, 2, \ldots, n\}$ such that $a_i = j$ (where $j \neq i$), while $a_k = k$ for all $k \neq i$.

(3a) List all the Err_2 sequences for $n = 2$.

Answer: There are $4 = 2^2$ possible sequences of length 2 using the set $\{1, 2\}$: $(1, 1)$, $(1, 2)$, $(2, 1)$, and $(2, 2)$. Given the identity sequence $S_2 = (1, 2)$, the Err_2 sequences are $(1, 1)$ (deviates at index 2) and $(2, 2)$ (deviates at index 1).

(3b) List all the Err_3 sequences for $n = 3$.

Answer: Out of the $27 = 3^3$ possible sequences of length 3 using the set $\{1, 2, 3\}$ the following six are Err_3 sequences:

$$(2,2,3) \quad (3,2,3) \quad (* \text{ The number in the } 1^{\text{st}} \text{ position is not } 1 *)$$

$$(1,1,3) \quad (1,3,3) \quad (* \text{ The number in the } 2^{\text{nd}} \text{ position is not } 2 *)$$

$$(1,2,1) \quad (1,2,2) \quad (* \text{ The number in the } 3^{\text{rd}} \text{ position is not } 3 *)$$

(3c) List all the Err_4 sequences for $n = 4$.

Answer: Out of the $256 = 4^4$ possible sequences of length 4 using the set $\{1, 2, 3, 4\}$ the following twelve are Err_4 sequences:

$$(2,2,3,4) \quad (3,2,3,4) \quad (4,2,3,4) \quad (* \text{ The number in the } 1^{\text{st}} \text{ position is not } 1 *)$$

$$(1,1,3,4) \quad (1,3,3,4) \quad (1,4,3,4) \quad (* \text{ The number in the } 2^{\text{nd}} \text{ position is not } 2 *)$$

$$(1,2,1,4) \quad (1,2,2,4) \quad (1,2,4,4) \quad (* \text{ The number in the } 3^{\text{rd}} \text{ position is not } 3 *)$$

$$(1,2,3,1) \quad (1,2,3,2) \quad (1,2,3,3) \quad (* \text{ The number in the } 4^{\text{th}} \text{ position is not } 4 *)$$

(3d) For $n \geq 2$, derive a formula for the total number of Err_n sequences as a function of n.

Answer: $n(n-1)$.

Explanation: Every Err_n sequence is uniquely determined by a pair of distinct numbers $i \neq j$ such that $\{i, j\} \subseteq \{1, 2, \ldots, n\}$, where i represents the index of the "incorrect" number and j is the value assigned to that position. There are n possible choices for i, and for each choice, there are $n-1$ available choices for the value j. Therefore, the total number of Err_n sequences is $n(n-1)$.

(3e) For $n \geq 2$, determine the number of Err_n sequences (as a function of n) in which the condition $a_i \neq i$ implies $a_i = i - 1$ (for $i > 1$) or $a_i = i + 1$ (for $i < n$).

Answer: $2n - 2$.

Explanation: There are $n - 1$ sequences where the deviation is an "increment" ($a_i = i + 1$ for $i \in \{1, \ldots, n-1\}$) and $n - 1$ sequences where the deviation is a "decrement" ($a_i = i - 1$ for $i \in \{2, \ldots, n\}$). Since by definition an Err_n sequence has exactly one such deviation, these two sets of sequences are disjoint, leading to a total of $(n-1) + (n-1) = 2n - 2$ such restricted Err_n sequences.

Problem 4

(4a) Prove the following identity.

$$\binom{8}{2}\binom{6}{2} = \binom{8}{4}\binom{4}{2}$$

Algebraic proof:

$$\begin{aligned}
\binom{8}{2}\binom{6}{2} &= \frac{8!}{2!6!} \cdot \frac{6!}{2!4!} \\
&= \frac{6!}{6!} \cdot \frac{8!}{2!} \cdot \frac{1}{2!4!} \\
&= \frac{4!}{4!} \cdot \frac{8!}{2!} \cdot \frac{1}{2!4!} \\
&= \frac{8!}{4!4!} \cdot \frac{4!}{2!2!} \\
&= \binom{8}{4}\binom{4}{2}
\end{aligned}$$

(4b) Prove the following identity for $x \geq 4$.

$$\binom{x}{2}\binom{x-2}{2} = \binom{x}{4}\binom{4}{2}$$

Algebraic proof:

$$\begin{aligned}
\binom{x}{2}\binom{x-2}{2} &= \frac{x!}{2!(x-2)!} \cdot \frac{(x-2)!}{2!(x-4)!} \\
&= \frac{(x-2)!}{(x-2)!} \cdot \frac{x!}{2!} \cdot \frac{1}{2!(x-4)!} \\
&= \frac{4!}{4!} \cdot \frac{x!}{2!} \cdot \frac{1}{2!(x-4)!} \\
&= \frac{x!}{4!(x-4)!} \cdot \frac{4!}{2!2!} \\
&= \binom{x}{4}\binom{4}{2}
\end{aligned}$$

(4c) Prove the following identity for $x \geq y \geq 2$.

$$\binom{x}{2}\binom{x-2}{y-2} = \binom{x}{y}\binom{y}{2}$$

Algebraic proof:

$$\begin{aligned}
\binom{x}{2}\binom{x-2}{y-2} &= \frac{x!}{2!(x-2)!} \cdot \frac{(x-2)!}{(y-2)!(x-y)!} \\
&= \frac{(x-2)!}{(x-2)!} \cdot \frac{x!}{2!} \cdot \frac{1}{(y-2)!(x-y)!} \\
&= \frac{y!}{y!} \cdot \frac{x!}{2!} \cdot \frac{1}{(y-2)!(x-y)!} \\
&= \frac{x!}{y!(x-y)!} \cdot \frac{y!}{2!(y-2)!} \\
&= \binom{x}{y}\binom{y}{2}
\end{aligned}$$

(4d) Prove the following identity for $x \geq y \geq z \geq 0$.

$$\binom{x}{z}\binom{x-z}{y-z} = \binom{x}{y}\binom{y}{z}$$

Algebraic proof:

$$\begin{aligned}
\binom{x}{z}\binom{x-z}{y-z} &= \frac{x!}{z!(x-z)!} \cdot \frac{(x-z)!}{(y-z)!(x-y)!} \\
&= \frac{(x-z)!}{(x-z)!} \cdot \frac{x!}{z!} \cdot \frac{1}{(y-z)!(x-y)!} \\
&= \frac{y!}{y!} \cdot \frac{x!}{z!} \cdot \frac{1}{(y-z)!(x-y)!} \\
&= \frac{x!}{y!(x-y)!} \cdot \frac{y!}{z!(y-z)!} \\
&= \binom{x}{y}\binom{y}{z}
\end{aligned}$$

Proving the (4c), (4b), and (4a) based on (4d): To establish the correctness of all four identities, it is sufficient to only prove the most general identity, **(4d)**. The other three identities are then shown to be special cases through successive substitutions as follows:

- Proving (4d) is the initial step.
- Setting $z = 2$ in **(4d)** yields **(4c)**.
- Setting $y = 4$ in **(4c)** yields **(4b)**.
- .Setting $x = 8$ in **(4b)** yields **(4a)**.

Combinatorial (counting) proof for the general identity (4d): For three nonnegative integers $x \geq y \geq z \geq 0$, the objective is to determine the total number of ways a club of x members can form a committee of y members, and subsequently, a subcommittee of z members chosen from within that formed committee. The following are two methods to determine this number:

Method 1: Subcommittee then Committee. First, select the z members for the subcommittee directly from the x club members. There are $\binom{x}{z}$ ways. Next, select the remaining $y - z$ members needed to complete the main committee from the $x - z$ members who have not yet been chosen. There are $\binom{x-z}{y-z}$ ways. Therefore, the total number of selections using this method is $\binom{x}{z}\binom{x-z}{y-z}$.

Method 2: Committee then subcommittee. First, select the y members for the main committee from the x club members. There are $\binom{x}{y}$ ways. Next, select the z members for the subcommittee from the y members of the already-chosen committee. There are $\binom{y}{z}$ ways. Therefore, the total number of selections using this method is $\binom{x}{y}\binom{y}{z}$.

Since both methods accurately count the same set of outcomes (the selection of the nested committee and subcommittee), the resulting counts must be equal, thus establishing the identity:

$$\binom{x}{z}\binom{x-z}{y-z} = \binom{x}{y}\binom{y}{z}$$

Problem 5: Solution

Two **fair** dice are thrown: one is a **5-sided** die labeled with the numbers $\{1, 2, 3, 4, 5\}$ and the other is a **3-sided** die labeled with the numbers $\{1, 2, 3\}$.

(5a) What is the probability that both dice show an even number?

Answer: $\mathbf{(2/15) = (2/5) \times (1/3)}$.

Explanation: With probability $(2/5)$ the first die shows an even number and with probability $(1/3)$ the second die shows an even number. Therefore, as the events are independent, with probability $(2/15)$ (calculated as $(2/5) \times (1/3)$) both dice show an even number.

(5b) What is the probability that both dice show an odd number?

Answer: $\mathbf{(2/5) = (3/5) \times (2/3)}$.

Explanation: With probability $(3/5)$ the first die shows an odd number and with probability $(2/3)$ the second die shows an odd number. Therefore, as the events are independent, with probability $(2/5)$ (calculated as $(3/5) \times (2/3)$) both dice show an odd number.

(5c) What is the probability that one die shows an even number while the other die shows an odd number?

Answer: $\mathbf{(7/15) = (2/5) \times (2/3) + (3/5) \times (1/3) = 1 - (2/15) - (2/5)}$.

Explanation I: First, with probability $(2/5)$ the first die shows an even number and with probability $(2/3)$ the second die shows an odd number. Therefore, as the events are independent, with probability $(4/15)$ (calculated as $(2/5) \times (2/3)$) the first die shows an even number while the second die shows an odd number. Second, with probability $(3/5)$ the first die shows an odd number and with probability $1/3$ the second die shows an even number. Therefore, as the events are independent, with probability $(1/5)$ (calculated as $(3/5) \times (1/3)$) the first die shows an odd number while the second die shows an even number. Summing these two probabilities together, it follows that with probability $(7/15)$ (calculated as $(4/15) + (1/5)$) one die shows an even number while the other die shows an odd number.

Explanation II: The probability that one die shows an even number while the other die shows an odd number is the complement probability to the event that both dice show an even number or both dice show an odd number. From the previous two calculations, it follows that this probability is $(7/15)$ (calculated as $1 - (2/15) - (2/5)$).

(5d) What is the probability that both dice show an even number, given that at least one of them shows an even number?

Answer: $\mathbf{(2/9) = \frac{(2/15)}{(2/15)+(7/15)}}$.

Explanation: The probability that at least one of them shows an even number is the sum of the probability that both of them show an even number and the probability that one of them shows an even number while the other shows an odd number. Based on previous computations, this probability is $(3/5)$ (calculated as $(2/15) + (7/15)$). Therefore, the conditional probability that both dice show an even number (calculated previously as $(2/15)$) given that at least one of them shows an even number is the ratio between these two probabilities, which is $(2/9)$ (calculated as $\frac{(2/15)}{(3/5)}$).

(5e) What is the probability that both dice show an odd number, given that at least one of them shows an odd number?

Answer: $\mathbf{(6/13) = \frac{(2/5)}{(2/5)+(7/15)}}$.

Explanation: The probability that at least one of them shows an odd number is the sum of the probability that both of them show an odd number and the probability that one of them shows an even number while the other shows an odd number. Based on previous computations, this probability is $(13/15)$ (calculated as $(2/5) + (7/15)$). Therefore, the conditional probability that both dice show an odd number (calculated previously as $(2/5)$) given that at least one of them shows an odd number is the ratio between these two probabilities, which is $(6/13)$ (calculated as $\frac{(2/5)}{(13/15)}$).