

# Discrete Structures: Counting and Combinatorics

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# Numbers

## How to count to 1000 on two hands

- <https://www.youtube.com/watch?v=1SMmc9gQmHQ>

## A better way to count?

- <https://www.youtube.com/watch?v=qID2B4MK7Y0>

## “Fascinating” properties of the numbers 1 to 9

- <https://www.youtube.com/watch?v=ByZLqOF-Jjk>

## There are 365 days and 12 months in a year

- $365 = 10^2 + 11^2 + 12^2 = 13^2 + 14^2$ .
- It is the only integer for which there exists an integer  $n$  such that

$$(n - 2)^2 + (n - 1)^2 + n^2 = (n + 1)^2 + (n + 2)^2$$

# The Number 142857

## The multiplication magic

- $142857 \times 1 = 142857$       $1/7 = 0.142857142857\dots$
- $142857 \times 2 = 285714$       $2/7 = 0.285714285714\dots$
- $142857 \times 3 = 428571$       $3/7 = 0.428571428571\dots$
- $142857 \times 4 = 571428$       $4/7 = 0.571428571428\dots$
- $142857 \times 5 = 714285$       $5/7 = 0.714285714285\dots$
- $142857 \times 6 = 857142$       $6/7 = 0.857142857142\dots$

## More surprises

- $142857 \times 7 = 999999$       $142 + 857 = 999$
- $(857)^2 - (142)^2 = 714285 = 5 \times 142857$

## Cyclic numbers

- <https://www.youtube.com/watch?v=WUlaUalgxqI>

# The Numbers 144 and 441

## 144 vs. 441

- Both are perfect square:  $144 = 12^2$  and  $441 = 21^2$ .
- 144 is the “reverse” of 441 and 12 is the “reverse” of 21.

## There are squares everywhere!

- All of their digits are perfect squares:  $1 = 1^2$  and  $4 = 2^2$ .
- The sum of their digits is a perfect square:  $1 + 4 + 4 = 9 = 3^2$ .
- The product of their digits is a perfect square:  $1 \times 4 \times 4 = 16 = 4^2$ .
- The sum of the digits is the square of the number of digits.
- The square of the sum of the digits of their square roots is the sum of the digits.

# Can Numbers be Interesting?

## The taxi-cub number

- 1729 is the smallest number that is the sum of two cubes in two different ways:

$$1729 = 12^3 + 1^3 = 10^3 + 9^3$$

## Story

- This number is called a taxicab number, because in a discussion between the mathematicians G. H. Hardy and Srinivasa Ramanujan about interesting and dull numbers, the former remarked that the number 1729 of the taxicab he had ridden seemed uninteresting, and the latter immediately answered that it is interesting, being the smallest number that is the sum of two cubes in two different ways.

# All the Positive Integers Are Interesting!

## Proof

- The "proof" is by contradiction.
- If there exists a non-empty set of uninteresting natural numbers, then there must be a smallest uninteresting number.
- But the smallest uninteresting number is itself interesting because it is the smallest uninteresting number: a contradiction.

## Online resources

- <https://www.youtube.com/watch?v=Ysd1XhqMbe8>
- [https://en.m.wikipedia.org/wiki/Interesting\\_number\\_paradox](https://en.m.wikipedia.org/wiki/Interesting_number_paradox)

# Euler's Identity

## The identity

$$e^{\pi i} + 1 = 0$$

$$e^{\pi i} = -1$$

## A joke???

*$\pi$  tells  $i$ : get real!*

*$i$  answers to  $\pi$ : be rational!*

*$e$  tells both of them: join me and we will be - one!*

<https://www.youtube.com/watch?v=IUTGFQpKaPU>

## Online resources

- <https://www.youtube.com/watch?v=sKtloBAuP74&t=233s>
- <https://www.youtube.com/watch?v=NXrBoWOBvIY>
- <https://www.youtube.com/watch?v=-dhHrg-KbJ0>

## Which is larger $e^\pi$ or $\pi^e$ ?

- [https://www.youtube.com/watch?v=JE\\_YeVlSqLY](https://www.youtube.com/watch?v=JE_YeVlSqLY)
- <https://www.youtube.com/watch?v=I7wis9rH2h0>

# Representing Numbers: Decimal, Binary, ...

## Different Bases For Numbers

- Online videos:

- <https://www.youtube.com/watch?v=1srwWeMe3BE>
- <https://www.youtube.com/watch?v=aW3qCcH6Dao>
- <https://www.youtube.com/watch?v=Fpm-E5v6ddc>

- A short tutorial:

- [https://www.tutorialspoint.com/computer\\_logical\\_organization/number\\_system\\_conversion.htm](https://www.tutorialspoint.com/computer_logical_organization/number_system_conversion.htm)

## The Josephus Problem

- <https://www.youtube.com/watch?v=uCsD3ZGzMgE>

# Combinations and Permutations

## Introduction with Cartoon Slides

- <http://tinytram.com/math/combinatorics/>

## Online resources

- <https://www.youtube.com/watch?v=uNS1QvDzCVw&feature=youtu.be>
- <https://www.youtube.com/watch?v=hVqq3nm0IHs>
- [https://youtu.be/LM5iOHKo\\_Fc?list=PLMyAzUai9V3ox\\_LDwl54GRkNxovx6NqQX](https://youtu.be/LM5iOHKo_Fc?list=PLMyAzUai9V3ox_LDwl54GRkNxovx6NqQX)

# Two Fundamental Counting Principles

## Additive Principle

- If event **A** can occur in  $m$  ways, and event **B** can occur in  $n$  **disjoint** ways (both events cannot occur at the same time), then the event **A or B** can occur in  $m + n$  ways.

## Multiplicative Principle

- If event **A** can occur in  $m$  ways, and each possibility for **A** allows for exactly  $n$  ways for event **B**, then the event **A and B** can occur in  $m \cdot n$  ways.

## Remarks

- Both principles generalize to more than two events.
- The **Principle of Inclusion Exclusion** is a generalization of the additive principle to many not necessarily disjoint event.

# Two Fundamental Counting Principles

## A simple demonstrating example

- A restaurant offers 8 appetizers and 14 main courses
- **Question 1:** How many choices do you have if you want to eat one dish, either an appetizer or a main course?
- **Question 2:** How many choices do you have if you want to eat one appetizer and one main course?

## Answers

- **Answer 1:** By the **additive principle** there are  $22 = 8 + 14$  choices.
- **Answer 2:** By the **multiplicative principle** there are  $112 = 8 \cdot 14$  choices.

# Four Counting Types

## Setting

- **Input:** A universal set  $\mathcal{S} = \{s_1, s_2, \dots, s_n\}$  with  $n$  distinct objects.
- **Goal:** Count the number of some structures with some parameters containing objects from  $\mathcal{S}$ .

## Structures

- **Permutations:** In how many ways can all the objects of  $\mathcal{S}$  be ordered?
- **Lists with repetitions:** How many lists with  $1 \leq k$  objects from  $\mathcal{S}$  are there when repetitions are allowed?
- **Lists without repetitions:** How many lists with  $1 \leq k \leq n$  objects from  $\mathcal{S}$  are there when repetitions are not allowed?
- **Subsets:** How many subsets of  $\mathcal{S}$  of size  $1 \leq k \leq n$  are there?

**Example:**  $\mathcal{S} = \{R, B, G, M\}$

There are  $3! = 6$  permutations of  $\{R, B, G\}$

*RBG RGB BRG BGR GRB GBR*

There are  $4! = 24$  permutations of  $\mathcal{S} = \{R, B, G, M\}$

*RBGM RBMG RGBM RGMB RMBG RMGB*  
*BRGM BRMG BGRM BGMR BMRG BMGR*  
*GRBM GRMB GBRM GBMR GMRB GMBR*  
*MRBG MRGB MBRG MBGR MGRB MGBR*

**Example:**  $\mathcal{S} = \{R, B, G, M\}$

There are  $4^2 = 16$  lists with repetitions of length 2

<i>RR</i>	<i>RB</i>	<i>RG</i>	<i>RM</i>
<i>BR</i>	<i>BB</i>	<i>BG</i>	<i>BM</i>
<i>GR</i>	<i>GB</i>	<i>GG</i>	<i>GM</i>
<i>MR</i>	<i>MB</i>	<i>MG</i>	<i>MM</i>

There are  $4^4 = 256$  lists with repetitions of length 4

	<i>RRRR</i>	<i>RRRB</i>	<i>RRRG</i>	<i>RRRM</i>	...
...	<i>BBBR</i>	<i>BBBB</i>	<i>BBBG</i>	<i>BBBM</i>	...
...	<i>GGGR</i>	<i>GGGB</i>	<i>GGGG</i>	<i>GGGM</i>	...
...	<i>MMMR</i>	<i>MMMB</i>	<i>MMMG</i>	<i>MMMM</i>	

**Example:**  $\mathcal{S} = \{R, B, G, M\}$

There are  $4 \cdot 3 = 12$  lists without repetitions of length 2

*RB*   *RG*   *RM*  
*BR*   *BG*   *BM*  
*GR*   *GB*   *GM*  
*MR*   *MB*   *MG*

There are  $4 \cdot 3 \cdot 2 = 24$  lists without repetitions of length 3

*RBG*   *RBM*   *RGB*   *RGM*   *RMB*   *RMG*  
*BRG*   *BRM*   *BGR*   *BGM*   *BMR*   *BMG*  
*GRB*   *GRM*   *GBR*   *GBM*   *GMR*   *GMB*  
*MRB*   *MRG*   *MBR*   *MBG*   *MGR*   *MGB*

**Example:**  $\mathcal{S} = \{R, B, G, M\}$

$\mathcal{S}$  has 4 subsets of size 1

$\{R\}$   $\{B\}$   $\{G\}$   $\{M\}$

$\mathcal{S}$  has 6 subsets of size 2

$\{RB\}$   $\{RG\}$   $\{RM\}$   
 $\{BG\}$   $\{BM\}$   $\{GM\}$

$\mathcal{S}$  has 4 subsets of size 3

$\{RBG\}$   $\{RBM\}$   $\{RGM\}$   $\{BGM\}$

# Permutations

## Definition

- An  $n$ -permutation  $\pi$  is a 1-1 function from the set of numbers  $\{1, 2, \dots, n\}$  to itself.
- $\pi(i) \neq \pi(j)$  for a permutation  $\pi = (\pi(1), \pi(2), \dots, \pi(n))$  for all  $1 \leq i \neq j \leq n$ .

## Counting permutations

- There are  $n! = n(n-1)(n-2) \cdots 2 \cdot 1$  permutations for  $n \geq 1$ .

## Proof

- There are  $n$  options for  $\pi(1)$ .
- There are  $n-1$  options for  $\pi(2)$ .
- $\vdots$
- There are 2 options for  $\pi(n-1)$ .
- There is only 1 option for  $\pi(n)$ .

## Permutations: Small Values of $n$

There is  $1! = 1$  permutation for  $n = 1$

(1)

There are  $2! = 2$  permutations for  $n = 2$

(12) (21)

There are  $3! = 6$  permutations for  $n = 3$

(123) (132) (213) (231) (312) (321)

There are  $4! = 24$  permutations for  $n = 4$

(1234) (1243) (1324) (1342) (1423) (1432)  
(2134) (2143) (2314) (2341) (2413) (2431)  
(3124) (3142) (3214) (3241) (3412) (3421)  
(4123) (4132) (4213) (4231) (4312) (4321)

There are  $5! = 120$  permutations for  $n = 5$

(12345) (12354)  $\dots$  (34512)  $\dots$  (54312) (54321)

# Permutations: Examples

## The 10 digits

- There are  $10! = 3628800$  different ways to arrange the 10 digits  $0, 1, \dots, 9$ .
- If numbers with less than 10 digits have leading zeros, then there are 10 billions ( $10000000000 = 10^{10}$ ) numbers with 10 digits.
- Only 0.036288% of these numbers contain all the 10 digits.

## The 26 letters

- There are  $26! = 403291461126605635584000000$  arrangements of the 26 letters of the English alphabet.
- $26! \approx 4 * 10^{27}$  which is about 400 millions billions of billions.

# Long Sequences Without Prime Numbers

## Theorem

- The following  $n$  integers are not prime numbers:

$$(n+1)! + 2, (n+1)! + 3, \dots, (n+1)! + (n+1)$$

## Proof idea

- For  $2 \leq k \leq n+1$ , the integer  $(n+1)! + k$  is divisible by  $k$  because  $k$  divides  $(n+1)!$  and  $k$  divides  $k$ .

## Corollary

- There exist sequences of consecutive integers of arbitrary length that contain no prime numbers.

## Examples

- 26, 27, 28 are not prime numbers.
- 122, 123, 124, 125 are not prime numbers.
- 722, 723, 724, 725, 726 are not prime numbers.

# There Are Exactly $10!$ Seconds in 6 Weeks

## Preprocessing

- There are  $60 = 3 \times 4 \times 5$  seconds in one minute.
- There are  $60 = 2 \times \sqrt{9} \times 10$  minutes in one hour.
- There are  $24 = \sqrt{9} \times 8$  hours in a day.
- There are 7 days in a week.
- There are 6 weeks.

## $10!$ seconds

- The number  $S$  of seconds in 6 weeks is therefore:

$$\begin{aligned} S &= 60 \times 60 \times 24 \times 7 \times 6 \\ &= (3 \times 4 \times 5) \times (2 \times \sqrt{9} \times 10) \times (\sqrt{9} \times 8) \times 7 \times 6 \\ &= 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &= 10! \end{aligned}$$

# Factorions and Fixed-Point Factorials

## Factorions

- A positive integer is called a **factorion** if it equals the sum of the factorials of its digits.
- There are only four factorions:
  - \*  $1 = 1!$
  - \*  $2 = 2!$
  - \*  $145 = 1! + 4! + 5! = 1 + 24 + 120$
  - \*  $40585 = 4! + 0! + 5! + 8! + 5! = 24 + 1 + 120 + 40320 + 120$

## Fixed-point factorials

- $n!$  is a **fixed-point** factorial if the number of digits in  $n!$  is  $n$ .
- There are only four fixed-point factorials:
  - \*  $1! = 1$
  - \*  $22! = 1, 124, 000, 727, 777, 607, 680, 000$
  - \*  $23! = 25, 852, 016, 738, 884, 976, 640, 000$
  - \*  $24! = 620, 448, 401, 733, 239, 439, 360, 000$

# Derangements

## Definition

- A **derangement** is a permutation  $\pi$  on the numbers  $1, 2, \dots, n$  such that  $\pi(i) \neq i$  for all  $1 \leq i \leq n$ .
- A **derangement** is a permutation  $\pi$  on the numbers  $1, 2, \dots, n$  without fixed points ( $\neg \exists_{1 \leq i \leq n} (\pi(i) = i)$ ).

## Examples

- (21) is the only derangement for  $n = 2$ .
- (231) and (312) are the only two derangements for  $n = 3$  while the other four permutations (123), (132), (213), and (321) are not derangements because each contains at least one fixed point.
- The **9** derangements for  $n = 4$  out of the 24 permutations:

(1234)	(1243)	(1324)	(1342)	(1423)	(1432)
(2134)	<b>(2143)</b>	(2314)	<b>(2341)</b>	<b>(2413)</b>	(2431)
(3124)	<b>(3142)</b>	(3214)	(3241)	<b>(3412)</b>	<b>(3421)</b>
<b>(4123)</b>	(4132)	(4213)	(4231)	<b>(4312)</b>	<b>(4321)</b>

# Counting Derangements

## Notation

- The number of derangements of  $n$  is  $!n$  (the **subfactorial** of  $n$ ).

## Small $n$

- $!1 = 0$ ,  $!2 = 1$ ,  $!3 = 2$ ,  $!4 = 9$ ,  $!5 = 44$ ,  $!6 = 265$ , ...

## Recursive formula

- $!1 = 0$ ,  $!2 = 1$ , and for  $n \geq 3$ :

$$!n = (n - 1)(!(n - 1) + !(n - 2))$$

- \*  $!3 = 2(!2 + !1) = 2(1 + 0) = 2$
- \*  $!4 = 3(!3 + !2) = 3(2 + 1) = 9$
- \*  $!5 = 4(!4 + !3) = 4(9 + 2) = 44$
- \*  $!6 = 5(!5 + !4) = 5(44 + 9) = 265$

# Derangements

## Non-recursive formulas

- $!n = \left[ \frac{n!}{e} \right] \approx \frac{n!}{2.718}$  where  $[x]$  is the nearest integer to  $x$ .
- $!n = n! \sum_{i=0}^n \frac{-1^i}{i!} = n!(1/2 - 1/6 + 1/24 - 1/120 + \dots)$ .

## Corollary

- About  $1/e \approx 0.367879$  of the permutations are derangements.
- About 63% of the permutations have at least one fixed point.

## An online resource

- <https://www.youtube.com/watch?v=pbXg5EI5t4c>

# Lists With Repetitions

## Definition

- $\mathcal{L} = (\ell_1, \ell_2, \dots, \ell_k)$  is an ordered list of  $k$  objects from the set  $\mathcal{S} = \{s_1, s_2, \dots, s_n\}$  if  $\ell_i \in \mathcal{S}$  for all  $1 \leq i \leq k$ .
- $\ell_i$  could be equal to  $\ell_j$  for  $1 \leq i < j \leq k$ .

## Counting the number of lists with repetitions

- There are  $n^k$  lists of length  $k$  from a set of size  $n$ .

## Proof

- For each  $1 \leq i \leq k$ , there are  $n$  options for  $\ell_i$ .
- For the  $k$  possible indices  $1 \leq i \leq k$ , there are  $n^k$  options for  $(\ell_1, \ell_2, \dots, \ell_k)$ .

# Lists With Repetitions: Examples

## Numbers

- Assuming numbers have leading zeros, then there are 10 billions ( $10000000000 = 10^{10}$ ) numbers (**lists**) with 10 digits.

## Letters

- There are  $26^3 = 17576$  possible three-letter words in English and  $26^4 = 456976$  possible four-letter words in English.
- There are less than 200000 words in the Oxford English Dictionary!

## Codes

- There are  $10^4 = 10000$  possible codes for a 4-digit lock.
- The codes: 0000, 0001 ... 4567 ... 7766 ... 9998, 9999.

# Nesting Loops

## Pseudocode

```
function  $f(n)$  (* integer  $n \geq 1$  *)  
   $c = 0$   
  for  $i = 1$  to  $n$  do  
    for  $j = 1$  to  $n$  do  
      for  $k = 1$  to  $n$  do  
        print  $(i, j, k)$   
         $c := c + 1$ 
```

## Observations

- The function  $f(n)$  prints all possible lists with repetitions  $(i, j, k)$  for which  $i, j, k \in \{1, 2, \dots, n\}$ .
- The value of  $c$  at the end is  $n^3$ .

# Strings

## Definition

- An  $n$ -ary string of length  $k$  is an ordered list  $\mathcal{D} = (d_1, d_2, \dots, d_k)$  such that  $d_i \in \{0, 1, \dots, n-1\}$  for all  $1 \leq i \leq k$ .
- In a binary string  $d_i = 0$  or  $d_i = 1$  for all  $1 \leq i \leq k$ .

## Counting strings

- There are  $n^k$  strings of length  $k$ .
- There are  $2^k$  binary strings of length  $k$ .

## Example: the 16 binary strings of length 4

0000	0001	0010	0011	0100	0101	0110	0111
1000	1001	1010	1011	1100	1101	1110	1111

## Example: the 27 ternary strings of length 3

000	001	002	010	011	012	020	021	022
100	101	102	110	111	112	120	121	122
200	201	202	210	211	212	220	221	222

# Non Homogeneous Lists

## Definition

- In a non-homogeneous list, for  $1 \leq i \leq k$ , the entry  $l_i$  gets its value from a different domain of objects denoted by  $S_i$ .
- $\mathcal{L} = (l_1, l_2, \dots, l_k)$  is an ordered non-homogeneous list of  $k$  objects if  $l_i \in S_i$  for all  $1 \leq i \leq k$ .

## Counting the number of non-homogeneous lists

- Assume  $n_i$  is the size  $S_i$ .
- Then there are  $n_1 n_2 \cdots n_k$  non-homogeneous lists of length  $k$ .

## Proof

- For each  $1 \leq i \leq k$ , there are  $n_i$  options for  $l_i$ .
- For the  $k$  possible  $1 \leq i \leq k$ , there are  $n_1 n_2 \cdots n_k$  options for  $(l_1, l_2, \dots, l_k)$ .

# Non Homogeneous Lists: Examples

## Passwords

- There are  $26^2 \cdot 10^4 = 6760000$  possible passwords of length 6 that must start with 2 letters and end with 4 digits.
- $S_1 = S_2 = \{A, B, \dots, Z\}$  and  $S_3 = S_4 = S_5 = S_6 = \{0, 1, \dots, 9\}$ .
- $AA0000, AA0001, \dots CZ9999, DA0000 \dots ZZ9998, ZZ9999$ .

## Taxi licenses

- There were only  $10 \cdot 26 \cdot 10^2 = 26000$  possible taxi license numbers in New York city that must start with a digit followed by a letter and end with two digits.
- $S_2 = \{A, B, \dots, Z\}$  and  $S_1 = S_3 = S_4 = \{0, 1, \dots, 9\}$ .
- $0A00, 0A01 \dots 5L99, 5M00 \dots 9Z98, 9Z99$ .
- To add licenses, there are now licenses like  $5L99_B$ .

# Nesting Loops

## Pseudocode

```
function  $f(r, s, t)$  (* integers  $r, s, t \geq 1$  *)  
   $c = 0$   
  for  $i = 1$  to  $r$  do  
    for  $j = 1$  to  $s$  do  
      for  $k = 1$  to  $t$  do  
        print  $(i, j, k)$   
         $c := c + 1$ 
```

## Observations

- The function  $f(r, s, t)$  prints all possible non-homogeneous lists  $(i, j, k)$  for which  $i \in \{1, 2, \dots, r\}$ ,  $j \in \{1, 2, \dots, s\}$ , and  $k \in \{1, 2, \dots, t\}$ .
- The value of  $c$  at the end is  $r \cdot s \cdot t$ .

# Lists Without Repetitions

## Definition

- $\mathcal{L} = (\ell_1, \ell_2, \dots, \ell_k)$  is an ordered list without repetitions of  $k \leq n$  objects from the set  $\mathcal{S} = \{s_1, s_2, \dots, s_n\}$ .
  - \*  $\ell_i \in \mathcal{S}$  for all  $1 \leq i \leq k$  and
  - \*  $\ell_i \neq \ell_j$  for  $1 \leq i < j \leq k$ .

## Counting the number of lists without repetitions

- There are  $n^k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$  lists without repetitions of length  $k$  on the numbers  $1, 2, \dots, n$ .

## Proof

- There are  $n$  options for  $\ell_1$ .
- There are  $n-1$  options for  $\ell_2$ .
- $\vdots$
- There are  $n-k+1$  options for  $\ell_k$ .
- In total there are  $n(n-1)(n-2)\cdots(n-k+1)$  options.

# Lists Without Repetitions: Remarks and Examples

## Remarks

- $k \leq n$  because there are only  $n$  options for each  $l_i$ .
- Permutations are lists without repetitions for which  $k = n$ .

## Three-digit numbers

- There are  $720 = 10 \cdot 9 \cdot 8$  three-digit numbers for which all the digits are different.
- 012, 013 ... 309, 310 ... 598, 601 ... 986, 987.

## Three-letter words

- There are  $26 \cdot 25 \cdot 24 = 15600$  possible three-letter words in English with three different letters.
- The Scrabble Dictionary (OWL2) recognizes only 1015 three-letter words (words with or without repetitions).

# Nesting Loops

## Pseudocode

```
function  $f(n)$  (* integer  $n \geq 1$  *)  
   $c = 0$   
  for  $i = 1$  to  $n$  do  
    for  $j = 1$  to  $n$  if  $j \neq i$  do  
      for  $k = 1$  to  $n$  if ( $k \neq i$  and  $k \neq j$ ) do  
        print ( $i, j, k$ )  
         $c := c + 1$ 
```

## Observations

- The function  $f(n)$  prints all possible lists without repetitions  $(i, j, k)$  for which  $i, j, k \in \{1, 2, \dots, n\}$  are three distinct numbers.
- The value of  $c$  at the end is  $n(n-1)(n-2)$ .

# Subsets

## Definition

- For  $n \geq 1$  and  $0 \leq k \leq n$ , a set of  $n$  objects has  $\binom{n}{k}$  different subsets of  $k$  objects.
- Equivalently, there are  $\binom{n}{k}$  different ways to select  $k$  objects from a set of  $n$  objects.

## Notations

- $\binom{n}{k}$  is called “**n choose k**”.
- Additional notations to  $\binom{n}{k}$  are  $C(n, k)$ ,  $C_{n,k}$ ,  $C_n^k$ , and  ${}^n C_k$ .

## Special cases

- $\binom{n}{0} = 1$  for  $k = 0$ : the empty set is the only subset with 0 objects and there is only one way to select 0 objects.
- $\binom{n}{n} = 1$  for  $k = n$ : the entire set is the only subset with  $n$  objects and there is only one way to select all the  $n$  objects.

# Example of Subsets

All the subsets of the set  $S = \{R, B, G, M\}$

- There is only  $\binom{4}{0} = 1$  way to choose zero colors from  $S$ :
  - \*  $\emptyset$
- There are  $\binom{4}{1} = 4$  ways to choose one color from  $S$ :
  - \*  $\{R\}, \{B\}, \{G\}, \{M\}$
- There are  $\binom{4}{2} = 6$  ways to choose two colors from  $S$ :
  - \*  $\{R, B\}, \{R, G\}, \{R, M\}, \{B, G\}, \{B, M\}, \{G, M\}$
- There are  $\binom{4}{3} = 4$  ways to choose three colors from  $S$ :
  - \*  $\{R, B, G\}, \{R, B, M\}, \{R, G, M\}, \{B, G, M\}$
- There is only  $\binom{4}{4} = 1$  way to choose four colors from  $S$ :
  - \*  $\{R, B, G, M\}$

# A Formula for $\binom{n}{k}$

## Theorem

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

## Equivalent formulas

$$\begin{aligned}\binom{n}{k} &= \frac{n^k}{k!} \\ &= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2 \cdot 1} \\ &= \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{(n-2)}{3} \cdots \frac{(n-k+1)}{k} \\ &= \prod_{i=1}^k \left( \frac{n+1-i}{i} \right)\end{aligned}$$

## Proof of Theorem

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- There are  $n$  ways to select the first object, there are  $n - 1$  ways to select the second object, and so on . . .
- There are  $(n - k + 1)$  ways to select the last  $k^{\text{th}}$  object.
- In total, there are

$$g(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1) = \frac{n!}{(n - k)!}$$

ways to select an **ordered** list of  $k$  objects from a set of  $n$  objects.

- Each subset of  $k$  objects is selected in  $k!$  different ordered lists and therefore, there are

$$\frac{g(n, k)}{k!} = \frac{n!}{k!(n - k)!}$$

ways to select a subset of  $k$  objects from a set of  $n$  objects.

# Nesting Loops

## Pseudocode

```
function  $f(n)$  (* integer  $n \geq 1$  *)  
   $c = 0$   
  for  $i = 1$  to  $n - 2$  do  
    for  $j = i + 1$  to  $n - 1$  do  
      for  $k = j + 1$  to  $n$  do  
        print  $(i, j, k)$   
       $c := c + 1$ 
```

## Observations

- The function  $f(n)$  prints all possible subsets  $\{i, j, k\}$  from the set  $\{1, 2, \dots, n\}$  such that  $i < j < k$ .
- The value of  $c$  at the end is  $\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$ .

## Calculating $\binom{n}{k}$ For Some Small Values of $n$ and $k$

$n = 4$  and  $k = 2$

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = \frac{24}{4} = 6$$

$n = 5$  and  $k = 3$

$$\binom{5}{3} = \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = \frac{120}{12} = 10$$

$n = 6$  and  $k = 3$

$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = \frac{720}{36} = 20$$

**Online calculator**

<https://www.omnicalculator.com/math/binomial-coefficient>

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{For Small } k$$

$$\binom{n}{0} = \frac{n!}{0!n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$$

$$\binom{n}{4} = \frac{n!}{4!(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{24}$$

$$\binom{n}{5} = \frac{n!}{5!(n-5)!} = \frac{n(n-1)(n-2)(n-3)(n-4)}{120}$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} \quad \text{For Small } n$$

$$\binom{2}{2} = \frac{2!}{2!0!} = \frac{2}{2 \cdot 1} = 1 = \frac{2 \cdot 1}{2}$$

$$\binom{3}{2} = \frac{3!}{2!1!} = \frac{6}{2 \cdot 1} = 3 = \frac{3 \cdot 2}{2}$$

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6 = \frac{4 \cdot 3}{2}$$

$$\binom{5}{2} = \frac{5!}{2!3!} = \frac{120}{2 \cdot 6} = 10 = \frac{5 \cdot 4}{2}$$

$$\binom{6}{2} = \frac{6!}{2!4!} = \frac{720}{2 \cdot 24} = 15 = \frac{6 \cdot 5}{2}$$

$$\binom{7}{2} = \frac{7!}{2!5!} = \frac{5040}{2 \cdot 120} = 21 = \frac{7 \cdot 6}{2}$$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6} \quad \text{For Small } n$$

$$\binom{3}{3} = \frac{3!}{3!0!} = \frac{6}{6 \cdot 1} = 1 = \frac{3 \cdot 2 \cdot 1}{6}$$

$$\binom{4}{3} = \frac{4!}{3!1!} = \frac{24}{6 \cdot 1} = 4 = \frac{4 \cdot 3 \cdot 2}{6}$$

$$\binom{5}{3} = \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = 10 = \frac{5 \cdot 4 \cdot 3}{6}$$

$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = 20 = \frac{6 \cdot 5 \cdot 4}{6}$$

$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{5040}{6 \cdot 24} = 35 = \frac{7 \cdot 6 \cdot 5}{6}$$

$$\binom{8}{3} = \frac{8!}{3!5!} = \frac{40320}{6 \cdot 120} = 56 = \frac{8 \cdot 7 \cdot 6}{6}$$

# Symmetry

## Theorem

$$\binom{n}{k} = \binom{n}{n-k}$$

## Algebraic Proof

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

# Symmetry

## Theorem

$$\binom{n}{k} = \binom{n}{n-k}$$

## Combinatorial Proof

- Selecting a subset of  $k$  objects from a set of  $n$  objects is **equivalent** to selecting the complement subset of the  $n - k$  objects not in the set.
- Therefore, the number of subsets of size  $k$  is equal to the number of subsets of size  $n - k$ .

**Example:**  $\mathcal{S} = \{C, R, B, G, M\}$

**Matching the  $\binom{5}{2} = 10$  two-subsets with the  $\binom{5}{3} = 10$  three-subsets**

$$\{G, M\} \longleftrightarrow \{C, R, B\}$$

$$\{B, M\} \longleftrightarrow \{C, R, G\}$$

$$\{B, G\} \longleftrightarrow \{C, R, M\}$$

$$\{R, M\} \longleftrightarrow \{C, B, G\}$$

$$\{R, G\} \longleftrightarrow \{C, B, M\}$$

$$\{R, B\} \longleftrightarrow \{C, G, M\}$$

$$\{C, M\} \longleftrightarrow \{R, B, G\}$$

$$\{C, G\} \longleftrightarrow \{R, B, M\}$$

$$\{C, B\} \longleftrightarrow \{R, G, M\}$$

$$\{C, R\} \longleftrightarrow \{B, G, M\}$$

# Subsets as Binary Strings

## Theorem

- The number of subsets of a set  $S$  with  $n$  objects is the same as the number of binary strings of length  $n$  which is  $2^n$ .

## Proof

- A subset  $R \subseteq S$  can be represented by the binary string  $(b_1, b_2, \dots, b_n)$  in which  $b_i = 1$  if  $s_i \in R$  and  $b_i = 0$  if  $s_i \notin R$ .
- A binary string  $(b_1, b_2, \dots, b_n)$  can be represented by a subset  $R \subseteq S$  such that  $s_i \in R$  if  $b_i = 1$  and  $s_i \notin R$  if  $b_i = 0$ .
- Thus, there is a **one-to-one mapping** from the set  $2^S$  of all the subsets of  $S$  to the set of all binary strings of length  $n$ .
- Therefore,  $|2^S| = 2^n$ .

## Example: The $2^4 = 16$ Subsets of $\{R, B, G, M\}$

$\emptyset$	$\equiv$	$(0, 0, 0, 0)$	$\{B, G\}$	$\equiv$	$(0, 1, 1, 0)$
$\{R\}$	$\equiv$	$(1, 0, 0, 0)$	$\{B, M\}$	$\equiv$	$(0, 1, 0, 1)$
$\{B\}$	$\equiv$	$(0, 1, 0, 0)$	$\{G, M\}$	$\equiv$	$(0, 0, 1, 1)$
$\{G\}$	$\equiv$	$(0, 0, 1, 0)$	$\{R, B, G\}$	$\equiv$	$(1, 1, 1, 0)$
$\{M\}$	$\equiv$	$(0, 0, 0, 1)$	$\{R, B, M\}$	$\equiv$	$(1, 1, 0, 1)$
$\{R, B\}$	$\equiv$	$(1, 1, 0, 0)$	$\{R, G, M\}$	$\equiv$	$(1, 0, 1, 1)$
$\{R, G\}$	$\equiv$	$(1, 0, 1, 0)$	$\{B, G, M\}$	$\equiv$	$(0, 1, 1, 1)$
$\{R, M\}$	$\equiv$	$(1, 0, 0, 1)$	$\{R, B, G, M\}$	$\equiv$	$(1, 1, 1, 1)$

# Subsets as Binary Strings

## Corollary

- For  $0 \leq k \leq n$ , there are  $\binom{n}{k}$  binary strings of length  $n$  with exactly  $k$  ones.

## Proof

- By definition, a set of size  $n$  has  $\binom{n}{k}$  subsets of size  $k$ .
- The one-to-one mapping in the proof of the theorem maps all the sets of size  $k$  to all the binary strings with exactly  $k$  ones.

## Special cases

- The null set  $\emptyset$  is equivalent to the all-0 binary string  $(0, 0, \dots, 0)$ .
- The set itself is equivalent to the all-1 binary string  $(1, 1, \dots, 1)$ .
- A binary string with a singleton 1 is equivalent to a singleton subset  $\{x\}$  that contains one of the objects  $x$  from the set.

# Recursive Formula

## Recursion

$$\text{for all integers } n \geq 0 \quad \binom{n}{0} = \binom{n}{n} = 1$$

$$\text{for all integers } 1 \leq k \leq n-1 \quad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

## Examples

- $6 = \binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6$
- $10 = \binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 6 + 4 = 10$
- $20 = \binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20$
- $15 = \binom{6}{4} = \binom{5}{3} + \binom{5}{4} = 10 + 5 = 15$
- $35 = \binom{7}{4} = \binom{6}{3} + \binom{6}{4} = 20 + 15 = 35$

# Recursive Formula: Combinatorial Proof I

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ : Informal proof by induction

- Consider a set with  $n$  objects  $S = \{s_1, s_2, \dots, s_n\}$ .
- There are two options for selecting a subset  $\mathcal{R}$  of  $S$  with  $k$  objects:
  - \*  $s_n \in \mathcal{R}$ : There are  $\binom{n-1}{k-1}$  different ways to select additional  $k-1$  objects out of  $s_1, s_2, \dots, s_{n-1}$ .
  - \*  $s_n \notin \mathcal{R}$ : There are  $\binom{n-1}{k}$  different ways to select  $k$  objects out of  $s_1, s_2, \dots, s_{n-1}$ .
- In total the number of ways to select  $k$  objects from a set of  $n$  objects  $\binom{n}{k}$  is also

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

## Subsets of $S = \{R, B, G, M\}$

### The six subsets of size 2 from the set $S = \{R, B, G, M\}$

- There are  $\binom{4}{2} = 6$  ways to choose two colors from  $S$ :
  - \*  $\{R, B\}, \{R, G\}, \{R, M\}, \{B, G\}, \{B, M\}, \{G, M\}$
- There are  $\binom{3}{1} = 3$  ways to choose two colors from  $S$  where one of them is **Magenta**:
  - \*  $\{R, M\}, \{B, M\}, \{G, M\}$
- There are  $\binom{3}{2} = 3$  ways to choose two colors from  $S$  none of them is **Magenta**:
  - \*  $\{R, B\}, \{R, G\}, \{B, G\}$

# Subsets of $S = \{C, R, B, G, M\}$

## The ten subsets of size 3 from the set $S = \{C, R, B, G, M\}$

- There are  $\binom{5}{3} = 10$  ways to choose three colors from  $S$ :
  - \*  $\{C, R, B\}, \{C, R, G\}, \{C, R, M\}, \{C, B, G\}, \{C, B, M\}, \{C, G, M\}$
  - \*  $\{R, B, G\}, \{R, B, M\}, \{R, G, M\}, \{B, G, M\}$
- There are  $\binom{4}{2} = 6$  ways to choose three colors from  $S$  where one of them is **Magenta**:
  - \*  $\{C, R, M\}, \{C, B, M\}, \{C, G, M\}, \{R, B, M\}, \{R, G, M\}, \{B, G, M\}$
- There are  $\binom{4}{3} = 4$  ways to choose three colors from  $S$  none of them is **Magenta**:
  - \*  $\{C, R, B\}, \{C, R, G\}, \{C, B, G\}, \{R, B, G\}$

# Recursive Formula: Algebraic Proof

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\begin{aligned} \binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(n-k)k!(n-k-1)!} \\ &= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\ &= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\ &= \frac{(k + (n-k))(n-1)!}{k!(n-k)!} \\ &= \frac{n(n-1)!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k} \end{aligned}$$

## Recursive Formula: Combinatorial Proof II

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} : \text{Informal proof by induction}$$

- $\binom{n}{k}$  is the number of binary strings of length  $n$  containing  $k$  1's.
- Some strings start with a 1 while the rest start with a 0.
- Binary strings which start with a 1:
  - \* After the 1, out of the remaining  $n - 1$  bits  $k - 1$  bits must be 1.
  - \* There are  $\binom{n-1}{k-1}$  such binary strings.
- Binary strings which start with a 0:
  - \* After the 0, out of the remaining  $n - 1$  bits  $k$  must be 1.
  - \* There are  $\binom{n-1}{k}$  such binary strings.
- Thus, the number of binary strings of length  $n$  containing  $k$  1's is

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

# Example

## The 10 binary strings of length 5 with exactly 2 ones

(11000) (10100) (10010) (10001) (01100)  
(01010) (01001) (00110) (00101) (00011)

- There are  $\binom{4}{1} = 4$  strings that start with 1:

(11000) (10100) (10010) (10001)

- There are  $\binom{4}{2} = 6$  strings that start with 0:

(01100) (01010) (01001) (00110) (00101) (00011)

- The total number of strings is  $\binom{5}{2} = \binom{4}{1} + \binom{4}{2} = 4 + 6 = 10$

# Example

## The 20 binary strings of length 6 with exactly 3 zeros

(000111) (001011) (001101) (001110) (010011)  
(010101) (010110) (011001) (011010) (011100)  
(100011) (100101) (100110) (101001) (101010)  
(101100) (110001) (110010) (110100) (111000)

- There are  $\binom{5}{2} = 10$  strings that start with 0:

(000111) (001011) (001101) (001110) (010011)  
(010101) (010110) (011001) (011010) (011100)

- There are  $\binom{5}{3} = 10$  strings that start with 1:

(100011) (100101) (100110) (101001) (101010)  
(101100) (110001) (110010) (110100) (111000)

- The total number of strings is  $\binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20$ .

## 2<sup>nd</sup> Recursive Formula

### Theorem

- For  $1 \leq k \leq n$ ,

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

### Proof

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n(n-1)!}{k(k-1)!(n-k)!} \\ &= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{n}{k} \binom{n-1}{k-1}\end{aligned}$$

## 3<sup>rd</sup> Recursive Formula

### Theorem

- For  $1 \leq k \leq n$ ,

$$\binom{n}{k} = \frac{n}{n-k} \binom{n-1}{k}$$

### Proof

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n(n-1)!}{k!(n-k)(n-k-1)!} \\ &= \frac{n}{n-k} \cdot \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{n}{n-k} \binom{n-1}{k} \end{aligned}$$

## 4<sup>th</sup> Recursive Formula

### Theorem

- For  $1 \leq k \leq n$ ,

$$\binom{n}{k} = \frac{n+1-k}{k} \binom{n}{k-1}$$

### Proof

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{(n+1-k)n!}{k(k-1)!(n+1-k)(n-k)!} \\ &= \frac{n+1-k}{k} \cdot \frac{n!}{(k-1)!(n+1-k)!} \\ &= \frac{n+1-k}{k} \binom{n}{k-1} \end{aligned}$$

# Examples

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

- $\binom{7}{3} = \binom{6}{2} + \binom{6}{3} = 15 + 20 = 35$
- $\binom{7}{3} = \frac{7}{3} \cdot \binom{6}{2} = \frac{7}{3} \cdot \frac{6 \cdot 5}{2 \cdot 1} = \frac{7}{3} \cdot 15 = 35$
- $\binom{7}{3} = \frac{7}{4} \cdot \binom{6}{3} = \frac{7}{4} \cdot \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = \frac{7}{4} \cdot 20 = 35$
- $\binom{7}{3} = \frac{5}{3} \cdot \binom{7}{2} = \frac{5}{3} \cdot \frac{7 \cdot 6}{2 \cdot 1} = \frac{5}{3} \cdot 21 = 35$

$$\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

- $\binom{8}{3} = \binom{7}{2} + \binom{7}{3} = 21 + 35 = 56$
- $\binom{8}{3} = \frac{8}{3} \cdot \binom{7}{2} = \frac{8}{3} \cdot \frac{7 \cdot 6}{2 \cdot 1} = \frac{8}{3} \cdot 21 = 56$
- $\binom{8}{3} = \frac{8}{5} \cdot \binom{7}{3} = \frac{8}{5} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \frac{8}{5} \cdot 35 = 56$
- $\binom{8}{3} = \frac{6}{3} \cdot \binom{8}{2} = 2 \cdot \frac{8 \cdot 7}{2 \cdot 1} = 2 \cdot 28 = 56$

## Another Proof for the Main Recursive Formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

- The 2<sup>nd</sup> recursive formula implies that

$$\binom{n-1}{k-1} = \frac{k}{n} \binom{n}{k}$$

- The 3<sup>rd</sup> recursive formula implies that

$$\binom{n-1}{k} = \frac{n-k}{n} \binom{n}{k}$$

- Therefore,

$$\begin{aligned} \binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{k}{n} \binom{n}{k} + \frac{n-k}{n} \binom{n}{k} \\ &= \frac{(k+n-k)}{n} \binom{n}{k} \\ &= \binom{n}{k} \end{aligned}$$

$$(x + y)^n$$

## Theorem

$$\begin{aligned}(x + y)^n &= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \\ &= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \dots + \binom{n}{k} x^{n-k} y^k + \dots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n \\ &= x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \dots + \frac{n(n-1)}{2}x^2y^{n-2} + nxy^{n-1} + y^n\end{aligned}$$

## The binomial coefficients

- Based on the above theorem,  $\binom{n}{k}$  is called **a binomial coefficient**.

$$(x + y)^n$$

## Proof

- By definition,

$$(x + y)^n = \underbrace{(x + y) \cdot (x + y) \cdots (x + y) \cdots (x + y) \cdot (x + y)}_n$$

- Using the distributive laws to get the product  $x^{n-k}y^k$ :
  - \* select  $k$  of the  $n$  terms to contribute a  $y$  to the product.
  - \* select the other  $n - k$  terms to contribute an  $x$  to the product.
- The coefficient of  $x^{n-k}y^k$  is therefore  $\binom{n}{k}$ :
  - \* the number of ways to select  $k$  objects from a set of size  $n$ .
- Summing over all possible values of  $k$  from 0 to  $n$  implies that

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$(x + y)^n$$

### Small $n$

$$\begin{aligned}(x + y)^1 &= \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 \\ &= x + y\end{aligned}$$

$$\begin{aligned}(x + y)^2 &= \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2 \\ &= x^2 + 2xy + y^2\end{aligned}$$

$$\begin{aligned}(x + y)^3 &= \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

$$\begin{aligned}(x + y)^4 &= \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

### Example

$$\begin{aligned}(3 + 2)^4 &= 3^4 + 4 \cdot 3^3 \cdot 2 + 6 \cdot 3^2 \cdot 2^2 + 4 \cdot 3 \cdot 2^3 + 2^4 \\ &= 81 + 216 + 216 + 96 + 16 = 625 = 5^4\end{aligned}$$

$$(1 + y)^n$$

## Corollary

$$\begin{aligned}(1 + y)^n &= \binom{n}{0}y^0 + \binom{n}{1}y^1 + \binom{n}{2}y^2 + \cdots + \binom{n}{n-1}y^{n-1} + \binom{n}{n}y^n \\ &= \sum_{k=0}^n \binom{n}{k}y^k\end{aligned}$$

## Example

$$\begin{aligned}(1 + y)^4 &= \binom{4}{0}y^0 + \binom{4}{1}y^1 + \binom{4}{2}y^2 + \binom{4}{3}y^3 + \binom{4}{4}y^4 \\ &= 1 + 4y + 6y^2 + 4y^3 + y^4\end{aligned}$$

$$\begin{aligned}(1 + 4)^4 &= 1 + 4 \cdot 4 + 6 \cdot 4^2 + 4 \cdot 4^3 + 4^4 \\ &= 1 + 16 + 96 + 256 + 256 = 625 = 5^4\end{aligned}$$

$$(x + 1)^n$$

## Corollary

$$\begin{aligned}(x + 1)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{n-1}x^1 + \binom{n}{n}x^0 \\ &= \sum_{k=0}^n \binom{n}{k}x^{n-k} \\ &= \sum_{k=0}^n \binom{n}{n-k}x^{n-k} \\ &= \sum_{\ell=0}^n \binom{n}{\ell}x^{\ell}\end{aligned}$$

## Example

$$\begin{aligned}(x + 1)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3 + \binom{4}{2}x^2 + \binom{4}{1}x^1 + \binom{4}{0}x^0 \\ &= x^4 + 4x^3 + 6x^2 + 4x + 1\end{aligned}$$

# Sum of All Binomial Coefficients for a Given $n$

## Examples

$$\binom{0}{0} = 1 = 2^0$$

$$\binom{1}{0} + \binom{1}{1} = 1 + 1 = 2 = 2^1$$

$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4 = 2^2$$

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8 = 2^3$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 1 + 4 + 6 + 4 + 1 = 16 = 2^4$$

# Sum of All Binomial Coefficients for a Given $n$

## Theorem

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

## Proof

$$\begin{aligned} 2^n &= (1 + 1)^n \\ &= \binom{n}{0} 1^n 1^0 + \binom{n}{1} 1^{n-1} 1^1 + \binom{n}{2} 1^{n-2} 1^2 + \cdots + \binom{n}{n} 1^0 1^n \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \\ &= \sum_{k=0}^n \binom{n}{k} \end{aligned}$$

# Number of Subsets

## Theorem

- A set of size  $n$  has  $2^n$  subsets.

## Proof

- By definition, a set of size  $n$  has  $\binom{n}{k}$  subsets of size  $k$  for  $0 \leq k \leq n$ .
- Therefore, the number of subsets of a set of size  $n$  is

$$\sum_{k=0}^{k=n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

- By the previous theorem this sum equals  $2^n$ .

# Computing $3^n$ with Powers of 2

## Corollary

$$\begin{aligned}3^n &= (1 + 2)^n \\&= \binom{n}{0} 1^n 2^0 + \binom{n}{1} 1^{n-1} 2^1 + \binom{n}{2} 1^{n-2} 2^2 + \dots + \binom{n}{n} 1^0 2^n \\&= \binom{n}{0} 2^0 + \binom{n}{1} 2^1 + \binom{n}{2} 2^2 + \dots + \binom{n}{n} 2^n \\&= \sum_{k=0}^n \binom{n}{k} 2^k\end{aligned}$$

## Computing $3^n$ with Powers of 2

$n = 3$

$$\begin{aligned}27 = 3^3 &= \binom{3}{0}2^0 + \binom{3}{1}2^1 + \binom{3}{2}2^2 + \binom{3}{3}2^3 \\ &= 1 \cdot 1 + 3 \cdot 2 + 3 \cdot 4 + 1 \cdot 8 \\ &= 1 + 6 + 12 + 8\end{aligned}$$

$n = 4$

$$\begin{aligned}81 = 3^4 &= \binom{4}{0}2^0 + \binom{4}{1}2^1 + \binom{4}{2}2^2 + \binom{4}{3}2^3 + \binom{4}{4}2^4 \\ &= 1 \cdot 1 + 4 \cdot 2 + 6 \cdot 4 + 4 \cdot 8 + 1 \cdot 16 \\ &= 1 + 8 + 24 + 32 + 16\end{aligned}$$

# Solving Problems

## Counting rectangles in a square grid

- Animation:

<https://www.youtube.com/watch?v=GfODdLHwWZw>

## Counting paths in a rectangular grid

- Lecture:

<https://www.youtube.com/watch?v=fpnNaAU0iPk&list=PLmdFyQYShrjFPLdHQxuNWvh2ct666Na3z>

- Animation:

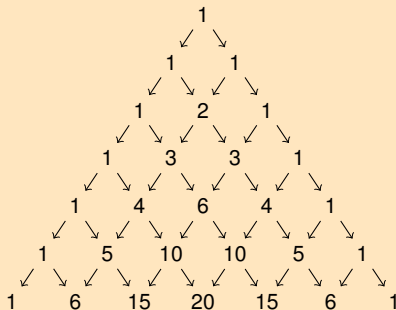
<https://www.youtube.com/watch?v=9YU10k2FYzc>

# Pascal's Triangle

## Definition

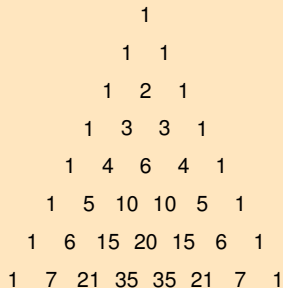
- A **triangular** array of positive integers.
- The left edge and the right edge are all 1.
- Construct the rows from top to bottom.
- Each number is the sum of the two numbers above it diagonally.

## The first 7 rows



# Pascal's Triangle

## The number triangle vs. the binomial coefficient triangle





# Pascal's Triangle

## The sum of the first 9 rows

$$\begin{array}{rcl} 1 & & = 1 = 2^0 \\ 1 + 1 & & = 2 = 2^1 \\ 1 + 2 + 1 & & = 4 = 2^2 \\ 1 + 3 + 3 + 1 & & = 8 = 2^3 \\ 1 + 4 + 6 + 4 + 1 & & = 16 = 2^4 \\ 1 + 5 + 10 + 10 + 5 + 1 & & = 32 = 2^5 \\ 1 + 6 + 15 + 20 + 15 + 6 + 1 & & = 64 = 2^6 \\ 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 & & = 128 = 2^7 \\ 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 & & = 256 = 2^8 \end{array}$$

# Pascal's Triangle

## Online video resources

- The mathematical secrets of Pascal's triangle  
<https://www.youtube.com/watch?v=XMriWTvPXHI>
- What You Don't Know About Pascal's Triangle?  
<https://www.youtube.com/watch?v=J0I1NuxUcpQ>
- Pascal's Triangle - Numberphile  
<https://www.youtube.com/watch?v=0iMtlus-af0>

## Online text resources

- Summary of facts  
<https://www.mathsisfun.com/pascals-triangle.html>
- Mysterious Patterns in Pascal's Triangle  
<https://www.cantorsparadise.com/mysterious-patterns-in-pascals-triangle-b8bad8d494e3>

# Triangular Numbers

## Definition

- The triangular number  $T_n$  counts objects arranged in an equilateral triangle whose sides each has  $n$  objects.

## Illustrations

- <https://cdn1.byjus.com/wp-content/uploads/2016/06/triangular-numbers.jpg>

## Animation

- <https://www.youtube.com/watch?v=TgQn8snKGtw>

## Recursive definition

$$T_n = \begin{cases} 1 & \text{for } n = 1 \\ T_{n-1} + n & \text{for } n > 1 \end{cases}$$

# Triangular Numbers

## Closed-form expression

$$\begin{aligned}T_n &= n + (n - 1) + (n - 2) + \cdots + 2 + 1 \\&= \frac{n(n + 1)}{2} \\&= \binom{n + 1}{2}\end{aligned}$$

## Proof

- By induction

**Visual proof for:  $n + (n - 1) + \cdots + 1 = \binom{n+1}{2}$**

- <https://www.youtube.com/watch?v=r5WloGGwVUg>



# Sum of Two Consecutive Triangular Numbers

Identity for integer  $n \geq 2$

$$T_{n-1} + T_n = \binom{n}{2} + \binom{n+1}{2} = n^2$$

Small values of  $n$

- $\binom{2}{2} + \binom{3}{2} = 1 + 3 = 4 = 2^2$
- $\binom{3}{2} + \binom{4}{2} = 3 + 6 = 9 = 3^2$
- $\binom{4}{2} + \binom{5}{2} = 6 + 10 = 16 = 4^2$
- $\binom{5}{2} + \binom{6}{2} = 10 + 15 = 25 = 5^2$
- $\binom{6}{2} + \binom{7}{2} = 15 + 21 = 36 = 6^2$

Demo

- <https://www.youtube.com/watch?v=ONCTEp-v1hU>

# Sum of Two Consecutive Triangular Numbers

Identity for integer  $n \geq 2$

$$T_{n-1} + T_n = \binom{n}{2} + \binom{n+1}{2} = n^2$$

Proof

$$\begin{aligned} \binom{n}{2} + \binom{n+1}{2} &= \frac{n(n-1)}{2} + \frac{(n+1)n}{2} \\ &= \frac{n(n-1) + (n+1)n}{2} \\ &= \frac{n((n-1) + (n+1))}{2} \\ &= \frac{n \cdot 2n}{2} \\ &= n^2 \end{aligned}$$

# Alternating Sum of First 2n Triangular Numbers

Identity for integer  $n \geq 1$

$$\sum_{k=1}^{2n} (-1)^k T_k = T_{2n} - T_{2n-1} + T_{2n-2} - T_{2n-3} + \cdots + T_2 - T_1 = 2T_n$$

Small values of  $n$

- $3 - 1 = 2 = 2T_1$
- $10 - 6 + 3 - 1 = 6 = 2T_2$
- $21 - 15 + 10 - 6 + 3 - 1 = 12 = 2T_3$
- $36 - 28 + 21 - 15 + 10 - 6 + 3 - 1 = 20 = 2T_4$

Visual proof

- [https://www.youtube.com/watch?v=Z\\_FNVP5eJrI](https://www.youtube.com/watch?v=Z_FNVP5eJrI)

# Alternating Sum of First $2n$ Triangular Numbers

Identity for integer  $n \geq 1$

$$T_{2n} - T_{2n-1} + T_{2n-2} - T_{2n-3} + \cdots + T_2 - T_1 = 2T_n$$

Proof

$$\begin{aligned}(T_{2n} - T_{2n-1}) + (T_{2n-2} - T_{2n-3}) + \cdots + (T_2 - T_1) &= 2n + (2n - 2) + \cdots + 2 \\ &= 2(n + (n - 1) + \cdots + 1) \\ &= 2T_n \\ &= 2 \cdot \frac{n(n+1)}{2} \\ &= n(n+1)\end{aligned}$$

# Alternating Sum of First $2n - 1$ Triangular Numbers

Identity for integer  $n \geq 1$

$$\sum_{k=1}^{2n-1} (-1)^k T_k = T_{2n-1} - T_{2n-2} + T_{2n-3} - T_{2n-4} + \cdots + T_3 - T_2 + T_1 = n^2$$

Small values of  $n$

- $1 = 1 = 1^2$
- $6 - 3 + 1 = 4 = 2^2$
- $15 - 10 + 6 - 3 + 1 = 9 = 3^2$
- $28 - 21 + 15 - 10 + 6 - 3 + 1 = 16 = 4^2$

Visual proof

- <https://www.youtube.com/watch?v=dRa3ItqEZwM>

# Alternating Sum of First $2n - 1$ Triangular Numbers

Identity for integer  $n \geq 1$

$$T_{2n-1} - T_{2n-2} + T_{2n-3} - T_{2n-4} + \cdots + T_3 - T_2 + T_1 = n^2$$

Proof

$$\begin{aligned}(T_{2n-1} - T_{2n-2}) + (T_{2n-3} - T_{2n-4}) + \cdots + (T_3 - T_2) + T_1 &= (2n-1) + (2n-3) + \cdots + 3 + 1 \\ &= ((2n-2) + 1) + ((2n-4) + 1) + (2+1) + 1 \\ &= ((2n-2) + (2n-4) + \cdots + 2) + n \\ &= 2((n-1) + (n-2) + \cdots + 1) + n \\ &= 2 \cdot \frac{(n-1)n}{2} + n \\ &= (n-1)n + n \\ &= n^2 - n + n \\ &= n^2\end{aligned}$$

# A Triangular Number Recurrence

Identity for integer  $n \geq 1$

$$T_{n+1} = \frac{n+2}{n} T_n$$

Proof sketch

$$nT_{n+1} = n \frac{(n+1)(n+2)}{2} = (n+2) \frac{n(n+1)}{2} = (n+2)T_n$$

Examples

- $\frac{3+2}{3} \cdot T_3 = \frac{5}{3} \cdot 6 = \frac{5 \cdot 6}{3} = 10 = T_4$
- $\frac{4+2}{4} \cdot T_4 = \frac{6}{4} \cdot 10 = \frac{6 \cdot 10}{4} = 15 = T_5$

Visual Proof

- <https://www.youtube.com/watch?v=69TvfNA0Lxc>

# Sum of First $n$ Cubes

Identity for integer  $n \geq 1$

$$\begin{aligned}\sum_{i=1}^n i^3 &= 1 + 8 + 27 + \cdots + (n-1)^3 + n^3 \\ &= T_n^2 \\ &= \binom{n+1}{2}^2 \\ &= \left(\frac{n(n+1)}{2}\right)^2 \\ &= (1 + 2 + 3 + \cdots + (n-1) + n)^2\end{aligned}$$

## Visual Proofs

- Figure: <https://i.sstatic.net/XHc4q.png>
- Animation 1: <https://www.youtube.com/watch?v=YQLicI8R4Gs>
- Animation 2: <https://www.youtube.com/watch?v=Ye9OPNqV9FA>
- Animation 3: [https://www.youtube.com/watch?v=NxOcT\\_VKQR0](https://www.youtube.com/watch?v=NxOcT_VKQR0)
- Animation 4: <https://www.youtube.com/watch?v=jWpyrXYZNiI>
- Animation 5: <https://www.youtube.com/watch?v=d1yM6Rq7Tfw>

# Sum of First $n$ Cubes

## Examples with Small $n$

$$1 = 1 = 1^2 = T_1^2$$

$$1 + 8 = 9 = 3^2 = T_2^2$$

$$1 + 8 + 27 = 36 = 6^2 = T_3^2$$

$$1 + 8 + 27 + 64 = 100 = 10^2 = T_4^2$$

$$1 + 8 + 27 + 64 + 125 = 225 = 15^2 = T_5^2$$

$$1 + 8 + 27 + 64 + 125 + 216 = 441 = 21^2 = T_6^2$$

$$1 + 8 + 27 + 64 + 125 + 216 + 343 = 784 = 28^2 = T_7^2$$

# Sum of First $n$ Cubes

Identity for integer  $n \geq 1$

$$\sum_{i=1}^n i^3 = T_n^2$$

## Proof sketch

- The proof is based on the following identity for  $i^3$

$$T_i^2 - T_{i-1}^2 = (T_i + T_{i-1})(T_i - T_{i-1}) = i^2 \cdot i = i^3$$

- As a result

$$\begin{aligned}\sum_{i=1}^n i^3 &= \sum_{i=1}^n (T_i^2 - T_{i-1}^2) \\ &= (T_n^2 - T_{n-1}^2) + (T_{n-1}^2 - T_{n-2}^2) + \cdots + (T_1^2 - T_0^2) \\ &= T_n^2\end{aligned}$$

# The Triangular Number of a Sum

Identity for integers  $n \geq 1$  and  $k \geq 1$

$$T_{n+k} = T_n + T_k + n \cdot k$$

## Examples

- $T_7 = T_3 + T_4 + 3 \cdot 4 = 6 + 10 + 12 = 28$
- $T_9 = T_2 + T_7 + 2 \cdot 7 = 3 + 28 + 14 = 45$

## Proof

$$\begin{aligned} T_{n+k} &= \frac{(n+k)(n+k+1)}{2} \\ &= \frac{n(n+1) + k(k+1) + n \cdot k + k \cdot n}{2} \\ &= \frac{n(n+1)}{2} + \frac{k(k+1)}{2} + \frac{n \cdot k + k \cdot n}{2} \\ &= T_n + T_k + n \cdot k \end{aligned}$$

# The Triangular Number of a Product

Identity for integers  $n > 1$  and  $k > 1$

$$T_{n \cdot k} = T_n \cdot T_k + T_{n-1} \cdot T_{k-1}$$

## Examples

- $T_8 = T_2 \cdot T_4 + T_1 \cdot T_3 = 3 \cdot 10 + 1 \cdot 6 = 30 + 6 = 36$
- $T_9 = T_3 \cdot T_3 + T_2 \cdot T_2 = 6 \cdot 6 + 3 \cdot 3 = 36 + 9 = 45$
- $T_{12} = T_4 \cdot T_3 + T_3 \cdot T_2 = 10 \cdot 6 + 6 \cdot 3 = 60 + 18 = 78$

## Visual proof

- <https://www.youtube.com/watch?v=p9hIwxLmCFk>

# The Triangular Number of a Product

Identity for integers  $n > 1$  and  $k > 1$

$$T_{n \cdot k} = T_n \cdot T_k + T_{n-1} \cdot T_{k-1}$$

**Proof**

$$\begin{aligned} T_{nk} &= \frac{nk(nk+1)}{2} = \frac{n^2k^2 + nk}{2} = \frac{2n^2k^2 + 2nk}{4} \\ &= \frac{2n^2k^2 + (n^2k - n^2k) + (nk^2 - nk^2) + 2nk}{4} \\ &= \frac{(n^2k^2 + n^2k + nk^2 + nk) + (n^2k^2 - n^2k - nk^2 + nk)}{4} \\ &= \frac{n^2k^2 + n^2k + nk^2 + nk}{4} + \frac{n^2k^2 - n^2k - nk^2 + nk}{4} \\ &= \frac{(n^2 + n)(k^2 + k)}{4} + \frac{(n^2 - n)(k^2 - k)}{4} \\ &= \frac{(n(n+1))(k(k+1))}{4} + \frac{((n-1)n)((k-1)k)}{4} \\ &= \frac{n(n+1)}{2} \frac{k(k+1)}{2} + \frac{(n-1)n}{2} \frac{(k-1)k}{2} \\ &= T_n T_k + T_{n-1} T_{k-1} \end{aligned}$$

# Odd Squares as Difference of Triangular Numbers

Identity for integer  $n \geq 1$

$$(2n + 1)^2 = T_{3n+1} - T_n$$

## Examples

- $(2 \cdot 3 + 1)^2 = 7^2 = 49 = 55 - 6 = T_{10} - T_3 = T_{3 \cdot 3 + 1} - T_3$
- $(2 \cdot 4 + 1)^2 = 9^2 = 81 = 91 - 10 = T_{13} - T_4 = T_{3 \cdot 4 + 1} - T_3$
- $(2 \cdot 5 + 1)^2 = 11^2 = 121 = 136 - 15 = T_{16} - T_5 = T_{3 \cdot 5 + 1} - T_3$

## Visual Proof

- <https://www.youtube.com/watch?v=hP5ExUA5P8A>

# Odd Squares as Difference of Triangular Numbers

Identity for integer  $n \geq 1$

$$(2n + 1)^2 = T_{3n+1} - T_n$$

Proof

$$\begin{aligned} T_{3n+1} - T_n &= \frac{(3n+1)(3n+2)}{2} - \frac{n(n+1)}{2} \\ &= \frac{9n^2 + 9n + 2}{2} - \frac{n^2 + n}{2} \\ &= \frac{8n^2 + 8n + 2}{2} \\ &= 4n^2 + 4n + 1 \\ &= (2n + 1)^2 \end{aligned}$$

# Sums of Powers of 9 are Triangular Numbers

Identity for integer  $n \geq 1$

$$\sum_{i=0}^{n-1} 9^i = \frac{9^n - 1}{8} = T_{\sum_{i=0}^{n-1} 3^i}$$

## Examples

- $1 + 9 = 10 = T_4 = T_{1+3}$
- $1 + 9 + 81 = 91 = T_{13} = T_{1+3+9}$

## Visual Proof


- <https://www.youtube.com/watch?v=Ch7GFdsc9pQ>

# Sum of the Reciprocals of All Triangular Numbers

## Identity and proof

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{T_n} &= \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \dots \\ &= \sum_{n=1}^{\infty} \frac{1}{\frac{n(n+1)}{2}} \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\ &= 2 \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= 2 \left( \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots \right) \\ &= 2\end{aligned}$$

## Visual proof

 <https://www.youtube.com/watch?v=I0juicS3Sic&t=119s>

# Tetrahedral Numbers

## Definition

- The tetrahedral number  $Te_n$  counts objects arranged in a four equilateral triangular faces pyramid in which each triangular face has  $T_n$  objects.

## Illustrations

- <https://www.geeksforgeeks.org/tetrahedral-numbers/>

## Animation

- [https://en.wikipedia.org/wiki/File:Pyramid\\_of\\_35\\_spheres\\_animation.gif](https://en.wikipedia.org/wiki/File:Pyramid_of_35_spheres_animation.gif)

## Recursive definition

$$Te_n = \begin{cases} 1 & \text{for } n = 1 \\ Te_{n-1} + T_n & \text{for } n > 1 \end{cases}$$

# Tetrahedral Numbers

## Closed-form expression

$$\begin{aligned}Te_n &= \sum_{i=1}^n T_i \\&= T_n + T_{n-1} + T_{n-2} + \cdots + T_2 + T_1 \\&= \sum_{i=1}^n \binom{n+1}{2} \\&= \binom{n+2}{3} \\&= \frac{n(n+1)(n+2)}{6}\end{aligned}$$

## Visual proof

<https://www.youtube.com/watch?v=N0ETyJ5K6j0>

# Tetrahedral Numbers

## Proof sketch

$$\begin{aligned}\binom{n+2}{3} &= \binom{n+1}{2} + \binom{n+1}{3} \\ &= \binom{n+1}{2} + \binom{n}{2} + \binom{n}{3} \\ &= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \binom{n-1}{3} \\ &\vdots \\ &= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \cdots + \binom{3}{2} + \binom{3}{3} \\ &= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \cdots + \binom{3}{2} + \binom{2}{2} \\ &= \sum_{i=1}^n \binom{n+1}{2}\end{aligned}$$

# Tetrahedral Numbers in Pascal's Triangle

			1	3	3	1										
		1	4	6	4	1										
		1	5	10	10	5	1									
		1	6	15	20	15	6	1								
		1	7	21	35	35	21	7	1							
		1	8	28	56	70	56	28	8	1						
		1	9	36	84	126	126	84	36	9	1					
		1	10	45	120	210	252	210	120	45	10	1				
		1	11	55	165	330	462	462	330	165	55	11	1			
		1	12	66	220	495	792	924	792	495	220	66	12	1		
		1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	
		1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1

## The Math of “The 12 Days Of Christmas”

<https://www.youtube.com/watch?v=fC8W4s6N9HQ>

# Tetrahedral Numbers as Sum of Products

Theorem for  $n \geq 1$

$$\binom{n+2}{3} = \sum_{h=1}^n h(n+1-h)$$

The binomial side

$$\binom{n+2}{3} = \frac{(n+2) \cdot (n+1) \cdot n}{3 \cdot 2 \cdot 1} = \frac{n^3 + 3n^2 + 2n}{6}$$

The sum of products side

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + h(n+1-h) + \cdots + (n-2) \cdot 3 + (n-1) \cdot 2 + n \cdot 1$$

Visual proof

 <https://www.youtube.com/watch?v=pucFDdbEyuE&t=3s>

# Tetrahedral Numbers as Sum of Products

## Examples

$$\begin{aligned}\binom{4}{3} &= \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} = 4 \\ &= 1 \cdot 2 + 2 \cdot 1 = 2 + 2 = 4\end{aligned}$$

$$\begin{aligned}\binom{5}{3} &= \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10 \\ &= 1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1 = 3 + 4 + 3 = 10\end{aligned}$$

$$\begin{aligned}\binom{6}{3} &= \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20 \\ &= 1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 4 + 6 + 6 + 4 = 20\end{aligned}$$

$$\begin{aligned}\binom{7}{3} &= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35 \\ &= 1 \cdot 5 + 2 \cdot 4 + 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 = 5 + 8 + 9 + 8 + 5 = 35\end{aligned}$$

# Tetrahedral Numbers as Sum of Products

**Combinatorial proof:**  $\binom{n+2}{3} = \sum_{h=1}^n h(n+1-h)$

- There are  $\binom{n+2}{3}$  ordered triplets  $(i < j < k)$  in the set  $\{1, \dots, n+2\}$ .
- Fix the middle index  $j$ .
- $j$  is neither 1 nor  $n+2$  and therefore  $2 \leq j \leq n+1$ .
- There are  $j-1$  ways to select  $i \in \{1, 2, \dots, j-1\}$  and there are  $n+2-j$  ways to select  $k \in \{j+1, j+2, \dots, n+2\}$ .
- Therefore, the number of triplets  $(i, j, k)$  with  $j$  as the middle index is  $(j-1)(n+2-j)$ .
- The total number of triplets is  $\sum_{j=2}^{n+1} (j-1)(n+2-j)$ .
- Replacing  $j$  with  $h+1$  implies that this number is  $\sum_{h=1}^n h(n+1-h)$ .

# Tetrahedral Numbers as Sum of Products

## The combinatorial proof using pseudocodes

- Pseudocode I: The value of  $c$  at the end is  $\binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}$ .
- Pseudocode II: The value of  $c$  is  $\sum_{j=2}^{n+1} (j-1)(n+2-j)$ .

### Pseudocode I

```
function  $f(n)$  (* integer  $n \geq 1$  *)  
   $c = 0$   
  for  $i = 1$  to  $n$  do  
    for  $j = i + 1$  to  $n + 1$  do  
      for  $k = j + 1$  to  $n + 2$  do  
         $c := c + 1$ 
```

### Pseudocode II

```
function  $f(n)$  (* integer  $n \geq 1$  *)  
   $c = 0$   
  for  $j = 2$  to  $n + 1$  do  
    for  $i = 1$  to  $j - 1$  do  
      for  $k = j + 1$  to  $n + 2$  do  
         $c := c + 1$ 
```

# Tetrahedral Numbers as Sum of Products

Example  $n = 3$ : lexicographic order vs. proof order

(123)	(123)
(124)	(124)
(125)	(125)
(134)	(134)
(135)	(135)
(145)	(234)
(234)	(235)
(235)	(145)
(245)	(245)
(345)	(345)

# Tetrahedral Numbers as Sum of Products

## Example $n = 4$ : lexicographic order vs. proof order

(123)	(123)	(234)	(145)
(124)	(124)	(235)	(146)
(125)	(125)	(236)	(245)
(126)	(126)	(245)	(246)
(134)	(134)	(246)	(345)
(135)	(135)	(256)	(346)
(136)	(136)	(345)	(156)
(145)	(234)	(346)	(256)
(146)	(235)	(356)	(356)
(156)	(236)	(456)	(456)

# Sum of Squares of Binomial Coefficients

## Theorem

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$$

## Example $n = 1$

$$\binom{2}{1} = \frac{2}{1} = 2$$

$$\binom{2}{1} = \sum_{k=0}^1 \binom{1}{k}^2$$

$$= \binom{1}{0}^2 + \binom{1}{1}^2$$

$$= 1^2 + 1^2 = 1 + 1 = 2$$

# Sum of Squares of Binomial Coefficients

## Theorem

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$$

## Example $n = 2$

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

$$\begin{aligned}\binom{4}{2} &= \sum_{k=0}^2 \binom{2}{k}^2 \\ &= \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 \\ &= 1^2 + 2^2 + 1^2 = 1 + 4 + 1 = 6\end{aligned}$$

# Sum of Squares of Binomial Coefficients

## Theorem

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$$

## Example $n = 3$

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$

$$\begin{aligned} \binom{6}{3} &= \sum_{k=0}^3 \binom{3}{k}^2 \\ &= \binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 \\ &= 1^2 + 3^2 + 3^2 + 1^2 = 1 + 9 + 9 + 1 = 20 \end{aligned}$$

# Sum of Squares of Binomial Coefficients

## Theorem

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$$

## Example $n = 4$

$$\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$$

$$\begin{aligned}\binom{8}{4} &= \sum_{k=0}^4 \binom{4}{k}^2 \\ &= \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 \\ &= 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 1 + 16 + 36 + 16 + 1 = 70\end{aligned}$$

# Sum of Squares of Binomial Coefficients

**Combinatorial proof:**  $\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$

- There are  $\binom{2n}{n}$  ways to select  $n$  numbers from the set  $\mathcal{S} = \{1, 2, \dots, 2n\}$ .
- Partition the set  $\mathcal{S}$  into two disjoint sets  $\mathcal{L} = \{1, 2, \dots, n\}$  and  $\mathcal{R} = \{n+1, n+2, \dots, 2n\}$ .
- Every selection of  $n$  numbers from  $\mathcal{S}$  is a selection of  $k$  numbers from  $\mathcal{L}$  and  $n-k$  numbers from  $\mathcal{R}$  for some  $0 \leq k \leq n$ .
- For a given  $k$ , there are  $f(n, k) = \binom{n}{k} \binom{n}{n-k}$  such selections.
- By the symmetry of the binomial coefficient,  $f(n, k) = \binom{n}{k}^2$ .
- Sum  $f(n, k)$  for all  $0 \leq k \leq n$  to get all the selections

$$\binom{2n}{n} = \sum_{k=0}^n f(n, k) = \sum_{k=0}^n \binom{n}{k}^2$$

# Sum of Squares of Binomial Coefficients

## Example $2n = 6$ : lexicographic order vs. proof order

(123)	(456)	(234)	(124)
(124)	(145)	(235)	(125)
(125)	(146)	(236)	(126)
(126)	(156)	(245)	(134)
(134)	(245)	(246)	(135)
(135)	(246)	(256)	(136)
(136)	(256)	(345)	(234)
(145)	(345)	(346)	(235)
(146)	(346)	(356)	(236)
(156)	(356)	(456)	(123)

# Sum of Products of Binomial Coefficients

## Identity I

$$\binom{r+s}{h} = \sum_{k=0}^h \binom{r}{k} \binom{s}{h-k} \quad \text{for given } h \leq r \text{ and } h \leq s$$

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2 \quad \text{for } h = r = s = n$$

## Identity II

$$\binom{n+1}{k+1} = \sum_{m=j}^{n-k+j} \binom{m}{j} \binom{n-m}{k-j} \quad \text{for given } 0 \leq j \leq k \leq n$$

$$\binom{n+1}{k+1} = \sum_{m=k}^n \binom{m}{k} \quad \text{for } j = k$$