# Discrete Structures: Probability

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# Discrete Probability

## What is probability?

- Probability is a measure of a **likelihood** of an **event** occurring.
  - ∗ The bigger the probability the more likely it is that the event occurs.
  - ∗ The lowest probability is 0 (0%), the highest probability is 1 (100%).
- Probability theory models the phenomenon of chance.
  - ∗ A probabilistic model of random phenomena is defined by assigning **probabilities** to all the possible **outcomes** of an **experiment**.

## Learning objectives

- Learn some basic rules, tools, and techniques without formal definitions and proofs.
- Be able to solve probability problems using common sense, intuition, and the basic concepts.

# Discrete Probability

## A song

- https://www.youtube.com/watch?v=au8hf0-A27s

## Likelihood

- https://www.youtube.com/watch?v=QpfMwA0z_1Y

## A problem with three methods to solve it

- https://youtu.be/fyv1i2YbNt0

# Discrete Probability

## Sample spaces

- A **sample space** set $\mathcal{S}$ contains a finite number of **outcomes**

$$\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$$

- Each outcome $s_i$ is associated with a **probability** value $p_i = p(s_i)$ which is a real number between 0 and 1 such that

$$\sum_{i=1}^{n} p_i = 1$$

## The uniform probability

- For $1 \leq i \leq n$, $p(s_i) = \frac{1}{n}$ for a probability space $\mathcal{S}$ with $n$ outcomes

$$\sum_{i=1}^{n} \frac{1}{n} = 1$$

# Discrete Probability

### Intuition and/or motivation and/or explanation

- Suppose the space is **sampled** $m$ times (for a very large $m$).
- Then $s_i$ is sampled approximately $p_i m$ times for each $1 \le i \le n$.

$$\sum_{i=1}^{n} p_i m = \left( \sum_{i=1}^{n} p_i \right) m = m$$

### The uniform probability

- Suppose the space is **sampled** $m$ times (for a very large $m$).
- Then $s_i$ is sampled approximately $\frac{m}{n}$ times for each $1 \le i \le n$.

$$\sum_{i=1}^{n} \frac{m}{n} = \left( \sum_{i=1}^{n} \frac{1}{n} \right) m = m$$

# Discrete Probability

## Generating sample spaces

- An **experiment** is defined that ends up with one outcome taken from the sample space.
- Each outcome is associated with a probability value.

## Four canonical examples

- **Flipping** (or **tossing**) a coin.
- **Throwing** (or **rolling**) a die.
- **Drawing** a card from a deck of cards.
- **Drawing** a marble from a bag of marbles.

# Flipping One Coin

## Model

- **Experiment:** Flip a coin that has two sides, one is **Heads** (**H**) and the other is **Tails** (**T**), to observe one of the sides.
- **Sample space:** $\{\mathbf{H}, \mathbf{T}\}$.

## Fair and biased coins

- In a **fair coin** the probabilities of the two outcomes are the same:

$$p(\mathbf{H}) = p(\mathbf{T}) = 1/2$$

- For a real number $0 \leq p \leq 1$, in a **biased coin** the probability of flipping **H** is $p$ while the probability of flipping **T** is $1 - p$:

$$p(\mathbf{H}) + p(\mathbf{T}) = p + (1 - p) = 1$$

## Online coins

- https://www.geogebra.org/m/LZbwMZtJ

# Flipping Several Coins or One Coin Several Times

## Flipping two coins together or one coin twice

- The sample space has **4** outcomes:

$$\{(\mathbf{H},\mathbf{H}),(\mathbf{H},\mathbf{T}),(\mathbf{T},\mathbf{H}),(\mathbf{T},\mathbf{T})\}$$

- With fair coins the probability of each outcome is $\mathbf{1/4}$.

## Flipping three coins together or one coin three times

- The sample space has **8** outcomes:

$$\{(\mathbf{H},\mathbf{H},\mathbf{H}),(\mathbf{H},\mathbf{H},\mathbf{T}),(\mathbf{H},\mathbf{T},\mathbf{H}),(\mathbf{H},\mathbf{T},\mathbf{T}),(\mathbf{T},\mathbf{H},\mathbf{H}),(\mathbf{T},\mathbf{H},\mathbf{T}),(\mathbf{T},\mathbf{T},\mathbf{H}),(\mathbf{T},\mathbf{T},\mathbf{T})\}$$

- With fair coins the probability of each outcome is $\mathbf{1/8}$.

## Flipping *k* coins together or one coin *k* times

- The sample space has $\mathbf{2^k}$ outcomes:

$$\{(\mathbf{H},\mathbf{H}\ldots,\mathbf{H},\mathbf{H}),(\mathbf{H},\mathbf{H},\ldots,\mathbf{H},\mathbf{T}),\ldots,(\mathbf{T},\mathbf{T},\ldots,\mathbf{T},\mathbf{H}),(\mathbf{T},\mathbf{T},\ldots,\mathbf{T},\mathbf{T})\}$$

- With fair coins the probability of each outcome is $\mathbf{(1/2)^k}$.

# Throwing One Die

## Model

- **Experiment:** Throw a die that has 6 faces (sides) labeled by the numbers $1, 2, 3, 4, 5, 6$ to observe one of the faces.
- **Sample space:** $\{1, 2, 3, 4, 5, 6\}$.
- In a fair die the probability of each outcome is $1/6$.

## A generalized die

- For an integer $n \geq 2$, the generalized die has $n$ faces labeled by the numbers $1, 2, \ldots, n$.
- The **sample space** is $\{1, 2, \ldots, n\}$.
- In a fair generalized die the probability of each outcome is $1/n$.
- A coin is a generalized die with $n = 2$ faces.

# Throwing Several Dice or One Die Several Times

## Throwing two dice together or one die twice

- There are $\mathbf{6^2} = \mathbf{36}$ outcomes:

$$\{(1,1),(1,2),\ldots,(3,6),(4,1),\ldots,(6,5),(6,6)\}$$

## Throwing $k$ dice together or one die $k$ times

- There are $\mathbf{6^k}$ outcomes:

$$\{(1,1,\ldots,1),(1,1,\ldots,2),\ldots,(2,1,\ldots,1),\ldots,(6,6,\ldots,5),(6,6,\ldots,6)\}$$

## Online generalized dice

- `https://rolladie.net/#!numbers=2&high=6&length=1&sets=&addfilters=&last_roll_only=false&totals_only=false`

# Drawing Cards

## Model

- **Sample space:**
  - A deck of cards contains 52 cards.
  - There are 4 suits: 13 Black Clubs ($\clubsuit$), 13 Red Diamonds ($\diamondsuit$), 13 Red Hearts ($\heartsuit$), and 13 Black Spades ($\spadesuit$).
  - Each suit has nine number cards $2, 3, 4, 5, 6, 7, 8, 9, 10$, three face cards Jack (J), Queen (Q), and King (K), and one Ace (A).
- **Experiment:** Blindly draw a random card from the deck .
- The probability of drawing any of the cards is $1/52$.

## Drawing more than one card from the deck

- When $k$ cards are drawn together the probability of each combination is $1/\binom{52}{k}$.

## Online cards

- https://www.calculatorsoup.com/calculators/statistics/random-card-generator.php

# Drawing Colored Pebbles from a Bag

## Model

- **Sample space:**
  - A bag contains $n$ pebbles each is colored with one color out of $k$ possible colors.
  - For $1 \leq i \leq k$, there are $n_i$ pebbles of color $i$.
  - $n = n_1 + n_2 + \cdots + n_k$.
- **Experiment:** Blindly draw a random pebble from the bag.
- The probability of each pebble to be drawn is $1/n$.

## Drawing more than one pebble from the bag

- **Model I:** The pebble is returned to the bag before drawing the next pebble.
- **Model II:** The pebble is left outside before drawing the next pebble.

# Events

## Definition

- An **event** $\mathcal{E}$ is a set of outcomes from the probability space:

$$\mathcal{E} \subseteq \mathcal{S}$$

- If an event $\mathcal{E}$ contains the $k$ outcomes $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ then its probability is

$$P(\mathcal{E}) = \sum_{j=1}^{n} p(s_{i_j})$$

- For a sample space with $n$ outcomes, when all the outcomes have the same probability then

$$P(\mathcal{E}) = \frac{|\mathcal{E}|}{|\mathcal{S}|} = \frac{k}{n}$$

# Events

## The null event

- The **null** event $\mathcal{E} = \emptyset$ contains no outcomes.
- The probability of the null event is $p(\emptyset) = 0$.

## The all event

- The **all** event $\mathcal{E} = \mathcal{S}$ contains all possible outcomes.
- The probability of the all event is $p(\mathcal{S}) = 1$.

## The singleton events

- A **singleton** event $\mathcal{E} = \{s_i\}$ contains only one outcome.
- The probability of the singleton event $\{s_i\}$ is $p(\{s_i\}) = p_i$.
- When all the $n$ outcomes of the sample space have the same probability then the probability of the singleton event $\{s_i\}$ is $1/n$.

# Examples of Events

## Coins

- Let $A$ be the event of observing at least one Heads when flipping two fair coins.
- $A = \{\textbf{HH}, \textbf{HT}, \textbf{TH}\}$.
- $p(A) = 3/4$ because $A$ contains 3 outcomes out of the 4 possible outcomes.

## Dice

- Let $B$ be the event of observing an even number when throwing a 6-face fair die.
- $B = \{2, 4, 6\}$.
- $p(B) = 3/6 = 1/2$ because $B$ contains 3 outcomes out of the 6 possible outcomes.

# Examples of Events

### Cards

- Let $C$ be the event of drawing an Ace from a standard 52-card deck.
- $C = \{A\clubsuit, A\diamondsuit, A\heartsuit, A\spadesuit\}$.
- $p(C) = 4/52 = 1/13$ because $C$ contains 4 outcomes out of the 52 possible outcomes.

### Pebbles

- Let $D$ be the event of drawing a **Red** pebble from a bag containing 3 **Red** pebbles, 4 **Blue** pebbles, and 5 **Green** pebbles.
- $D = \{1\mathbf{R}, 2\mathbf{R}, 3\mathbf{R}\}$.
- $p(D) = 3/12 = 1/4$ because $D$ contains 3 outcomes out of the $12 = 3 + 4 + 5$ possible outcomes.

# The Complement of an Event

**Definition**

- Let $\mathcal{E}$ be an event.
- Event $\overline{\mathcal{E}}$ occurs if and only if event $\mathcal{E}$ does not occur.

**Observation**

- $p(\overline{\mathcal{E}}) = 1 - p(\mathcal{E})$.
- Proof idea: Any outcome either belongs to $\mathcal{E}$ or belongs to $\overline{\mathcal{E}}$.

**Example**

- $A$: The event that a die shows an odd number.
- $\overline{A}$: The event that a die shows an even number.
- $p(A) = p(\overline{A}) = 1/2$.

# The Union of Two Events

**Definition**

- The union event $A \cup B$ occurs if and only if event $A$ occurs **or** event $B$ occurs (or both).

**Disjoint events**

- Events $A$ and $B$ are **disjoint** (**mutually exclusive**) if $A \cap B = \emptyset$ .

**Probability of disjoint events $A$ and $B$**

$$p(A \cup B) = p(A) + p(B)$$

# The Union of Two Events

**Example: throw a 6-face fair die**

- Let $A = \{2, 4, 6\}$ be the event of observing an even number, its probability is $p(A) = 3/6 = 1/2$.
- Let $B = \{3, 5\}$ be the event of observing an odd prime number, its probability is $p(B) = 2/6 = 1/3$.
- Let $A \cup B = \{2, 3, 4, 5, 6\}$ be the event of observing either an even number or an odd prime number, its probability is $p(A \cup B) = 5/6$.
- $A$ and $B$ are disjoint events because $A \cap B = \emptyset$ and therefore

$$p(A \cup B) = p(A) + p(B) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

# The Intersection of Events

**Definition**

- The intersection event $A \cap B$ occurs if and only if both event $A$ **and** event $B$ occur.

**Dependent and Independent events**

- Informally, events $A$ and $B$ are **dependent** if the occurrence of one of them "influences" the probability of the other one occurring.
- Events $A$ and $B$ are **independent** if the probability of event $B$ occurring in the sample space $A$ is the same as its probability of occurring in the overall sample space.

**Probability of independent events $A$ and $B$**

$$p(B) = \frac{p(A \cap B)}{p(A)} \implies p(A \cap B) = p(A) \cdot p(B)$$

# The Intersection of Events

**Example: throw two 6-face fair dice**

- Let $A = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$ be the event of observing 1 on the first die, its probability is $p(A) = 6/36 = 1/6$.
- Let $B = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6)\}$ be the event of observing 6 on the second die, its probability is $p(B) = 6/36 = 1/6$.
- The intersection event $A \cap B = \{(1,6)\}$ is the event of observing both 1 on the first die and 6 on the second die, its probability is $p(A \cap B) = 1/36$.
- Intuitively both events are independent because they cannot "influence" each other.
- Indeed,

$$p(A \cap B) = p(A) \cdot p(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

# The Intersection of Events

**Example: throw a $6$-face fair die**

- Let $A = \{1, 2, 3\}$ be the event of observing a smaller than 4 number, its probability is $p(A) = 3/6 = 1/2$.
- Let $B = \{1, 2, 3, 4\}$ be the event of observing a smaller than 5 number, its probability is $p(B) = 4/6 = 2/3$.
- Let $C = \{2, 3, 5\}$ be the event of observing a prime number, its probability is $p(C) = 3/6 = 1/2$.
- $A$ and $C$ are dependent events because $p(A \cap C) = p(\{2, 3\}) = 2/6 = 1/3$ while $p(A) \cdot p(C) = (1/2) \cdot (1/2) = 1/4$.
- $B$ and $C$ are independent events because $p(B \cap C) = p(\{2, 3\}) = 2/6 = 1/3$ and also $p(B) \cdot p(C) = (2/3) \cdot (1/2) = 1/3$.

# The Principle of Inclusion exclusion

**Theorem**

- $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ for any two events $A$ and $B$.

**Example: throw a 6-face fair die**

- Let $A = \{1, 3, 5\}$ be the event of observing an odd number, its probability is $p(A) = 3/6 = 1/2$.
- Let $B = \{2, 3, 5\}$ be the event of observing a prime number, its probability is $p(B) = 3/6 = 1/2$.
- Let $A \cup B = \{1, 2, 3, 5\}$ be the event of observing either an odd number or a prime number, its probability is $p(A \cup B) = 4/6 = 2/3$.
- Let $A \cap B = \{3, 5\}$ be the event of observing an odd prime number, its probability is $p(A \cap B) = 2/6 = 1/3$.
- By the principle of inclusion exclusion

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$$

# Remarks

## Many events and laws among sets

- Intersection and union of more than two events are generalized in a "natural" way.
- Events are sets and therefore the laws of sets apply to their probabilities. For example, the associative laws, the distributive laws, and De Morgan's laws.

## Two main techniques for calculating probabilities of events

- When all the outcomes have the same probability, the **counting** method counts the number of outcomes that belong to the event and divides it by the sample space size.
- The **probability** method applies the probability rules like the complement, union, and intersection among events.

# Some Online Examples

## Probability and counting: guessing in T/F exams

- https://www.youtube.com/watch?v=3V2omKRX9gc

## A dice problem

- https://www.youtube.com/watch?v=Kgudt4PXs28&feature=youtu.be

# The Law of Large Numbers

## Setting

- Let $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ be a sample space in which the probability of each outcome $s_i$ is $p_i$ for all $1 \leq i \leq n$.
- Suppose the space $\mathcal{S}$ is **sampled** $m$ times (for a very large $m$).

## Informal statement of the law

- $s_i$ is sampled **approximately** $p_i m$ times for each $1 \leq i \leq n$.
- With the uniform probability, $s_i$ is sampled **approximately** $\frac{m}{n}$ times for each $1 \leq i \leq n$.

## The other direction

- For $1 \leq i \leq n$, suppose $s_i$ is sampled $h_i$ times.
- Then $p_i = \frac{h_i}{m}$.

# The Law of Large Numbers and Simulations

## Introducing the concept

- https://www.youtube.com/watch?v=MntX3zWNWec

## Experimental versus theoretical probability

- https://www.youtube.com/watch?v=Nos-xOCpQqg&feature=youtu.be

## Collecting six coupons via simulations

- https://www.youtube.com/watch?v=2AVLfSRpmfg&feature=youtu.be

# Conditional Probability

## Motivation

- The probabilities assiciated with the outcomes are **pure** in the sense that they predict their likelihood assuming nothing is known about the saple space.
- These predicting probabilities may change once some information is known about some events that had happened.

## Definition

- **Conditional probability** is defined as the likelihood of an event occurring based on the occurrence of a previous event.

## Notation

- Let $A$ and $B$ be two events in the sample space $\mathcal{S}$. The event $B|A$ in $\mathcal{S}$ is the event $B$ **given** that event $A$ happened.

# Conditional Probability

**Formula for two events $A$ and $B$**

$$p(B|A) = \frac{p(A \cap B)}{p(A)} \qquad p(A|B) = \frac{p(A \cap B)}{p(B)}$$

**Bayes' Theorem for two events $A$ and $B$**

$$p(B|A) \cdot P(A) = p(A|B) \cdot P(B)$$

**Two independent events $A$ and $B$**

- If $A$ and $B$ are independent events then $p(A \cap B) = p(A) \cdot p(B)$.
- Therefore

$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{p(A) \cdot p(B)}{p(A)} = p(B)$$

$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{p(A) \cdot p(B)}{p(B)} = p(A)$$

# Conditional Probability

## Example: throw a 6-face fair die

- Let $A = \{1, 2, 3, 4, 5\}$ be the event of observing a number smaller than 6, its probability is $p(A) = 5/6$.
- Let $B = \{1, 3, 5\}$ be the event of observing an odd number, its probability is $p(B) = 3/6 = 1/2$.
- The intersection event $A \cap B = \{1, 3, 5\}$ is observing an odd number smaller than 6, its probability is $p(A \cap B) = 3/6 = 1/2$.
- Let $B|A$ be the event of observing an odd number **given** that event $A$ happened:
$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{1/2}{5/6} = \frac{3}{5}$$
- Let $A|B$ be the event of observing a number smaller than 6 **given** that event $B$ happened:
$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{1/2}{1/2} = 1$$

# Conditional Probability

**Example: flip two fair coins (sample space is $\{HH, HT, TH, TT\}$)**

- Let $A = \{HH, HT, TH\}$ be the event of flipping at least one heads, its probability is $p(A) = 3/4$.
- Let $B = \{TH, TT\}$ be the event of the first coin showing tails, its probability is $p(B) = 2/4 = 1/2$.
- The intersection event $A \cap B = \{TH\}$ is flipping at least one heads while the first coing showing tails, its probability is $p(A \cap B) = 1/4$.
- Let $B|A$ be the event of the first coin showing tails **given** that event $A$ happened:
$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$
- Let $A|B$ be the event of flipping at least one heads **given** that event $B$ happened:
$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

# Conditional Probability: Online Examples

## Statistics example: independence and conditional probability

- https://youtu.be/pIfpHdGVwLU

## Chance of guessing correctly

- https://www.youtube.com/watch?v=7DSA4qQv3oc

## Coin example: conditional probability

- https://youtu.be/Zxm4Xxvzohk

## The false positive “paradox”

- https://youtu.be/1csFTDXXULY

## Conditional probability via false positive and false negative

- https://youtu.be/hxEdXUB_IdQ

# Intuition Or Not?

## The Boy and a Girl Paradox

- https://www.youtube.com/watch?v=YtK4R66_YAk
- https://www.youtube.com/watch?v=GYpvh5NBYuI
- https://www.youtube.com/watch?v=O9YkN2UdfFE

## Variations

- Three red-blue cards: https://www.youtube.com/watch?v=1IFM1ngwEb0
- 2 Aces and 2 twos: https://www.youtube.com/watch?v=TGukPgViEkg
- A dice version: https://www.youtube.com/watch?v=lTJydnfGyyk

## The Monty Hall Problem

- https://youtu.be/_X5erR9LKUs
- https://youtu.be/4Lb-6rxZxx0

# Problems With "Counter Intuitive" Answers

## Relatively Easier Problems

- Random walkers meeting on a grid
  - https://www.youtube.com/watch?v=F_kt51Qj1RI
- Guessing randomly on A matching test
  - https://www.youtube.com/watch?v=h2VMMlpEtRU

## Relatively Harder Problems

- What is the chance twin brothers are identical?
  - https://www.youtube.com/watch?v=xBPMoavS5Hs
- HH needs 6 and HT needs 4 flips via markov chains
  - https://www.youtube.com/watch?v=IAiNqQi30-Y
  - https://www.youtube.com/watch?v=--mxW3jDlGk
- Conditional probability with boys and girls
  - https://www.youtube.com/watch?v=UQnBGm1aLjY

# Beyond Basic Probability

## Fermet needs 2 wins Euler needs 3 wins

- https://www.youtube.com/watch?v=C_nV3cVNjog

## Weird dice with the same distribution

- https://www.youtube.com/watch?v=kG4Fh_PrN-E

## The reason casinos always win meet the law of large numbers

- https://www.youtube.com/watch?v=RXY-WN0ahiw

## An alien extinction riddle

- https://www.youtube.com/watch?v=A5-Q2GdD5xw