Discrete Structures: Logic

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Boolean Functions With One Variables

Definition

- Let x be a boolean variable: $x \in \{$**TRUE,FALSE**$\}$.
- A boolean function $F(x) \in \{$**TRUE,FALSE**$\}$ is determined by its values for each one of the two possible assignments to x:
  - $*$ x is **TRUE**
  - $*$ x is **FALSE**

Notation

- $\{\mathbf{T}, \mathbf{F}\}$ for $\{$**TRUE,FALSE**$\}$

The four ($4 = 2^2$) functions

- The $\mathcal{IDENTITY}$ function: $F(x) = x$
- The $\mathcal{NOT}$ function: $F(x) = \neg x$
- The $\mathcal{TRUE}$ function: $F(x) =$ **TRUE**
- The $\mathcal{FALSE}$ function: $F(x) =$ **FALSE**

Boolean Functions With Two Variables

Definition

- Let x and y be two boolean variables: $x, y \in \{$**TRUE,FALSE**$\}$.
- A boolean function $F(x, y) \in \{$**TRUE,FALSE**$\}$ is determined by its values for each of the four possible assignments to x and y:
  - Both x and y are **TRUE**
  - x is **TRUE** and y is **FALSE**
  - x is **FALSE** and y is **TRUE**
  - Both x and y are **FALSE**

Number of two-variable functions

- There are $16 = 2^4$ possible boolean functions with two variables.

Boolean Functions With k Variables

Counting the number of k-variable boolean functions

- Let $x_1, x_2, \ldots, x_k$ be k boolean variables:
  - $(x_i \in \{$**TRUE,FALSE**$\})$ For all $1 \leq i \leq n$
- There are 2^k possible **TRUE** or **FALSE** assignments to the k variables $x_1, x_2, \ldots, x_k$.
- A boolean function $F(x_1, \ldots, x_k) \in \{$**TRUE,FALSE**$\}$ is determined by its values for each one of these 2^k assignments.
- Therefore, there are $2^{(2^k)}$ different k-variable bollean functions.

$2^{(2^k)}$ grows very fast!

- $4, 16, 256, 65536, 4294967296, 2^{64}, 2^{128} \ldots$
- There are more than 4 billions 5-variable boolean functions.

Six "Trivial" Two Variable Functions

x	y	$\mathcal{TRUE}$ T	$\mathcal{FALSE}$ F	$\mathcal{ID}(\mathcal{X})$ x	$\mathcal{ID}(\mathcal{Y})$ y	$\mathcal{NOT}(\mathcal{X})$ $\neg x$	$\mathcal{NOT}(\mathcal{Y})$ $\neg y$
T	T	T	F	T	T	F	F
T	F	T	F	T	F	F	T
F	T	T	F	F	T	T	F
F	F	T	F	F	F	T	T

Six “Useful” Or “Popular” Two Variable Functions

x	y	$\mathcal{AND}$ $x \wedge y$	$\mathcal{OR}$ $x \vee y$	$\mathcal{NAND}$ $\neg(x \wedge y)$	$\mathcal{NOR}$ $\neg(x \vee y)$	$\mathcal{XOR}$ $x \oplus y$	$\mathcal{EQUIV}$ $x \equiv y$
T	T	T	T	F	F	F	T
T	F	F	T	T	F	T	F
F	T	F	T	T	F	T	F
F	F	F	F	T	T	F	T

Representing $\mathcal{XOR}$ and $\mathcal{EQUIV}$ with $\mathcal{AND}$ and $\mathcal{OR}$

- $\mathcal{XOR}$: $(x \vee y) \wedge (\neg x \vee \neg y)$
- $\mathcal{EQUIV}$: $(x \wedge y) \vee (\neg x \wedge \neg y)$

Four Conditional Two Variable Functions

x	y	$\mathcal{X}$–$\mathcal{IMPLIES}$–$\mathcal{Y}$ $x \to y \equiv \neg x \vee y$	$\mathcal{Y}$–$\mathcal{IMPLIES}$–$\mathcal{X}$ $y \to x \equiv x \vee \neg y$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

x	y	$\mathcal{NOT}(\mathcal{X}$–$\mathcal{IMPLIES}$–$\mathcal{Y})$ $\neg(x \to y) \equiv x \wedge \neg y$	$\mathcal{NOT}(\mathcal{Y}$–$\mathcal{IMPLIES}$–$\mathcal{X})$ $\neg(y \to x) \equiv \neg x \wedge y$
T	T	F	F
T	F	T	F
F	T	F	T
F	F	F	F

All 16 Two Variables Functions

Function(x, y)	(T, T)	(T, F)	(F, T)	(F, F)
$\mathcal{TRUE}$	T	T	T	T
$\mathcal{OR}$	T	T	T	F
$\mathcal{Y}$–$\mathcal{IMPLIES}$–$\mathcal{X}$	T	T	F	T
$\mathcal{ID}(\mathcal{X})$	T	T	F	F
$\mathcal{X}$–$\mathcal{IMPLIES}$–$\mathcal{Y}$	T	F	T	T
$\mathcal{ID}(\mathcal{Y})$	T	F	T	F
$\mathcal{EQUIV}$	T	F	F	T
$\mathcal{AND}$	T	F	F	F
$\mathcal{NAND}$	F	T	T	T
$\mathcal{XOR}$	F	T	T	F
$\mathcal{NOT}(\mathcal{Y})$	F	T	F	T
$\mathcal{NOT}(\mathcal{X}$–$\mathcal{IMPLIES}$–$\mathcal{Y})$	F	T	F	F
$\mathcal{NOT}(\mathcal{X})$	F	F	T	T
$\mathcal{NOT}(\mathcal{Y}$–$\mathcal{IMPLIES}$–$\mathcal{X})$	F	F	T	F
$\mathcal{NOR}$	F	F	F	T
$\mathcal{FALSE}$	F	F	F	F

Laws of Logic: One Variable

The identity laws:

- $x \vee F \equiv x$
- $x \wedge T \equiv x$

The domination laws:

- $x \vee T \equiv T$
- $x \wedge F \equiv F$

The idempotent laws:

- $x \vee x \equiv x$
- $x \wedge x \equiv x$

The complement laws:

- $x \vee \neg x \equiv T$
- $x \wedge \neg x \equiv F$

The double negation law:

- $\neg\neg x \equiv x$

Laws of Logic: Two and Three Variables

The commutative Laws:

- $x \vee y \equiv y \vee x$
- $x \wedge y \equiv y \wedge x$

The associative laws:

- $(x \vee y) \vee z \equiv x \vee (y \vee z) \equiv (x \vee y \vee z)$
- $(x \wedge y) \wedge z \equiv x \wedge (y \wedge z) \equiv (x \wedge y \wedge z)$

The distributive laws:

- $x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$
- $x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$

The absorption laws:

- $x \vee (x \wedge y) \equiv x$
- $x \wedge (x \vee y) \equiv x$

Boolean Formulas

Definition

- A boolean **formula** (**expression**, **proposition**) is a sequence, with or without parentheses, of one or two variable boolean functions.
- The value of a boolean formula is **TRUE** (**T**) or **FALSE** (**F**) depending on the **TRUE/FALSE** assignments to all the boolean variables that appear in the formula.

Evaluating a boolean formula

- A formula is processed in order from left to right.
- Negations are evaluted first.
- Priorities are given to parentheses.

Evaluating Boolean Formulas: Examples

$(\mathbf{x} \vee \mathbf{y}) \wedge (\neg \mathbf{y} \wedge \mathbf{z})$

- The formula is **TRUE** only if both $(x \vee y)$ and $(\neg y \wedge z)$ are **TRUE**.
- $(\neg y \wedge z)$ is **TRUE** only if y is **FALSE** and z is **TRUE**.
- x must be **TRUE** to force $(x \vee y)$ to be **TRUE** when y is **FALSE**.
- In summary, the formula is **TRUE if and only if** x is **TRUE**, y is **FALSE**, and z is **TRUE**.

$(\mathbf{x} \vee \mathbf{y}) \wedge (\neg \mathbf{y} \oplus \mathbf{z}) \wedge (\neg \mathbf{z} \equiv \mathbf{w})$

- When all the variables are **TRUE** then the formula is **FALSE** because the sub-formula $(\neg z \equiv w)$ is **FALSE**.
- When all the variables except w are **TRUE** then the formula is **TRUE** because all the sub-formulas are **TRUE**.

Truth Tables

Definition

- A formula with k variables has a value for each one of the 2^k possible **TRUE/FALSE** assignments to its k variables.
- The **truth table** of a formula with k variables is a table with 2^k rows which gives the value of the formula for each one of the 2^k assignments.
- The first k columns of the truth table represent the **TRUE/FALSE** assignments to the k variables.
- The last column of the truth table represents the value of the formula.
- Usually there are additional columns that represent the values of sub-formulas helping verifying the correctness of the last column.

Truth Table for: $(x \vee y) \wedge (\neg y \wedge z)$

x	y	z	$(x \vee y)$	$\neg y$	$(\neg y \wedge z)$	formula
T	T	T	T	F	F	F
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	F	F	F
F	T	F	T	F	F	F
F	F	T	F	T	T	F
F	F	F	F	T	F	F

Truth Table for: $(x \vee y) \wedge (\neg y \oplus z) \wedge (\neg z \equiv w)$

x	y	z	w	$(x \vee y)$	$(\neg y \oplus z)$	$(\neg z \equiv w)$	formula
T	T	T	T	T	T	F	F
T	T	T	F	T	T	T	T
T	T	F	T	T	F	T	F
T	T	F	F	T	F	F	F
T	F	T	T	T	F	F	F
T	F	T	F	T	F	T	F
T	F	F	T	T	T	T	T
T	F	F	F	T	T	F	F
F	T	T	T	T	T	F	F
F	T	T	F	T	T	T	T
F	T	F	T	T	F	T	F
F	T	F	F	T	F	F	F
F	F	T	T	F	F	F	F
F	F	T	F	F	F	T	F
F	F	F	T	F	T	T	F
F	F	F	F	F	T	F	F

Satisfiability

Definition

- A boolean formula is **satisfied** if there exists at least one **TRUE/FALSE** assignment to its variables for which the value of the formula is **TRUE**.
- Such assignments are the **truth assignments** of the formula.

Example: $(x \vee y) \wedge (\neg y \wedge z)$

- This formula is satisfied and it has only one truth assignments:
  - $*$ $x = T$, $y = F$, and $z = T$

Example: $(x \vee y) \wedge (\neg y \oplus z) \wedge (\neg z \equiv w)$

- This formula is satisfied and has three truth assignments:
  - $*$ $x = T$, $y = T$, $z = T$ and $w = F$
  - $*$ $x = T$, $y = F$, $z = F$ and $w = T$
  - $*$ $x = F$, $y = T$, $z = T$ and $w = F$

Tautologies and Contradictions

Definitions

- A boolean formula is a **tautology** if its value is **TRUE** for any **TRUE/FALSE** assignment to its variables.
- A boolean formula is a **contradiction** if its value is **FALSE** for any **TRUE/FALSE** assignment to its variables.

Trivial examples

- $(x \vee \neg x)$ is a tautology.
- $(x \wedge \neg x)$ is a contradiction.

Observations

- A tautology is satisfied by all possible assignments to its variables and the entries in the last column of its truth table are all T.
- A contradiction is a non-satisfied formula and the entries in the last column of its truth table are all F.

A Tautology Example

Theorem

- $P \equiv x \vee \neg(x \wedge y)$ is a tautology.

Proof

- If x is **TRUE** then P is **TRUE**.
- If x is **FALSE** then $x \wedge y$ is **FALSE** implying that $\neg(x \wedge y)$ is **TRUE** implying that P is **TRUE**.
- It follows that P is always **TRUE**.

Truth table

x	y	$x \wedge y$	$\neg(x \wedge y)$	$x \vee \neg(x \wedge y)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

A Contradiction Example

Theorem

- $P \equiv x \wedge \neg(x \vee y)$ is a contradiction.

Proof

- If x is **FALSE** then P is **FALSE**.
- If x is **TRUE** then $x \vee y$ is **TRUE** implying that $\neg(x \vee y)$ is **FALSE** implying that P is **FALSE**.
- It follows that P is always **FALSE**.

Truth table

x	y	$x \vee y$	$\neg(x \vee y)$	$x \wedge \neg(x \vee y)$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

Proving the Associative Laws Using Truth Tables

The $\mathcal{AND}$ associative law

- $(x \wedge y) \wedge z \equiv x \wedge (y \wedge z)$

Proof

- The last columns in both truth tables are identical:

x	y	z	$x \wedge y$	$(x \wedge y) \wedge z$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

x	y	z	$y \wedge z$	$x \wedge (y \wedge z)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Proving the Associative Laws Using Truth Tables

The $\mathcal{OR}$ associative law

- $(x \vee y) \vee z \equiv x \vee (y \vee z)$

Proof

- The last columns in both truth tables are identical:

x	y	z	$x \vee y$	$(x \vee y) \vee z$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

x	y	z	$y \vee z$	$x \vee (y \vee z)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

The De Morgan's Laws

The two variables case

$$\begin{aligned} \neg(x \wedge y) &\equiv \neg x \vee \neg y \\ \neg(x \vee y) &\equiv \neg x \wedge \neg y \end{aligned}$$

The three variables case

$$\begin{aligned} \neg(x \wedge y \wedge z) &\equiv (\neg x) \vee (\neg y) \vee (\neg z) \\ \neg(x \vee y \vee z) &\equiv (\neg x) \wedge (\neg y) \wedge (\neg z) \end{aligned}$$

The four variables case

$$\begin{aligned} \neg(x \wedge y \wedge z \wedge w) &\equiv (\neg x) \vee (\neg y) \vee (\neg z) \vee (\neg w) \\ \neg(x \vee y \vee z \vee w) &\equiv (\neg x) \wedge (\neg y) \wedge (\neg z) \wedge (\neg w) \end{aligned}$$

The De Morgan's Laws

A "binary" world

- The outside is **rainy** (**R**) or **sunny** (**S**).
- The outside is **cold** (**C**) or **hot** (**H**).

Person I

- Alice does not leave the house if it is (**rainy and cold**) outside.
- Alice left the house: $\neg(\mathbf{R} \wedge \mathbf{C}) \equiv (\neg\mathbf{R} \vee \neg\mathbf{C}) \equiv (\mathbf{S} \vee \mathbf{H})$.
- It is **sunny or hot** outside.

Person II

- Bob does not leave the house if it is (**rainy or cold**) outside.
- Bob left the house: $\neg(\mathbf{R} \vee \mathbf{C}) \equiv (\neg\mathbf{R} \wedge \neg\mathbf{C}) \equiv (\mathbf{S} \wedge \mathbf{H})$.
- It is **sunny and hot** outside.

Proof with a Truth Table: $\neg(x \wedge y) \equiv \neg x \vee \neg y$

$\neg(x \wedge y)$

x	y	$x \wedge y$	$\neg(x \wedge y)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

$\neg x \vee \neg y$

x	y	$\neg x$	$\neg y$	$\neg x \vee \neg y$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Proof with a Truth Table: $\neg(x \vee y) \equiv \neg x \wedge \neg y$

$\neg(x \vee y)$

x	y	$x \vee y$	$\neg(x \vee y)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

$\neg x \wedge \neg y$

x	y	$\neg x$	$\neg y$	$\neg x \wedge \neg y$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Proof with a Truth Table: $\neg(x \wedge y \wedge z) \equiv \neg x \vee \neg y \vee \neg z$

$\neg(x \wedge y \wedge z)$

x	y	z	$x \wedge y \wedge z$	$\neg(x \wedge y \wedge z)$
T	T	T	T	F
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

$\neg x \vee \neg y \vee \neg z$

x	y	z	$\neg x$	$\neg y$	$\neg z$	$\neg x \vee \neg y \vee \neg z$
T	T	T	F	F	F	F
T	T	F	F	F	T	T
T	F	T	F	T	F	T
T	F	F	F	T	T	T
F	T	T	T	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	F	T
F	F	F	T	T	T	T

Proof with a Truth Table: $\neg(x \vee y \vee z) \equiv \neg x \wedge \neg y \wedge \neg z$

$\neg(x \vee y \vee z)$

x	y	z	$x \vee y \vee z$	$\neg(x \vee y \vee z)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F
T	F	F	T	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

$\neg x \wedge \neg y \wedge \neg z$

x	y	z	$\neg x$	$\neg y$	$\neg z$	$\neg x \wedge \neg y \wedge \neg z$
T	T	T	F	F	F	F
T	T	F	F	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	T	F	F	F
F	T	F	T	F	T	F
F	F	T	T	T	F	F
F	F	F	T	T	T	T

The General De Morgan's Laws

Complement of Conjunctions $\equiv$ Disjunction of Complements

$$\left(\bigwedge_{i=1}^{n} x_i\right)' \equiv (x_1 \wedge x_2 \wedge \cdots \wedge x_n)' \equiv (x_1' \vee x_2' \vee \cdots \vee x_n') \equiv \bigvee_{i=1}^{n} x_i'$$

Complement of Disjunctions $\equiv$ Conjunction of Complements

$$\left(\bigvee_{i=1}^{n} x_i\right)' \equiv (x_1 \vee x_2 \vee \cdots \vee x_n)' \equiv (x_1' \wedge x_2' \wedge \cdots \wedge x_n') \equiv \bigwedge_{i=1}^{n} x_i'$$

The General De Morgan's Laws

Theorem

- Complement of Conjunctions $\equiv$ Disjunction of Complements.

Proof

- Let $L = (x_1 \wedge x_2 \wedge \cdots \wedge x_n)'$ and $R = (x_1' \vee x_2' \vee \cdots \vee x_n')$
- Prove that L is **TRUE** if and only if R is **TRUE**

$$\begin{aligned}
L \equiv \textbf{TRUE} \quad & \Leftrightarrow \quad (x_1 \wedge x_2 \wedge \cdots \wedge x_n)' \equiv \textbf{TRUE} \\
& \Leftrightarrow \quad (x_1 \wedge x_2 \wedge \cdots \wedge x_n) \equiv \textbf{FALSE} \\
& \Leftrightarrow \quad (x_i \equiv \textbf{FALSE}) \textbf{ for at least one } i \in \{1, \ldots, n\} \\
& \Leftrightarrow \quad (x_i' \equiv \textbf{TRUE}) \textbf{ for at least one } i \in \{1, \ldots, n\} \\
& \Leftrightarrow \quad (x_1' \vee x_2' \vee \cdots \vee x_n') \equiv \textbf{TRUE} \\
& \Leftrightarrow \quad R \equiv \textbf{TRUE}
\end{aligned}$$

The General De Morgan's Laws

Theorem

- Complement of Disjunctions $\equiv$ Conjunction of Complements.

Proof

- Let $L = (x_1 \vee x_2 \vee \cdots \vee x_n)'$ and $R = (x_1' \wedge x_2' \wedge \cdots \wedge x_n')$
- Prove that L is **TRUE** if and only if R is **TRUE**

$$\begin{aligned}
L \equiv \textbf{TRUE} &\Leftrightarrow (x_1 \vee x_2 \vee \cdots \vee x_n)' \equiv \textbf{TRUE} \\
&\Leftrightarrow (x_1 \vee x_2 \vee \cdots \vee x_n) \equiv \textbf{FALSE} \\
&\Leftrightarrow (x_i \equiv \textbf{FALSE}) \textbf{ for all } i \in \{1, \ldots, n\} \\
&\Leftrightarrow (x_i' \equiv \textbf{TRUE}) \textbf{ for all } i \in \{1, \ldots, n\} \\
&\Leftrightarrow (x_1' \wedge x_2' \wedge \cdots \wedge x_n') \equiv \textbf{TRUE} \\
&\Leftrightarrow R \equiv \textbf{TRUE}
\end{aligned}$$

Logic puzzles

The lady or the tiger puzzle

- https://www.youtube.com/watch?v=eUYWXqk5Ags

Knights and knaves

- https://www.youtube.com/watch?v=C6PeX4iKJbU
- A good introduction to propositional logic

Fork in the road

- https://www.youtube.com/watch?v=MsV3XRLxX6Q
- https://www.youtube.com/watch?v=F8s03KBjeY0

The Function $\mathcal{IMPLY}$

x implies y

x	y	$x \rightarrow y$	$\neg x \vee y$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Four interpretations of $x \rightarrow y$

- If x then y.
- x only if y.
- x is a sufficient condition for y.
- y is a necessary condition for x.

The Function $\mathcal{IMPLY}$

$\mathcal{IMPLY}$ for propositions P, Q

- **Conditional:** $P \rightarrow Q$ is equivalent to **Contrapositive:** $\neg Q \rightarrow \neg P$
- **Inverse:** $\neg P \rightarrow \neg Q$ is equivalent to **Converse:** $Q \rightarrow P$

The Wason Selection task

- https://www.youtube.com/watch?v=XjEEEVXqCMo&feature=youtu.be
- https://www.youtube.com/watch?v=hzS3BOwvT4I

Why $x \rightarrow y$ is TRUE when x is FALSE?

- https://www.youtube.com/watch?v=xag9TEAORK4&feature=youtu.be

If, Only If, and If and only If (iff)

The propositions P and Q

- P: "Passing".
- Q: "Good grades".

The sufficient condition

- $Q \rightarrow P$: You pass **If** you have good grades.
  - $\mathcal{IMPLY}(T, T) = T$: You pass with good grades.
  - $\mathcal{IMPLY}(T, F) = F$: You cannot fail with good grades.
  - $\mathcal{IMPLY}(F, T) = T$: You might pass with bad grades.
  - $\mathcal{IMPLY}(F, F) = T$: You might fail with bad grades.

The equivalent necessary condition

- You have good grades **only if** you pass.

If, Only If, and If and only If (iff)

The propositions P and Q

- P: “Passing”.
- Q: “Good grades”.

The necessary condition

- $P \to Q$: You pass **only if** you have good grades.
  - $\mathcal{IMPLY}(T, T) = T$: You might pass with good grades.
  - $\mathcal{IMPLY}(T, F) = F$: You cannot pass with bad grades.
  - $\mathcal{IMPLY}(F, T) = T$: You might fail with good grades.
  - $\mathcal{IMPLY}(F, F) = T$: You fail with bad grades.

The equivalent sufficient condition

- You have good grades **if** you pass.

If, Only If, and If and only If (iff)

The propositions P and Q

- P: "Passing".
- Q: "Good grades".

The necessary and sufficient condition

- $P \equiv Q$: You pass **if and only if** you have good grades.
  - $\mathcal{EQUIV}(T, T) = T$: You pass with good grades.
  - $\mathcal{EQUIV}(T, F) = F$: You cannot pass with bad grades.
  - $\mathcal{EQUIV}(F, T) = F$: You cannot fail with good grades.
  - $\mathcal{EQUIV}(F, F) = T$: You fail with bad grades.

Be Careful with Your Words!

Bad Defense Attorney

- Prosecutor: “If the defendants are guilty, then they had accomplices”.
- Defense attorney: “This is not true!!”

What is the defense attorney claiming?

- P: “The defendants are guilty”.
- Q: “The defendants had accomplices”.
- Prosecutor: $P \rightarrow Q$.
- Defense attorney: $P \rightarrow Q$ is **FALSE**.

Conclusion

- $P \rightarrow Q$ is **FALSE** only when P is **TRUE** and Q is **FALSE**.
- $P \rightarrow Q$ is **FALSE** only when the defendants are guilty and they did not have accomplices.

Laws With Two Propositions

Notations

- Let P and Q be boolean propositions.

$\mathcal{IMPLY}$

- $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
- $\neg P \rightarrow \neg Q \equiv Q \rightarrow P$

$\mathcal{EQUIV}$

- $(P \equiv Q) \equiv (\neg Q \equiv \neg P)$

$\mathcal{IMPLY}$ **and** $\mathcal{EQUIV}$

- $(P \rightarrow Q) \wedge (Q \rightarrow P) \equiv (P \equiv Q)$

$\mathcal{IMPLY}$ and $\mathcal{EQUIV}$

Theorem

- $(P \rightarrow Q) \wedge (Q \rightarrow P) \equiv (P \equiv Q)$

Proof with truth tables

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

P	Q	$P \equiv Q$
T	T	T
T	F	F
F	T	F
F	F	T

Laws With Three Propositions

Notations:

- Let P and Q and R be three boolean propositions.

The $\mathcal{IMPLY}$ and $\mathcal{EQUIV}$ transitive laws

- $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$
- $((P \equiv Q) \wedge (Q \equiv R)) \rightarrow (P \equiv R)$

The $\mathcal{IMPLY}$ distributive laws

- $((P \rightarrow Q) \wedge (P \rightarrow R)) \equiv (P \rightarrow (Q \wedge R))$
- $((P \rightarrow Q) \vee (P \rightarrow R)) \equiv (P \rightarrow (Q \vee R))$
- $((Q \rightarrow P) \wedge (R \rightarrow P)) \equiv ((Q \vee R) \rightarrow P)$
- $((Q \rightarrow P) \vee (R \rightarrow P)) \equiv ((Q \wedge R) \rightarrow P)$

Truth Tables: $((P \to Q) \land (P \to R)) \equiv (P \to (Q \land R))$

$(P \to Q) \land (P \to R)$

P	Q	R	$P \to Q$	$P \to R$	$(P \to Q) \land (P \to R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

$P \to (Q \land R)$

P	Q	R	$Q \land R$	$P \to (Q \land R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Truth Tables: $((P \to Q) \lor (P \to R)) \equiv (P \to (Q \lor R))$

$(P \to Q) \lor (P \to R)$

P	Q	R	$P \to Q$	$P \to R$	$(P \to Q) \lor (P \to R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

$P \to (Q \lor R)$

P	Q	R	$Q \lor R$	$P \to (Q \lor R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

Truth Tables: $((Q \rightarrow P) \wedge (R \rightarrow P)) \equiv ((Q \vee R) \rightarrow P)$

$(Q \rightarrow P) \wedge (R \rightarrow P)$

P	Q	R	$Q \rightarrow P$	$R \rightarrow P$	$(Q \rightarrow P) \wedge (R \rightarrow P)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	T	F	F	T	F
F	F	T	T	F	F
F	F	F	T	T	T

$(Q \vee R) \rightarrow P$

P	Q	R	$Q \vee R$	$(Q \vee R) \rightarrow P$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Truth Tables: $((Q \to P) \vee (R \to P)) \equiv ((Q \wedge R) \to P)$

$(Q \to P) \vee (R \to P)$

P	Q	R	$Q \to P$	$R \to P$	$(Q \to P) \vee (R \to P)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	T	F	T
F	F	F	T	T	T

$(Q \wedge R) \to P$

P	Q	R	$Q \wedge R$	$(Q \wedge R) \to P$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	F
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

The $((P \to Q) \wedge (P \to R)) \equiv (P \to (Q \wedge R))$ law

A "binary" world

- The outside is (**rainy** or **sunny**) and (**cold** or **hot**).
- Alice does not leave the house if it is (**rainy and cold**) outside.

Notations

- P: "staying in the house".
- Q: "rainy outside".
- R: "cold outside".

The interpretations of the two equivalent sides of the law

- (If Alice stayed in the house then it is **rainy** outside) **and** (If Alice stayed in the house then it is **cold** outside).
- If Alice stayed in the house then it is (**rainy and cold**) outside.

The $((P \to Q) \vee (P \to R)) \equiv (P \to (Q \vee R))$ law

A "binary" world

- The outside is (**rainy** or **sunny**) and (**cold** or **hot**).
- Bob does not leave the house if it is (**rainy or cold**) outside.

Notations

- P: "staying in the house".
- Q: "rainy outside".
- R: "cold outside".

The interpretations of the two equivalent sides of the law

- (If Bob stayed in the house then it is **rainy** outside) **or** (If Bob stayed in the house then it is **cold** outside).
- If Bob stayed in the house then it is (**rainy or cold**) outside.

The $((Q \rightarrow P) \wedge (R \rightarrow P)) \equiv ((Q \vee R) \rightarrow P)$ law

A "binary" world

- The outside is (**rainy** or **sunny**) and (**cold** or **hot**).
- Alice does not leave the house if it is (**rainy and cold**) outside.

Notations

- P: "leaving the house".
- Q: "sunny outside".
- R: "hot outside".

The interpretations of the two equivalent sides of the law

- (If it is **sunny** outside then Alice will leave the house) **and** (If it is **hot** outside then Alice will leave the house).
- If it is (**sunny or hot**) outside then Alice will leave the house.

The $((Q \to P) \vee (R \to P)) \equiv ((Q \wedge R) \to P)$ law

A “binary” world

- The outside is (**rainy** or **sunny**) and (**cold** or **hot**).
- Bob does not leave the house if it is (**rainy or cold**) outside.

Notations

- P: “leaving the house”.
- Q: “sunny outside”.
- R: “hot outside”.

The interpretations of the two equivalent sides of the law

- (If it is **sunny** outside then Bob will leave the house) **or** (If it is **hot** outside then Bob will leave the house).
- If it is (**sunny and hot**) outside then Bob will leave the house.

Sport (P), Happiness (Q), and Good Health (R)

$((P \rightarrow Q) \wedge (P \rightarrow R)) \equiv P \rightarrow (Q \wedge R)$

- "Sport is sufficient for happiness **and** sport is sufficient for good health" **is equivalent to** "sport is sufficient for happiness **and** good health".

$((P \rightarrow Q) \vee (P \rightarrow R)) \equiv P \rightarrow (Q \vee R)$

- "Sport is sufficient for happiness **or** sport is sufficient for good health" **is equivalent to** "sport is sufficient for happiness **or** good health".

$((P \rightarrow R) \wedge (Q \rightarrow R)) \equiv (P \vee Q) \rightarrow R$

- "Sport is sufficient for good health **and** happiness is sufficient for good health" **is equivalent to** "sport **or** happiness are sufficient for good health.

$((P \rightarrow R) \vee (Q \rightarrow R)) \equiv (P \wedge Q) \rightarrow R$

- "Sport is sufficient for good health **or** happiness is sufficient for good health **is equivalent to** "sport **and** happiness are sufficient for good health.

Some Online Resources

Necessary and Sufficient Conditions

- https://www.youtube.com/watch?v=QtMFyTV8jfg
- https://www.youtube.com/watch?v=KCMGWRoPuMY
- https://www.youtube.com/watch?v=FHeimgMWiog
- https://www.youtube.com/watch?v=OnbJ8S1Ainc

An Online Tutorial on Conditional Reasoning

- Conditional reasoning and logical equivalence
  https://www.khanacademy.org/test-prep/lsat/lsat-lessons/logic-toolbox-new/a/logic-toolbox--article--conditional-reasoning-logical-equivalence
- If X, then Y — Sufficiency and necessity
  https://www.khanacademy.org/test-prep/lsat/lsat-lessons/logic-toolbox-new/a/logic-toolbox--article--if-x-then-y--sufficiency-and-necessity
- The Logic of “If” vs. “Only if”
  https://www.khanacademy.org/test-prep/lsat/lsat-lessons/logic-toolbox-new/a/logic-toolbox--if-and-only-if
- A quick guide to conditional logic
  https://www.khanacademy.org/test-prep/lsat/lsat-lessons/logic-toolbox-new/a/logic-toolbox--article--quick-guide-conditional-logic

Complete Sets of Logic Functions

Definition

- A set of logic functions is **complete** if any boolean formula can be expressed by using only the functions from that set.

The canonical complete set

- The set $\{\mathcal{NOT}, \mathcal{AND}, \mathcal{OR}\}$ ($\{\neg, \wedge, \vee\}$) is complete.
- All $\{1, 2\}$-variable functions can be expressed using $\{\neg, \wedge, \vee\}$.
- Any function with more variables can be expressed by a formula containing one-variable and two-variable functions.

Expressing Two-Variable Functions With $\{\neg, \vee, \wedge\}$

Function(x, y)	(T,T)	(T,F)	(F,T)	(F,F)	Formula
$\mathcal{TRUE}$	T	T	T	T	T
$\mathcal{OR}$	T	T	T	F	$x \vee y$
$\mathcal{Y}$–$\mathcal{IMP}$–$\mathcal{X}$	T	T	F	T	$x \vee \neg y$
$\mathcal{ID}(\mathcal{X})$	T	T	F	F	x
$\mathcal{X}$–$\mathcal{IMP}$–$\mathcal{Y}$	T	F	T	T	$\neg x \vee y$
$\mathcal{ID}(\mathcal{Y})$	T	F	T	F	y
$\mathcal{EQUIV}$	T	F	F	T	$(x \wedge y) \vee (\neg x \wedge \neg y)$
$\mathcal{AND}$	T	F	F	F	$x \wedge y$
$\mathcal{NAND}$	F	T	T	T	$\neg(x \wedge y)$
$\mathcal{XOR}$	F	T	T	F	$(x \vee y) \wedge (\neg x \vee \neg y)$
$\mathcal{NOT}(\mathcal{Y})$	F	T	F	T	$\neg y$
$\neg(\mathcal{X}$–$\mathcal{IMP}$–$\mathcal{Y})$	F	T	F	F	$x \wedge \neg y$
$\mathcal{NOT}(\mathcal{X})$	F	F	T	T	$\neg x$
$\neg(\mathcal{Y}$–$\mathcal{IMP}$–$\mathcal{X})$	F	F	T	F	$\neg x \wedge y$
$\mathcal{NOR}$	F	F	F	T	$\neg(x \vee y)$
$\mathcal{FALSE}$	F	F	F	F	F

Other Complete Sets of Logic functions

How to prove that a set $\mathcal{S}$ of functions is complete?

- **First method:** Show how the 4 one-variable and 16 two-variable functions can be expressed with functions from $\mathcal{S}$.
- **Second method:** Show that $\mathcal{NOT}$, $\mathcal{AND}$, $\mathcal{OR}$ can be expressed with functions from $\mathcal{S}$ and then transitivity implies that the 4 one-variable and 16 two-variable functions can be expressed with functions from $\mathcal{S}$.

$\{\mathcal{NOT}, \mathcal{AND}\}$ is a complete set

- Trivially: $(x \wedge y) \equiv (x \wedge y)$ and $\neg x \equiv \neg x$.
- By one of the De Morgan's laws: $(x \vee y) \equiv \neg(\neg x \wedge \neg y)$.

$\{\mathcal{NOT}, \mathcal{OR}\}$ is a complete set

- Trivially: $(x \vee y) \equiv (x \vee y)$ and $\neg x \equiv \neg x$.
- By one of the De Morgan's laws: $(x \wedge y) \equiv \neg(\neg x \vee \neg y)$.

$\mathcal{NOR}$ ($\downarrow$) is a Complete Function

The function $\mathcal{NOR}$:

x	y	$x \downarrow y$
T	T	F
T	F	F
F	T	F
F	F	T

Expressing $\mathcal{NOT}$, $\mathcal{AND}$, and $\mathcal{OR}$ with $\mathcal{NOR}$:

- $\neg x \equiv x \downarrow x$
- $(x \wedge y) \equiv (x \downarrow x) \downarrow (y \downarrow y)$
- $(x \vee y) \equiv (x \downarrow y) \downarrow (x \downarrow y)$

Expressing $\mathcal{XOR}$ and $\mathcal{EQUIV}$ with $\mathcal{NOR}$:

- $(x \oplus y) \equiv ((x \downarrow x) \downarrow (y \downarrow y)) \downarrow (x \downarrow y)$
- $(x \equiv y) \equiv (x \downarrow (x \downarrow y)) \downarrow (y \downarrow (x \downarrow y))$

$\mathcal{NAND}$ (↑) is a Complete Function

The function $\mathcal{NAND}$:

x	y	$x \uparrow y$
T	T	F
T	F	T
F	T	T
F	F	T

Expressing $\mathcal{NOT}$, $\mathcal{AND}$, and $\mathcal{OR}$ with $\mathcal{NAND}$:

- $\neg x \equiv x \uparrow x$
- $(x \wedge y) \equiv (x \uparrow y) \uparrow (x \uparrow y)$
- $(x \vee y) \equiv (x \uparrow x) \uparrow (y \uparrow y)$

Expressing $\mathcal{XOR}$ and $\mathcal{EQUIV}$ with $\mathcal{NAND}$:

- $(x \oplus y) \equiv (x \uparrow (x \uparrow y)) \uparrow (y \uparrow (x \uparrow y))$
- $(x \equiv y) \equiv ((x \uparrow x) \uparrow (y \uparrow y)) \uparrow (x \uparrow y)$

Quantifiers

Notations

- Let $P(x)$ be a boolean proposition (**TRUE** or **FALSE**) defined on all the objects $x \in U$.

The universal quantifier

- **P(x) is TRUE for every $x \in U$** is denoted by:

$$\forall_{x \in U} P(x)$$

The existential quantifier

- **There exists $x \in U$ such that P(x) is TRUE** is denoted by:

$$\exists_{x \in U} P(x)$$

Relationships between the quantifiers

Generalizing the De Morgan's laws

$$\neg(\forall_{x \in U} P(x)) \equiv \exists_{x \in U} \neg P(x)$$

$$\neg(\exists_{x \in U} P(x)) \equiv \forall_{x \in U} \neg P(x)$$

Example: $P(x)$ means x is smart

- $\neg(\forall_{x \in U} P(x))$: Not everyone is smart.
- $\exists_{x \in U} \neg P(x)$: Someone is not smart.
- $\neg(\exists_{x \in U} P(x))$: There is no one who is smart.
- $\forall_{x \in U} \neg P(x)$: Everyone is not smart.

Relationships between the quantifiers

Theorem

$$\neg(\forall_{x \in U} P(x)) \equiv \exists_{x \in U} \neg P(x)$$

Proof

- The left side of the equivalence:
  - $\forall_{x \in U} P(x)$ means that $P(x)$ is **TRUE** for all $x \in U$.
  - $\neg(\forall_{x \in U} P(x))$ means that $P(x)$ is **FALSE** for at least one $x \in U$.
- The right side of the equivalence:
  - $\exists_{x \in U} \neg P(x)$ means that $\neg P(x)$ is **TRUE** for at least one $x \in U$.
  - This is equivalent to $P(x)$ is **FALSE** for at least one $x \in U$.

Remark

- This is a generalization of the De Morgan's law:

$$\neg\left(\bigwedge_{i=1}^{n} P_i\right) \equiv \bigvee_{i=1}^{n} (\neg P_i)$$

Relationships between the quantifiers

Theorem

$$\neg(\exists_{x \in U} P(x)) \equiv \forall_{x \in U} \neg P(x)$$

Proof

- The left side of the equivalence:
  - $\exists_{x \in U} P(x)$ means that $P(x)$ is **TRUE** for at least one $x \in U$.
  - $\neg(\exists_{x \in U} P(x))$ means that $P(x)$ is **FALSE** for all $x \in U$.
- The right side of the equivalence:
  - $\forall_{x \in U} \neg P(x)$ means that $\neg P(x)$ is **TRUE** for all $x \in U$.
  - This is equivalent to $P(x)$ is **FALSE** for all $x \in U$.

Remark

- This is a generalization of the De Morgan's law:

$$\neg(\bigvee_{i=1}^{n} P_i) \equiv \bigwedge_{i=1}^{n} (\neg P_i)$$

Order Among Quantifiers

Notations

- Let $P(x, y)$ be a boolean proposition (**TRUE** or **FALSE**) defined on all the objects $x, y \in U$.

Order does not matter

- $\forall_{x \in U} \forall_{y \in U} P(x, y) \equiv \forall_{y \in U} \forall_{x \in U} P(x, y) \equiv \forall_{x,y \in U} P(x, y)$.
- $\exists_{x \in U} \exists_{y \in U} P(x, y) \equiv \exists_{y \in U} \exists_{x \in U} P(x, y) \equiv \exists_{x,y \in U} P(x, y)$.

Order matters

- $\forall_{x \in U} \exists_{y \in U} P(x, y) \not\equiv \exists_{y \in U} \forall_{x \in U} P(x, y)$.
- A **TRUE** proposition: "For every integer n there exists an integer m such that $m = n^2$" — $\forall_{n \in \mathbb{Z}} \exists_{m \in \mathbb{Z}} (m = n^2)$.
- A **FALSE** proposition: "There exists an integer m such that $m = n^2$ for every integer n" — $\exists_{m \in \mathbb{Z}} \forall_{n \in \mathbb{Z}} (m = n^2)$.

Quantifications of Two Variables

Summary

Statement	When true?	When False?
$\forall_x \forall_y P(x,y)$ $\forall_y \forall_x P(x,y)$	$P(x,y) = T$ for every pair x, y	There is a pair x, y for which $P(x,y) = F$
$\exists_x \exists_y P(x,y)$ $\exists_y \exists_x P(x,y)$	There is a pair x, y for which $P(x,y) = T$	$P(x,y) = F$ for every pair x, y
$\forall_x \exists_y P(x,y)$	For every x there is a y for which $P(x,y) = T$	There is an x such that $P(x,y) = F$ for every y
$\exists_x \forall_y P(x,y)$	There is an x such that $P(x,y) = T$ for every y	For every x there is a y for which $P(x,y) = F$

Quantifications of Two Variables

Example: $P(x, y)$ means that x and y are friends

- $\forall_{x,y} P(x, y)$:
  - **TRUE**: Everyone is a friend with everyone else.
  - **FALSE**: There exists at least one pair who are not friends.
- $\exists_{x,y} P(x, y)$:
  - **TRUE**: At least one pair are friends.
  - **FALSE**: There are no friends at all.
- $\forall_x \exists_y P(x, y)$:
  - **TRUE**: Everyone has at least one friend.
  - **FALSE**: There exists someone who has no friends.
- $\exists_x \forall_y P(x, y)$:
  - **TRUE**: There exists someone who is a friend with everyone else.
  - **FALSE**: Everyone has at least someone who is not their friend.

Be Careful and Precise with Logic

The unexpected hanging paradox

- Paradox with a logical school solution:
  https://www.youtube.com/watch?v=vxlCiV_axQ0
- Paradox with a logical and an epistemological school solutions:
  https://www.youtube.com/watch?v=EPOXhFJsqlM
- Wikipedia:
  https://en.wikipedia.org/wiki/Unexpected_hanging_paradox

An Online Tutorial

8 Logic lectures from TrevTutor

- Propositional Logic: www.youtube.com/watch?v=itrXYg41-V0&feature=youtu.be
- Truth Tables: www.youtube.com/watch?v=UiGu57JzLkE&feature=youtu.be
- Proofs with Truth Tables: www.youtube.com/watch?v=9fX6n0_MDic&feature=youtu.be
- Logic laws: www.youtube.com/watch?v=eihhu72YdpQ&feature=youtu.be
- Conditionals: www.youtube.com/watch?v=xag9TEAORK4&feature=youtu.be
- Proof by Contraposition: https://www.youtube.com/watch?v=X-hJ7krLBn0
- Rules of Inference: www.youtube.com/watch?v=8DW0K3mnc-0&feature=youtu.be
- Predicate Logic: www.youtube.com/watch?v=gyoqX0W-NH4&feature=youtu.be

Additional Logic Puzzles

Knights and knaves

- https://www.youtube.com/watch?v=Imgus1ispQk

Prisoners with hats

- https://youtu.be/N5vJSNXPEwA
- https://www.youtube.com/watch?v=RtidKw-qDxY

How many liars are at the party?

- https://www.youtube.com/watch?v=jqX9nnRiD9g

Can you crack the code?

- https://www.youtube.com/watch?v=-etLb-8sHBc&feature=youtu.be