# Discrete Structures: Introduction to Proofs

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# What is a Proof?

## Five out of many definitions

16 words: *The cogency of evidence that compels acceptance by the mind of a truth or a fact.*

18 words: *A formal series of statements showing that if one thing is true something else necessarily follows from it.*

20 words: *A chain of reasoning using rules of inference, ultimately based on a set of axioms, that lead to a conclusion.*

24 words: *The process or an instance of establishing the validity of a statement, especially by derivation from other statements in accordance with principles of reasoning.*

39 words: *A sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the conclusion, is the statement of which the truth is thereby established.*

# Proofs from the Book

## A story

- One of the greatest 20$^{\text{th}}$ century mathematicians, Paul Erdös, although an atheist, spoke of **“The Book”**, an imaginary book in which God had written down the best and **most elegant proofs** for **all** mathematical theorems.
- He said,“You don’t have to believe in God, but you should believe in **The Book**”.
- He accused God of keeping the **most elegant mathematical proofs** to itself.
- When he saw a particularly **beautiful mathematical proof** he would exclaim: “This one is from **The Book**”.

# Proofs and End of Proofs

## Some history

- https://www.youtube.com/watch?v=S0DSM-EkQE8

## Proofs without words: five "classic" examples

- https://www.youtube.com/watch?v=6H6nCpGW1nU

## End of Proof

- Q.E.D. or $\square$ or $\blacksquare$
- Latin: Quod Erat Demonstrandum.
- English: Which Was to be Demonstrated.
- What is QED? https://www.youtube.com/watch?v=U6FmQP7fQw8
- Quite Easily Done: https://www.youtube.com/watch?v=oTkQD65XMwg

# Cartoons

- https://s3.amazonaws.com/lowres.cartoonstock.com/science-proof-theory-desk_organiser-desk_organizer-in_tray-shrn2749_low.jpg
- https://s3.amazonaws.com/lowres.cartoonstock.com/science-proof-proves-philosopher-philosophy-philosophize-shrn1786_low.jpg
- https://s3.amazonaws.com/lowres.cartoonstock.com/science-theory-scientist-proof-disprove-disproved-shrn1763_low.jpg
- https://s3.amazonaws.com/lowres.cartoonstock.com/science-statistician-statistics-proof-proves-proofs-shrn5222_low.jpg
- https://s3.amazonaws.com/lowres.cartoonstock.com/business-commerce-statisticians-statistician-evidence-belief_systems-proof-shrn2330_low.jpg
- https://lowres.cartooncollections.com/maths-science-scientists-mathematicians-proofs-science-CX300153_low.jpg
- https://lowres.cartooncollections.com/academics-job-existence-scientific_proofs-scientists-science-CX901967_low.jpg

# Some Major Techniques

**Direct proofs: by construction and/or by exhaustion**

**Proof by contradiction**

**Proof by contrapositive**

**Proof by induction**

# Some Major Techniques

## Direct proofs: by construction and/or by exhaustion

- **Goal:** prove that $Q$ is **TRUE**.
- Start with a **TRUE** $P$.
- Demonstrate that $Q$ **must** follow from $P$.

## Proof by contradiction

## Proof by contrapositive

## Proof by induction

# Some Major Techniques

### Direct proofs: by construction and/or by exhaustion

### Proof by contradiction

- **Goal:** prove that $Q$ is **TRUE**.
- Assume that $Q$ is **FALSE**.
- Demonstrate a **contradiction**.

### Proof by contrapositive

### Proof by induction

# Some Major Techniques

**Direct proofs: by construction and/or by exhaustion**

**Proof by contradiction**

**Proof by contrapositive**

- **Goal:** prove that $P$ is **TRUE** implies that $Q$ is **TRUE**.
- Assume that $Q$ is **FALSE**.
- Demonstrate that $P$ is **FALSE** **must** follow from $Q$ is **FALSE**.
- Observe that ($P$ is **TRUE** implies $Q$ is **TRUE**) **if-and-only-if** ($Q$ is **FALSE** implies $P$ is **FALSE**).

**Proof by induction**

# Some Major Techniques

**Direct proofs: by construction and/or by exhaustion**

**Proof by contradiction**

**Proof by contrapositive**

**Proof by induction**

- **Goal:** prove $Q$ is **TRUE** as a function of some ordered set $S$.
- **Basis:** show $Q$ is **TRUE** for a specific initial element $k \in S$.
- **Inductive Hypothesis:** Assume $Q$ is **TRUE** for some element $n \in S$ such that $n \geq k$.
- **Inductive step:** Demonstrate $Q$ is **TRUE** for the element $n + 1$.
- **Conclude:** $Q$ is **TRUE** for all elements in $S$ that are greater than or equal to $k$.

# Resources

## Lectures about logic&proofs from "Introduction to Higher Math"

- Problem Solving:
  www.youtube.com/watch?v=CMWFmjlB8v0&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX&index=1
- Introduction to Proofs:
  www.youtube.com/watch?v=_x65OJU8Uq4&index=2&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX
- Propositional Logic:
  www.youtube.com/watch?v=3kzhDsSzKCU&index=3&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX
- Proof Techniques:
  www.youtube.com/watch?v=WSb9q4Rj2Bg&index=4&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX

## Lectures about proofs from TrevTutor

- Direct Proofs:
  https://www.youtube.com/watch?v=YFZzLQN5qOU&feature=youtu.be
- Proof by Case:
  https://www.youtube.com/watch?v=YDKBEOuMuy8&feature=youtu.be
- Proof by Contraposition:
  https://www.youtube.com/watch?v=X-hJ7krLBn0&feature=youtu.be
- Proof by Contradiction:
  https://www.youtube.com/watch?v=sRDwsfNDXak&feature=youtu.be

# Resources

## A text book

- “Discrete Mathematics and its Applications,” by Kenneth H. Rosen.
  https:
  //www.mheducation.com/highered/product/Discrete-Mathematics-and-Its-Applications-Rosen
  - ∗ Section 1.7: “Introduction to Proofs”’ (pages 80–90).
  - ∗ Section 1.8: “Proof Methods and Strategy”’ (pages 92–107).

## Two articles about proofs

- Basic Proof Techniques:
  https://www.cse.wustl.edu/~cytron/547Pages/f14/IntroToProofs_Final.pdf
- Mathematical proof (Wikipedia):
  https://en.wikipedia.org/wiki/Mathematical_proof

# A "Trivial" Proof

## Theorem

- $n^2 \geq n$ for all integers $n$.

## Examples

- $0^2 = 0 \geq 0$
- $(-1)^2 = 1^2 = 1 \geq 1 > -1$
- $(-2)^2 = 2^2 = 4 > 2 > -2$

## Proof

- Prove by case analysis:
  - $n^2 > 0 > n$ for $n \leq -1$
  - $n^2 = 0^2 = 0 \geq 0 = n$ for $n = 0$
  - $n \geq 1 \Rightarrow n \cdot n \geq n \cdot 1 \Rightarrow n^2 \geq n$ for $n \geq 1$
- Q.E.D.

# A “Non-Trivial” Proof

### Theorem

- $n^2$ is the maximum possible product of two positive integers $h$ and $k$ such that $h + k = 2n$.

### Remark

- Without loss of generality assume that $h \leq k$ because otherwise $h$ and $k$ may be exchanged.

### Example: $n = 5$

- For $h = 1$ and $k = 9$ the product is $9 = 1 \times 9$
- For $h = 2$ and $k = 8$ the product is $16 = 2 \times 8$
- For $h = 3$ and $k = 7$ the product is $21 = 3 \times 7$
- For $h = 4$ and $k = 6$ the product is $24 = 4 \times 6$
- For $h = 5$ and $k = 5$ the product is $25 = 5 \times 5$

# A "Non-Trivial" Proof

**Theorem**

- $n^2$ is the maximum possible product of two positive integers $h \leq k$ such that $h + k = 2n$.

**Proof**

- Observe that $k \geq n$ since $k \geq h$ and $k \leq 2n - 1$ since $h \geq 1$.
- Assign $k = n + i$ for $0 \leq i \leq n - 1$.
- Since $h + k = 2n$, it follows that $h = n - i$.
- Therefore the product $h \cdot k$ is

$$(n - i)(n + i) = n^2 - i^2 \leq n^2$$

- The last inequality follows because $i^2$ is never negative.
- Q.E.D.

# A More General Theorem

**Theorem**

- $\lfloor m/2 \rfloor \times \lceil m/2 \rceil$ is the maximum possible product of two positive integers $h$ and $k$ such that $h + k = m$.

**Remark**

- Without loss of generality assume that $h \leq k$ because otherwise $h$ and $k$ may be exchanged.

**Observation**

- For an even $m$ this theorem is reduced to the previous theorem.
- Substituting $m$ with $2n$ implies that $\lfloor m/2 \rfloor = \lceil m/2 \rceil = n$ and therefore $\lfloor m/2 \rfloor \times \lceil m/2 \rceil = n^2$.

# A More General Theorem

**Example:** $m = 10$

- For $h = 1$ and $k = 9$ the product is $9 = 1 \times 9$
- For $h = 2$ and $k = 8$ the product is $16 = 2 \times 8$
- For $h = 3$ and $k = 7$ the product is $21 = 3 \times 7$
- For $h = 4$ and $k = 6$ the product is $24 = 4 \times 6$
- For $h = 5$ and $k = 5$ the product is $25 = 5 \times 5$

**Example:** $m = 11$

- For $h = 1$ and $k = 10$ the product is $10 = 1 \times 10$
- For $h = 2$ and $k = 9$ the product is $18 = 2 \times 9$
- For $h = 3$ and $k = 8$ the product is $24 = 3 \times 8$
- For $h = 4$ and $k = 7$ the product is $28 = 4 \times 7$
- For $h = 5$ and $k = 6$ the product is $30 = 5 \times 6$

# Visual Proofs

## Are visual proofs “real” proofs?

- Equivalent questions:
  - How can visual proofs be valid?
  - To what extent are visual proofs acceptable?
  - Are visual proofs considered rigorous enough to be accepted by the math community?
- Answer: Yes and No!?

## Four short “iconic” visual proofs

- `https://www.youtube.com/shorts/gnGxF0XcGZ8`

# A Proof With a “Trivial” Visualization

## Theorem

- Let $a$, $b$, and $c$ be positive integers. If $a$ divides $b$ and $a$ divides $c$, then $a$ divides the sum $b + c$.

## Proof

- $a$ divides $b$ implies that $b = a \cdot k$ for some integer $k$.
- $a$ divides $c$ implies that $c = a \cdot \ell$ for some integer $\ell$.
- $\implies \quad b + c = a \cdot k + a \cdot \ell = a \cdot (k + \ell)$.
- Therefore, $a$ divides $b + c$ because $k + \ell$ is an integer.
- Q.E.D.

## Visual proof

- https://www.youtube.com/watch?v=BKS0KIq6kTY&t=4s

# A Proof With a “Non-Trivial” Visualizations

**Theorem**

- A positive number plus its reciprocal is always $\geq 2$.
- $x + \frac{1}{x} \geq 2$ for any positive number $x$.

**Discussion**

- Trivial for $x \geq 2$ because $x + \frac{1}{x} \geq x \geq 2$.
- Equality exists for $x = 1$ because $x + \frac{1}{x} = 1 + \frac{1}{1} = 1 + 1 = 2$.
- Trivial for $x \leq \frac{1}{2}$ because $x + \frac{1}{x} \geq \frac{1}{x} \geq \frac{1}{1/2} = 2$.
- What about the range $\frac{1}{2} < x < 2$ when $x \neq 1$?

# A Proof With a “Non-Trivial” Visualizations

## Proof

$$
\begin{array}{rcll}
(x-1)^2 & \geq & 0 & (*\text{ squares are always nonnegative }*) \\
x^2-2x+1 & \geq & 0 & (*\text{ algebra }*) \\
x^2+1 & \geq & 2x & (*\text{ algebra }*) \\
\frac{x^2+1}{x} & \geq & \frac{2x}{x} & (*\text{ algebra }*) \\
\frac{x^2}{x}+\frac{1}{x} & \geq & 2 & (*\text{ algebra }*) \\
x+\frac{1}{x} & \geq & 2 & (*\text{ algebra }*)
\end{array}
$$

## Visual proofs for $x \geq 1$

- https://www.youtube.com/watch?v=zBUK8C6wSqs
- https://www.youtube.com/watch?v=IghOHBl0Do8
- https://www.youtube.com/watch?v=BkQvnriVblY

# The Arithmetic Mean vs. the Geometric Mean

## Definitions

For two positive numbers $x > 0$ and $y > 0$:

- $\frac{\mathbf{x+y}}{\mathbf{2}}$ is the **arithmetic mean ($\mathbf{AM(x, y)}$)** of $x$ and $y$.
- $\sqrt{\mathbf{xy}}$ is the **geometric mean ($\mathbf{GM(x, y)}$)** of $x$ and $y$.

## Theorem

$\mathbf{AM(x, y)} \geq \mathbf{GM(x, y)}$ for any two positive numbers $x > 0$ and $y > 0$:

$$\frac{\mathbf{x+y}}{\mathbf{2}} \geq \sqrt{\mathbf{xy}}$$

## Remarks

- The inequality is strict ("$>$" and not "$\geq$") when $x \neq y$.
- When $x = y$, both means are equal to $x = y$.

# The Arithmetic Mean vs. the Geometric Mean

**Examples**

$$\begin{array}{ccccccccc} \frac{8+2}{2} & = & 5 & > & 4 & = & \sqrt{16} & = & \sqrt{8 \cdot 2} \\ \frac{7+5}{2} & = & 6 & > & 5.916 & \approx & \sqrt{35} & = & \sqrt{7 \cdot 5} \\ \frac{9+9}{2} & = & 9 & = & 9 & = & \sqrt{81} & = & \sqrt{9 \cdot 9} \end{array}$$

**Corollary: $\mathbf{y} = \frac{\mathbf{1}}{\mathbf{x}}$**

- **AM$(\mathbf{x}, \frac{\mathbf{1}}{\mathbf{x}})$:** $\frac{x+\frac{1}{x}}{2}$
- **GM$(\mathbf{x}, \frac{\mathbf{1}}{\mathbf{x}})$:** $\sqrt{x \cdot \frac{1}{x}} = \sqrt{1} = 1$
- **AM$(\mathbf{x}, \frac{\mathbf{1}}{\mathbf{x}}) \geq$ GM$(\mathbf{x}, \frac{\mathbf{1}}{\mathbf{x}})$:** $\frac{x+\frac{1}{x}}{2} \geq 1 \implies x + \frac{1}{x} \geq 2$

# The Arithmetic Mean vs. the Geometric Mean

## Theorem

**AM(x, y) ≥ GM(x, y)** for any two positive numbers $x > 0$ and $y > 0$:

$$\frac{\mathbf{x+y}}{\mathbf{2}} \geq \sqrt{\mathbf{xy}}$$

## Proof

$$\begin{aligned}
(x-y)^2 &\geq 0 \\
x^2 - 2xy + y^2 &\geq 0 \\
x^2 + 2xy + y^2 &\geq 4xy \\
(x+y)^2 &\geq 4xy \\
\sqrt{(x+y)^2} &\geq \sqrt{4xy} \\
x+y &\geq 2\sqrt{xy} \\
\frac{x+y}{2} &\geq \sqrt{xy}
\end{aligned}$$

# The Arithmetic Mean vs. the Geometric Mean

## Visual proofs

- The algebraic proof
  - https://www.youtube.com/watch?v=62G9fak1vyk
  - https://www.youtube.com/watch?v=02yM5LqMWUs
- The traditional geometric proof
  - https://www.youtube.com/watch?v=dUhleJQ6QV4
  - https://www.youtube.com/watch?v=Pt8IX_Q5U1A&t=5s
- Additional proofs
  - https://www.youtube.com/watch?v=IJGwvvQdfgg
  - https://www.youtube.com/watch?v=jNx05gXSblE
  - https://www.youtube.com/watch?v=HFGSsPbYc-s
  - https://www.youtube.com/watch?v=SEPkdPWKdO0

# Which Mean Do You Mean?

**Theorem**

- Let $x > 0$ and $y > 0$ be two positive numbers.
- Let $\sqrt{(x^2+y^2)/2}$ be the **root mean square** of $x$ and $y$.
- Let $(x+y)/2$ be the **arithmetic mean** of $x$ and $y$.
- Let $\sqrt{xy}$ be the **geometric mean** of $x$ and $y$.
- Let $2/(1/x+1/y)$ be the **harmonic mean** of $x$ and $y$.
- Then

$$\max\{x,y\} \geq \sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2} \geq \sqrt{xy} \geq \frac{2}{1/x+1/y} \geq \min\{x,y\}$$

**Remark**

- The inequalities are strict ("$>$" and not "$\geq$") when $x \neq y$.
- When $x = y$, all means equal to $x = y = \max\{x,y\} = \min\{x,y\}$.

# Which Mean Do You Mean?

### The case $x = y$

$$x = \sqrt{\frac{x^2 + x^2}{2}} = \frac{x + x}{2} = \sqrt{x \cdot x} = \frac{2}{1/x + 1/x}$$

### Example: $x = 8$ and $y = 6$

$$\begin{array}{cccccc} \sqrt{\frac{8^2+6^2}{2}} & = & \sqrt{50} & \approx & 7.07 \\ \frac{8+6}{2} & = & 7 & = & 7.0 \\ \sqrt{8 \cdot 6} & = & \sqrt{48} & \approx & 6.928 \\ \frac{2}{1/8+1/6} & = & \frac{48}{7} & \approx & 6.857 \end{array}$$

### On line resource

- https://www.youtube.com/watch?v=_gTClIbHreI&t=11s

# Three Classic Proofs

**From Hippasus about 2500 years ago**

- The square root of 2 is an irrational number.

**From Euclid about 2300 years ago**

- There are infinitely many prime numbers.

**From Pythagoras about 2550 years ago**

- The Pythagorean Theorem.

# The square root of 2 is an irrational number

## Hippasus' proof

- Assume towards **contradiction** that $\frac{p}{q} = \sqrt{2}$ for two positive integers $p$ and $q$ that have no common divisor.
- The assumption is equivalent to $p = \sqrt{2}q$.
- Squaring the equation implies $p^2 = 2q^2$.
- $p$ must be even, because a square of an odd integer is odd and $2q^2$ is even.
- Let $p = 2r$ for a positive integer $r$.
- It follows that $2q^2 = p^2 = (2r)^2 = 4r^2$.
- This is equivalent to $q^2 = 2r^2$.
- $q$ must be even, because a square of an odd integer is odd and $2r^2$ is even.
- A **contradiction** since $2$ is a common divisor of both $p$ and $q$.

# The square root of 2 is an irrational number

## Another proof

- Assume towards **contradiction** that $\frac{p}{q} = \sqrt{2}$ for two positive integers $p$ and $q$ with the **smallest** possible positive integer $q$.
- $\sqrt{2} > 1 \implies \frac{p}{q} > 1 \implies p > q$.
- $\sqrt{2} < 2 \implies \frac{p}{q} < 2 \implies p < 2q$.
- The above inequality is equivalent to $p - q < q$.
- Hence, the fraction $r = \frac{2q-p}{p-q}$ is a fraction of two positive integers whose denominator is smaller than the fraction $\frac{p}{q}$.
- The **contradiction** is established by proving that $r = \sqrt{2}$.

$$r = \frac{2q-p}{p-q} = \frac{2-\frac{p}{q}}{\frac{p}{q}-1} = \frac{2-\sqrt{2}}{\sqrt{2}-1} = \sqrt{2}$$

# Resources

## The original proof

- https://www.youtube.com/watch?v=sbGjr_awePE

## Proofs with animation

- The second proof: https://www.youtube.com/watch?v=-dbzvN4jnfY
- 5 proofs: https://www.youtube.com/watch?v=zEXcsZo4hOQ

## The story of $\sqrt{2}$

- https://www.youtube.com/watch?v=5sKah3pJnHI&feature=youtu.be

## Videos with longer explanations

- Original proof: https://www.youtube.com/watch?v=rOGqq1O1rzI&feature=youtu.be
- Original proof: https://www.youtube.com/watch?v=TwNfWgMHFmI
- Second proof: https://www.youtube.com/watch?v=VeZ1eYkcFus

# The number $\sqrt{2}^{\sqrt{2}}$

## What is known about $\sqrt{2}^{\sqrt{2}}$

- $\sqrt{2}^{\sqrt{2}}$ is a **transcendental** number: a number that is not the root of any non-zero polynomial equation with integer coefficients.
- $\sqrt{2}^{\sqrt{2}}$ is an **irrational** number (**transcendental** numbers are **irrational**).

## Theorem

- There exist **irrational** numbers **s** and **t** such that $\mathbf{s^t}$ is an **integer**.

## Proof

- Set $\mathbf{s} = \sqrt{\mathbf{2}}^{\sqrt{\mathbf{2}}}$ and $\mathbf{t} = \sqrt{\mathbf{2}}$.

$$\mathbf{s^t} = \left(\sqrt{\mathbf{2}}^{\sqrt{\mathbf{2}}}\right)^{\sqrt{\mathbf{2}}} = \left(\sqrt{\mathbf{2}}\right)^{\left(\sqrt{\mathbf{2}}\sqrt{\mathbf{2}}\right)} = \left(\sqrt{\mathbf{2}}\right)^{\mathbf{2}} = \mathbf{2}$$

# There are infinitely many prime numbers

## Euclid's proof

- Let $p_1 < p_2 < \cdots < p_n$ be $n$ prime numbers.
- Let $Q = p_1 p_2 \cdots p_n + 1$.
- If $Q$ is a prime number, then a new prime number is found.
- Otherwise, $Q$ is a product of two or more prime numbers.
  - **The Fundamental Theorem of Arithmetic**.
- None of these prime numbers can be $p_1, \ldots, p_n$ because each one of them is a factor of $Q - 1$.
- Therefore, the prime factors of $Q$ are new prime numbers.
- The above process can continue forever to find infinitely many prime numbers.

# There are infinitely many prime numbers

## Examples for the Euclid's proof

- Set of prime numbers $\{2, 3\}$:
  - $Q = 2 \cdot 3 + 1 = 7$
  - The new prime number is 7
- Set of prime numbers $\{3, 5\}$:
  - $Q = 3 \cdot 5 + 1 = 16 = 2 \cdot 2 \cdot 2 \cdot 2$
  - The new prime number is 2
- Set of prime numbers $\{2, 3, 5\}$:
  - $Q = 2 \cdot 3 \cdot 5 + 1 = 31$
  - The new prime number is 31
- Set of prime numbers $\{2, 3, 5, 7, 11, 13\}$:
  - $Q = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \cdot 509$
  - The new prime numbers are 59 and 509

# There are infinitely many prime numbers

## A similar proof

- Assume towards **contradiction** that $\mathbf{p}$ is the largest prime number.
- Let $\mathbf{Q} = \mathbf{p}! + \mathbf{1}$.
- If $\mathbf{Q}$ is a prime number, then a new prime number larger than $\mathbf{p}$ exists. A **contradiction**.
- Otherwise, $\mathbf{Q}$ is a product of two or more prime numbers.
  - ∗ **The Fundamental Theorem of Arithmetic**.
- None of the prime numbers $\mathbf{2}, \mathbf{3}, \ldots, \mathbf{p}$ can be a factor of $\mathbf{Q}$ because each one of them is a factor of $\mathbf{Q} - \mathbf{1}$.
- Therefore, the prime factors of $\mathbf{Q}$ are larger than $\mathbf{p}$.
- A **contradiction** to the assumption about $\mathbf{p}$.
- There is no largest prime number $\implies$ there are infinitely many prime numbers.

# There are infinitely many prime numbers

## Examples for the second proof

- The largest prime number is 3:
  - $Q = 3! + 1 = 7$
  - The new larger prime number is 7
- The largest prime number is 5:
  - $Q = 5! + 1 = 121 = 11 \cdot 11$
  - The new larger prime number is 11
- The largest prime number is 7:
  - $Q = 7! + 1 = 5041 = 71 \cdot 71$
  - The new larger prime number is 71
- The largest prime number is 11:
  - $Q = 11! + 1 = 39916801$
  - The new larger prime number is 39916801

# Resources

## The original proof by Euclid

- A 2.5 minute video:
  https://www.youtube.com/watch?v=dQmdHpvfyJs
- A 4 minute video:
  https://www.youtube.com/watch?v=ZYkZws-23R8
- A 7 minute video:
  https://www.youtube.com/watch?v=inUkhh8-h-I
- A 7 minute video with more historical background:
  https://www.youtube.com/watch?v=ctC33JAV4FI

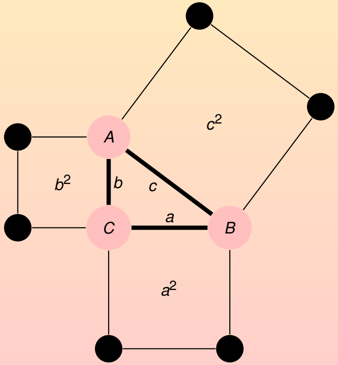
## The second proof

- A 1.5 minute video:
  https://www.youtube.com/watch?v=fOXZgcAsrP8

# The Pythagorean Theorem

## Theorem

- Let $\triangle ABC$ be a right triangle with hypotenuse of length $c$ and two sides of lengths $a$ and $b$.
- Then $c^2 = a^2 + b^2$.



# Sample Proofs

## An algebraic proof

- https://www.youtube.com/watch?v=BNCj-K2hd_k

## Three proofs in 5 minutes

- https://www.youtube.com/watch?v=YompsDlEdtc

## A visual proof

- https://www.youtube.com/shorts/1gFB7dG_t00

# “Famous” Proofs

## Pythagoras’ proof

- https://www.youtube.com/watch?v=4yEyLZUEmQ8&feature=youtu.be

## Euclid’s Proof

- https://www.youtube.com/watch?v=nxi8gV6_50o

## Leonardo da Vinci’s Proof

- https://www.youtube.com/watch?v=Z1GaQdNRdqA

## Albert Einstein’s Proof

- https://www.youtube.com/watch?v=FT71K25L5ZY

## James Garfield’s Proof

- https://www.youtube.com/watch?v=TsKTteGJpZU

# Additional Proofs

## Moving objects proofs

- https://www.youtube.com/watch?v=6HbmSRUOuDE
- https://www.youtube.com/watch?v=odfhFSnLaaw
- https://www.youtube.com/watch?v=o1L86rJWkBc

## Six proofs without words all of them in 1 minute

- https://www.youtube.com/watch?v=COkhrDbNcuA

## More 1-minute proofs

- https://www.youtube.com/watch?v=JRjPsuJ7S-g
- https://www.youtube.com/watch?v=q28H5NY5UVU
- https://www.youtube.com/watch?v=pVTLiO1nl54

## A collection of 10 visual proofs in 8 minutes

- https://www.youtube.com/watch?v=o2mbN452ywA

# Proofs?

## An “experimental” proof

- https://www.youtube.com/watch?v=CAkMUdeB06o
- https://www.youtube.com/watch?v=_e6w5GtkcGI

## Origami Pythagorean Proofs

- https://www.youtube.com/watch?v=ncNt15SNlZE
- https://www.youtube.com/watch?v=b_IV0YDPCm0

# Beyond the Traditional Theorem

## Replacing a right triangle with two lunes

- https://www.youtube.com/watch?v=p3_7bOjiWm8

## A Pythagorean Theorem for Pentagons

- https://www.youtube.com/watch?v=_PyPOXXPIJQ

## Visualization and generalizations

- https://www.youtube.com/watch?v=p-0SOWbzUYI

# Texts About Proofs

## 370 proofs from 900 B.C. to 1940 A.D.

- An 1940 book: “The Pythagorean Propsition” by Elisha S. Loonis

  https://files.eric.ed.gov/fulltext/ED037335.pdf

## Online list of 122 proofs

- http://www.cut-the-knot.org/pythagoras/

## Wikipedia: More about the Theorem and its proofs

- https://en.wikipedia.org/wiki/Pythagorean_theorem

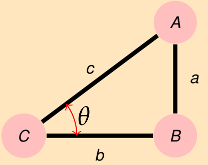
# The Pythagorean Trigonometric Identity

## Theorem

- Let $\triangle ABC$ be a right triangle with hypotenuse $c$ and sides $a$ and $b$
- Let $\theta$ be the angle between the side $b$ and the hypotenuse $c$
- Then $\sin^2(\theta) + \cos^2(\theta) = 1$

## Proof

- $\sin(\theta) = \frac{a}{c} \quad \cos(\theta) = \frac{b}{c}$
- $\sin^2(\theta) + \cos^2(\theta) = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2+b^2}{c^2}$
- $c^2 = a^2 + b^2 \iff \sin^2(\theta) + \cos^2(\theta) = 1$




## Resources

- https://www.youtube.com/watch?v=MyO2MFJbfi4
- https://www.youtube.com/watch?v=Zb0An8dxDas

# The Pythagorean Tree

## Constructing the tree

The construction of the Pythagoras tree begins with a square. Upon this square are constructed two squares, each scaled down by a linear factor of $\sqrt{2}/2$, such that the corners of the squares coincide pairwise. The same procedure is then applied recursively to the two smaller squares, ad infinitum.

## Resources

- https://en.wikipedia.org/wiki/Pythagoras_tree_(fractal)#/media/File:Pythagoras_tree_1_1_13.svg
- https://www.youtube.com/watch?v=0Ih8LuIJ_go

# False Proofs

**$1 = 2$**

$$
\begin{aligned}
a &= b \\
a^2 &= ab \\
a^2 + a^2 &= a^2 + ab \\
2a^2 &= a^2 + ab \\
2a^2 - 2ab &= a^2 + ab - 2ab \\
2a^2 - 2ab &= a^2 - ab \\
2(a^2 - ab) &= 1(a^2 - ab) \\
2 &= 1
\end{aligned}
$$

# False Proofs

**1 = 2**

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 + a^2 &= a^2 + ab \\ 2a^2 &= a^2 + ab \\ 2a^2 - 2ab &= a^2 + ab - 2ab \\ 2a^2 - 2ab &= a^2 - ab \\ 2(a^2 - ab) &= 1(a^2 - ab) \\ 2 &= 1 \end{aligned}$$

**What is wrong?**

- Division by zero is **forbidden**!

# False Proofs

**$2 + 2 = 5$**

$$\begin{aligned}
-20 &= -20 \\
16 - 36 &= 25 - 45 \\
16 - 36 + \frac{81}{4} &= 25 - 45 + \frac{81}{4} \\
\left(4 - \frac{9}{2}\right)^2 &= \left(5 - \frac{9}{2}\right)^2 \\
4 - \frac{9}{2} &= 5 - \frac{9}{2} \\
4 &= 5 \\
2 + 2 &= 5
\end{aligned}$$

# False Proofs

**$2 + 2 = 5$**

$$\begin{aligned} -20 &= -20 \\ 16 - 36 &= 25 - 45 \\ 16 - 36 + \frac{81}{4} &= 25 - 45 + \frac{81}{4} \\ \left(4 - \frac{9}{2}\right)^2 &= \left(5 - \frac{9}{2}\right)^2 \\ 4 - \frac{9}{2} &= 5 - \frac{9}{2} \\ 4 &= 5 \\ 2 + 2 &= 5 \end{aligned}$$

**What is wrong?**

- $(-a)^2 = a^2$ **does not imply** that $-a = a$

# False Proofs

**$1 = -1$**

$$
\begin{aligned}
1 &= \sqrt{1} \\
1 &= \sqrt{-1 \cdot -1} \\
1 &= \sqrt{-1} \cdot \sqrt{-1} \\
1 &= i \cdot i \\
1 &= i^2 \\
1 &= -1
\end{aligned}
$$

# False Proofs

**$1 = -1$**

$$\begin{aligned} 1 &= \sqrt{1} \\ 1 &= \sqrt{-1 \cdot -1} \\ 1 &= \sqrt{-1} \cdot \sqrt{-1} \\ 1 &= i \cdot i \\ 1 &= i^2 \\ 1 &= -1 \end{aligned}$$

**What is wrong?**

- $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ **only if** $a \geq 0$ and $b \geq 0$

**Online resource**

- **$2 = 0$:** https://www.youtube.com/watch?v=1irvvZzbJkU

# Visual False Proofs

## $60 = 59 = 58$?

- https://www.youtube.com/watch?v=iMgFDhpa000

## The staircase paradox

- Illustration: https://upload.wikimedia.org/wikipedia/commons/2/2d/Quadrat_Diagonale.svg
- Wikipedia: https://en.wikipedia.org/wiki/Staircase_paradox
- $5 = 7$? https://www.youtube.com/watch?v=LWPOlZBXtD8
- $\pi = 4$? https://www.youtube.com/shorts/qPjhZL6cyUI