Some Conclusions

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Probability, expectation, and Fairness

The problem of points

- In the 17th century, Pierre de Fermat and Blaise Pascal were playing a coin toss game.
- In each round a fair coin was flipped.
- Fermat scored a point for every Heads while Pascal scored a point for every Tails.
- The first player to reach three points would win 120 coins.
- The game was interrupted with Fermat leading 2 points to 1.
- How should the 120 coins be divided fairly between them?

Probability, expectation, and Fairness

Resolution I: 60/60 split

- This ignores the fact that Fermat was closer to winning.
- Clearly, Fermat deserves a larger share.

Resolution II: Fermat gets all 120

- While less likely, Pascal still had a chance to come back and win.
- Pascal deserves some coins to reflect that possibility.

Resolution III: 80/40 split

- Intuitively, such a proportional allocation seems to be fair as Fermat was leading 2 to 1.
- However, it implies that Fermat would take all if his lead was 1 to 0.

Probability, expectation, and Fairness

A “fair” resolutions: 90/30 split

- To determine a fair way to divide the 120 coins, analyze the potential outcomes if the game had continued.
- Pascal needed to score two consecutive Tails to win. The probability of flipping two Tails in a row is 1/4.
- This suggests that Pascal had a 1/4 chance of winning the game, and therefore, he should receive 1/4 of the pot, which is 30 coins.

Explanation with probability expectation

- If Fermat and Pascal had continued playing the game 100 times from the same point, Pascal would have likely won approximately $9000 = 100 \times 0.75 \times 120$ coins.
- As a result, Fermat's average winnings per game would be $90 = 9000/100$ coins.

Probability, expectation, and Fairness

Generalization I

- The winner is the first to score 10 points. When the game is interrupted, Fermat had 8 points while Pascal had 7 points.
- How should the 120 coins be divided fairly between them?
- https://www.youtube.com/watch?v=C_nV3cVNjog

Generalization II

- A, B, and C play a similar game with a three-sided fair dice labeled with A, B, and C.
- Players get one point when their letter shows up.
- The game ends when one of the players scored four points.
- When the game was interrupted A had 3 points, B had 2 points, while C had 1 point.
- How should the 120 coins be divided fairly among the players?

Expressing positive integers using the number 2

A puzzle with some history

- In 1929, mathematicians and physicists at the University of Göttingen occupied their free time by solving the following puzzle:
  - Express any positive integer using the number 2 exactly four times.
  - Use only mathematical symbols (+, -, *, /, powers, radicals, ...)

Examples

- $1 = (2/2) * (2/2) = (2 * 2)/(2 + 2)$
- $2 = \sqrt{2} * \sqrt{2} * (2/2) = ((2 - 2) * 2) + 2)$
- $3 = (2^2) - (2/2) = \log_2(2 * 2 * 2)$
- $16384 = 2^{2^{2^2}}$

Expressing positive integers using the number 2

Solution with three 2s

- $n = -\log_2 \log_2 \sqrt{\sqrt{\sqrt{\cdots\sqrt{2}}}}$
- The squareroot is applied n times.

Solution with four 2s

- $n = -\log_2 \log_2 \sqrt{\sqrt{\sqrt{\cdots\sqrt{2^2}}}}$
- The squareroot is applied $n+1$ times.

Solution with k 2s

- $n = -\log_2 \log_2 \sqrt{\sqrt{\sqrt{\cdots\sqrt{((2^2)^2)^{\cdots 2}}}}}$
- The power is applied $k-3$ times and the squareroot is applied $n+k-3$ times.

Infinite Sums As Constants!

Three Divergent Sequences

- $1 - 1 + 1 - 1 + 1 - \cdots = 1/2$
- $1 - 2 + 3 - 4 + 5 - \cdots = 1/4$
- $1 + 2 + 3 + 4 + 5 + \cdots = -1/12$

$\mathbf{0} = \mathbf{1}$?

- $S = (1 - 1) + (1 - 1) + \cdots = 0 + 0 + \cdots = 0$
- $S = 1 + (-1 + 1) + (-1 + 1) + \cdots = 1 + 0 + 0 + \cdots = 1$

$S = 1 - 1 + 1 - 1 + 1 - \cdots$ Can be Any Integer!

$S = n$ for any positive integer n

$$\begin{aligned} S &= 1_1 - 1_2 + 1_3 - 1_4 + 1_5 - 1_6 + \cdots + 1_{2n-1} - 1_{2n} + \cdots \\ &= (1_1 + 1_3 + \cdots + 1_{2n-1}) + (1_{2n+1} - 1_2) + (1_{2n+3} - 1_4) + \cdots \\ &= n + 0 + 0 + \cdots \\ &= n \end{aligned}$$

$S = -n$ for any positive integer n

$$\begin{aligned} S &= 1_1 - 1_2 + 1_3 - 1_4 + 1_5 - 1_6 + \cdots + 1_{2n-1} - 1_{2n} + \cdots \\ &= (-1_2 - 1_4 - \cdots - 1_{2n}) + (1_1 - 1_{2n+2}) + (1_3 - 1_{2n+4}) + \cdots \\ &= -n + 0 + 0 + \cdots \\ &= -n \end{aligned}$$

$S = 1 - 1 + 1 - 1 + 1 - \cdots = 1/2$

Proof 1

$$\begin{aligned} S &= 1 - 1 + 1 - 1 + 1 - \cdots \\ 1 - S &= 1 - (1 - 1 + 1 - 1 + \cdots) \\ 1 - S &= 1 - 1 + 1 - 1 + 1 - \cdots \\ 1 - S &= S \\ S &= 1/2 \end{aligned}$$

Proof 2

- The partial sums sequence: $1, 0, 1, 0, 1, 0 \ldots, 1, 0, \ldots$
- The partial average sequence: $1, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{3}{5}, \frac{1}{2}, \ldots \frac{i}{2i-1}, \frac{1}{2} \ldots$
- The partial average sequence **converges** to $\frac{1}{2}$

$1 - 2 + 3 - 4 + 5 - 6 + \cdots = 1/4$

Proof

$$\begin{aligned}
Y &= 1 - 2 + 3 - 4 + 5 - 6 + \cdots \\
Y &= 0 + 1 - 2 + 3 - 4 + 5 - \cdots \\
2Y &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\
2Y &= S \\
2Y &= 1/2 \\
Y &= 1/4
\end{aligned}$$

$1+2+3+4+5+6+\cdots=-1/12$

Proof

$$\begin{aligned}
X &= 1+2+3+4+5+6+\cdots \\
4X &= 0+4+0+8+0+12+\cdots \\
X-4X &= 1-2+3-4+5-6+\cdots \\
-3X &= Y \\
-3X &= 1/4 \\
X &= -1/12
\end{aligned}$$

“Playing” With Numbers and Shapes

The $3x + 1$ problem

- Start with any positive number greater than 1. As long as the number is greater than 1: Divide it by 2 if it is even or multiply it by 3 and add 1 if it is odd. Experimentally the process always ends with 1.
- https://www.youtube.com/watch?v=m4CjXk_b8zo&list=UUjwOWaOX-c-NeLnj_YGiNEg

The Ducci challenge

- Start with 4 positive integers arranged as a circle. As long as one of the number is positive: replace the 4 numbers with their 4 non-negative differences. The process always ends up with 4 zeros.
- https://www.youtube.com/watch?v=SivHMQDnbWM

The game of life

- https://www.youtube.com/watch?v=jvSp6VHt_Pc

Puzzles

Crossing a Bridge

- https://ed.ted.com/lessons/can-you-solve-the-bridge-riddle-alex-gendler

Ants on a log

- https://www.youtube.com/watch?v=TDp-dGnaxU4&feature=youtu.be

The game of Nim

- https://ed.ted.com/lessons/can-you-solve-the-rogue-ai-riddle-dan-finkel

Cups and robbers

- https://ed.ted.com/lessons/can-you-solve-the-seven-planets-riddle-edwin-f-meyer

Autobiographical numbers

- https://ed.ted.com/lessons/can-you-solve-the-leonardo-da-vinci-riddle-tanya-khovanova

Ant on a rubber rope

- https://www.youtube.com/watch?v=lbjTGqZspjE

More Puzzles

Six "classic" relatively easy puzzles

- https://www.youtube.com/watch?v=CCV1YVyEI74

More "classic" puzzles

- https://www.youtube.com/watch?v=HCp_eN6JSac

"Interview" puzzles

- https://www.youtube.com/watch?v=JzyrbnFOaZQ
- https://www.youtube.com/watch?v=yxGf3Knon2I
- https://www.youtube.com/watch?v=N4wBTXcx9eI
- https://www.youtube.com/watch?v=NgH5MAqOtoo

Probability Paradoxes

The Simpson's paradox

- https://youtu.be/sxYrzzy3cq8
- https://youtu.be/6-y5dRDSv_A?list=PLZh9gzIvXQUvA98b5678zXrcxQyBH-Mys
- https://www.youtube.com/watch?v=ebEkn-BiW5k
- https://www.youtube.com/watch?v=E_ME4P9fQbo&vl=en
- https://www.youtube.com/watch?v=cOQFSmjljv0
- https://www.youtube.com/watch?v=ZDinnCwP3dg

Bertrand's paradox

- https://youtu.be/xy6xXEhbGa0

Illusions

Vanishing objects

- The vanishing Leprechaun: https://www.youtube.com/watch?v=9Pt8Ot3tZMI
- The vanishing line: https://www.youtube.com/watch?v=3LOCUgquM7M

The staircase paradox

- Illustration: https://upload.wikimedia.org/wikipedia/commons/2/2d/Quadrat_Diagonale.svg
- Wikipedia: https://en.wikipedia.org/wiki/Staircase_paradox
- $5 = 7$? https://www.youtube.com/watch?v=LWPOlZBXtD8
- $\pi = 4$? https://www.youtube.com/shorts/qPjhZL6cyUI

Interesting Topics

The magic of Fibonacci numbers

- https://www.youtube.com/watch?v=SjSHVDfXHQ4&vl=ja

The Golden Ratio

- https://www.youtube.com/watch?v=6nSfJEDZ_WM

The Josephus Problem

- https://www.youtube.com/watch?v=uCsD3ZGzMgE

The birthday “paradox”

- https://youtu.be/KtT_cgMzHx8

The Lazy Mathematician

- https://youtu.be/FdmApk9V2-w

The prisoner dillema

- https://www.youtube.com/watch?v=t9Lo2fgxWHw
- https://www.youtube.com/watch?v=TJCGTNIwmv8

Covid-19: Do we understand what we are told?

Three great lectures (approximately 40 minutes) from May 2020

- Exponential growth and epidemics

  https://www.youtube.com/watch?v=Kas0tIxDvrg&list=PLZHQObOWTQDOcxqQ36Vow3TdTRjkdSvT-&index=1&t=5s

- Simulating an epidemic

  https://www.youtube.com/watch?v=gxAaO2rsdIs&list=PLZHQObOWTQDOcxqQ36Vow3TdTRjkdSvT-&index=2

- The DP-3T algorithm for contact tracing

  https://www.youtube.com/watch?v=D__UaR5MQao&list=PLZHQObOWTQDOcxqQ36Vow3TdTRjkdSvT-&index=3