Discrete Structures: Sets

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Sets Basics

Definition

- A **set** is a collection of **objects** (elements, members, items,. . .).

Roster notation

- $S = \{x_1, x_2, \ldots, x_n\}$: The set S **contains** the objects $x_1, x_2, \ldots, x_n$.

Membership

- $\mathbf{x \in S}$: S contains x because the object x is one of the objects of the set S.
- $\mathbf{x \notin S}$: S does not contain x because the object x is not one of the objects of the set S.

Example: $S = \{1, 2, 3, 4, 5\}$ **and** $T = \{1, 3, 5, 7, 9\}$

- $1 \in S$ and $1 \in T$
- $2 \in S$ and $2 \notin T$
- $7 \notin S$ and $7 \in T$
- $8 \notin S$ and $8 \notin T$

Sets Basics

Two fundamental rules

- Objects cannot appear **more than once** in a set.
  - $\{1, 2, 3\}$ is a **set** while $\{1, 2, 2, 3\}$ is a **multi-set** but not a set.
- The **order** among the objects is irrelevant.
  - $\{1, 2, 3\} = \{1, 3, 2\} = \{2, 1, 3\} = \{2, 3, 1\} = \{3, 1, 2\} = \{3, 2, 1\}$

The null (empty) set

- The null set has **no objects** and is denoted by $\emptyset$ or by $\{\}$.

Cardinality (size) of sets

- If S contains $n \geq 1$ objects then its cardinality is n: $|S| = n$.
- The cardinality of the null set is zero: $|\emptyset| = 0$.
- If S contains infinite number of objects then $|S| = \infty$.

Sets Basics

Set-builder notation

- Let P be a property (attribute) that is either **TRUE** or **FALSE** for all possible objects.
- The set $\{x \mid P(x)\}$ or $\{P(x)\}$ contains all the objects x for which $P(x)$ is **TRUE**.

Example: $S = \{1, 2, 3, 4, 5\}$

- $S = \{x \mid x \text{ is an integer and } 1 \leq x \leq 5\}$
- $S = \{\text{all the integers } \geq 1 \text{ and } \leq 5\}$

Example: $T = \{1, 3, 5, 7, 9\}$

- $T = \{x \mid x \text{ is an odd integer and } 1 \leq x \leq 9\}$
- $T = \{\text{all the odd integers } \geq 1 \text{ and } \leq 9\}$

Set Operations

Union

- The union of two sets S and T is the set containing all the objects that belong to at least one of the sets.

$$S \cup T = \{x \mid x \in S \textbf{ or } x \in T\}$$

Intersection

- The intersection of two sets S and T is the set containing all the objects that belong to both sets.

$$S \cap T = \{x \mid x \in S \textbf{ and } x \in T\}$$

Example: $S = \{1, 2, 3, 4, 5\}$ **and** $T = \{1, 3, 5, 7, 9\}$

- $S \cup T = \{1, 2, 3, 4, 5, 7, 9\}$
- $S \cap T = \{1, 3, 5\}$

Union and Intersection Laws

The idempotent laws

- $S \cup S = S$
- $S \cap S = S$

The null set laws

- $S \cup \emptyset = S$
- $S \cap \emptyset = \emptyset$

The commutative laws

- $S \cup T = T \cup S$
- $S \cap T = T \cap S$

Union and Intersection Laws

The Associative laws

- $(R \cup S) \cup T = R \cup (S \cup T) = R \cup S \cup T$
  - ∗ Any order is correct to find $R \cup S \cup T$ for example $S \cup (R \cup T)$.
  - ∗ For $n \geq 2$, any order is correct to find the union of n sets

$$A_1 \cup A_2 \cup \cdots \cup A_n$$

- $(R \cap S) \cap T = R \cap (S \cap T) = R \cap S \cap T$
  - ∗ Any order is correct to find $R \cap S \cap T$ for example $(T \cap R) \cap S$.
  - ∗ For $n \geq 2$, any order is correct to find the union of n sets

$$A_1 \cap A_2 \cap \cdots \cap A_n$$

The Distributive laws

- $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$
- $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$

Set Operations

Difference

- The difference between two sets S and T is the set containing all the objects that belong to S but do not belong to T.

$$S \setminus T = \{x \mid x \in S \textbf{ and } x \notin T\}$$

Symmetric difference

- The symmetric difference between two sets S and T is the set containing all the objects that belong to one of the sets but not to the other set.

$$S \triangle T = (S \setminus T) \cup (T \setminus S) = \{x \mid (x \in S) \oplus (x \in T)\}$$

Example: $S = \{1, 2, 3, 4, 5\}$ **and** $T = \{1, 3, 5, 7, 9\}$

- $S \setminus T = \{2, 4\}$
- $T \setminus S = \{7, 9\}$
- $S \triangle T = \{2, 4, 7, 9\}$

Outside a Set

Complement

- The complement of a set S is the set $S^C = \{x \mid x \notin S\}$ containing all the objects that are not contained in S.
- S^C is usually defined **relatively** to a **universal set** $\mathcal{U}$ that contains all the objects in S and possibly other objects.

$$S_{\mathcal{U}}^C = \mathcal{U} \setminus S = \{x \mid x \in \mathcal{U} \text{ and } x \notin S\}$$

- When $\mathcal{U}$ is known the subscript $\mathcal{U}$ is omitted: $S^C \equiv S_{\mathcal{U}}^C$.

The complement laws

- $\left(S^C\right)^C = S$
- $S \cup S^C = \mathcal{U}$
- $S \cap S^C = \emptyset$
- $\mathcal{U}^C = \emptyset$ and $\emptyset^C = \mathcal{U}$

Outside a Set

Additional Notations

$$S^C \equiv S' \equiv \neg S \equiv \overline{S}$$

Example: $S = \{1,2,3,4,5\}$ **and** $T = \{1,3,5,7,9\}$

- $\mathcal{U} = \{0,1,\ldots,9\}$

$$\begin{aligned} \overline{S} &= \{0,6,7,8,9\} \\ \neg T &= \{0,2,4,6,8\} \\ (S \cup T)' &= \{0,6,8\} \\ (S \cap T)^C &= \{0,2,4,6,7,8,9\} \end{aligned}$$

- $\mathcal{U} = \{1,2,\ldots,10\}$

$$\begin{aligned} \overline{S} &= \{6,7,8,9,10\} \\ \neg T &= \{2,4,6,8,10\} \\ (S \cup T)' &= \{6,8,10\} \\ (S \cap T)^C &= \{2,4,6,7,8,9,10\} \end{aligned}$$

Subsets

Definitions

- T is a **subset** of S if all the objects of T are also objects of S.
- If T is a subset of S than S is **superset** of T.
- A subset T of S is a **proper subset** of S if $T \neq S$.

Notations

- $T \subseteq S$: T is a subset of S.
- $T \subset S$: T is a proper subset of S.

Special subsets

- Any set S is a subset of itself ($S \subseteq S$) but is not a proper subset of itself $S \not\subset S$.
- The null set $\emptyset$ is a subset of every set including itself ($\emptyset \subseteq \emptyset$) and it is a proper subset of every non-empty set S ($\emptyset \subset S$).

Subsets

Equality between sets

- $S = T$ **if and only if** $x \in S \Leftrightarrow x \in T$
- $S = T$ **if and only if** $T \subseteq S$ and $S \subseteq T$

Observations

- $T \subset S \quad \Rightarrow \quad T \subseteq S$
- $T \subseteq S \quad \not\Rightarrow \quad T \subset S$

Examples

- $\{1, 2\} \subseteq \{1, 2, 3\}$ and $\{1, 2\} \subset \{1, 2, 3\}$
- $\{3, 4\} \not\subseteq \{1, 2, 3\}$ and $\{3, 4\} \not\subset \{1, 2, 3\}$
- $\{1, 2, 3\} \subseteq \{1, 2, 3\}$ and $\{1, 2, 3\} \not\subset \{1, 2, 3\}$

Subsets

Propositions

- $(S \cap T) \subseteq S$ and $(S \cap T) \subseteq T$
- $S \subseteq (S \cup T)$ and $T \subseteq (S \cup T)$
- $(S \setminus T) \subseteq S$ and $(T \setminus S) \subseteq T$
- $(S \cap T) \subseteq (S \cup T)$

Transitivity

- If $T \subseteq S$ and $S \subseteq R$ then $T \subseteq R$
- If $T \subseteq S$ and $S \subset R$ then $T \subset R$
- If $T \subset S$ and $S \subseteq R$ then $T \subset R$
- If $T \subset S$ and $S \subset R$ then $T \subset R$

Propositions

- $T \subseteq S$ **if and only if** $T \cap S = T$
- $T \subseteq S$ **if and only if** $T \cup S = S$

The Power Set

Definition

- The **power set** of a set S, denoted by $P(S)$ or 2^S, is a set containing all the subsets of S.

Observation

- The power set of S contains S and the null set: $\emptyset, S \in P(S)$.

Examples

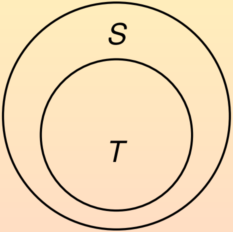
$$\begin{aligned} P(\emptyset) &= \{\emptyset\} \\ P(\{1\}) &= \{\emptyset, \{1\}\} \\ P(\{1,2\}) &= \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \\ P(\{1,2,3\}) &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} \end{aligned}$$

Theorem

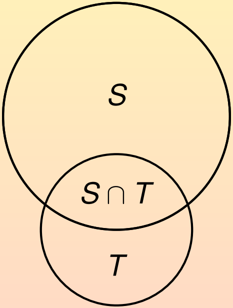
- The cardinality of the power set of a set with n objects is 2^n:

$$|P(S)| = 2^{|S|}$$

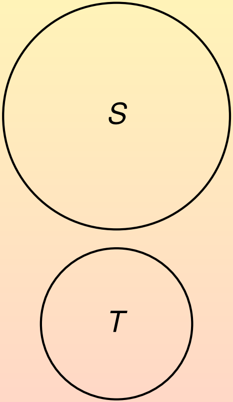
Venn Diagrams for Relationships Between Sets



A **set** and a **subset**:
$T \subseteq S$

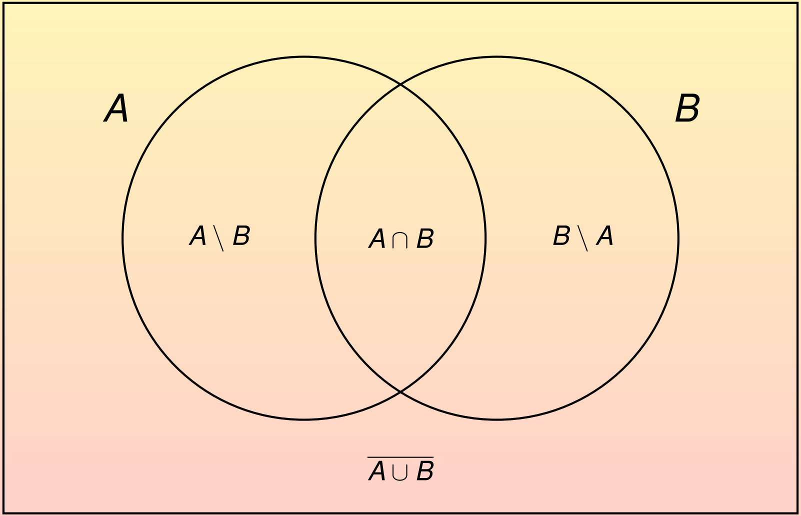


Two **intersecting** sets:
$S \cap T \neq \emptyset$

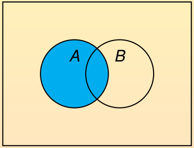


Two **disjoint** sets:
$S \cap T = \emptyset$

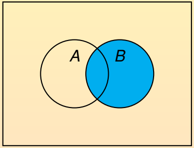
Venn Diagrams for Two Intersecting Sets



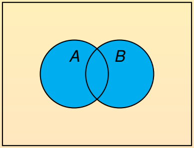
Venn Diagrams for Two Intersecting Sets



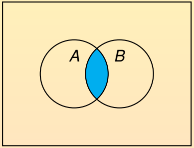
A



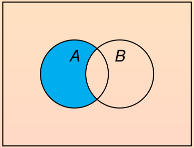
B



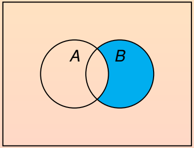
$A \cup B$



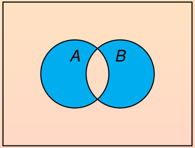
$A \cap B$



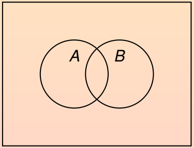
$A \setminus B$



$B \setminus A$

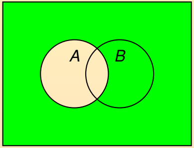


$A \triangle B$

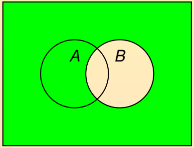


$\emptyset$

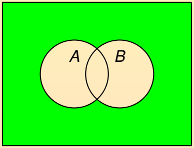
Venn Diagrams for Two Intersecting Sets



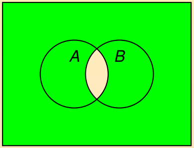
$\overline{A}$



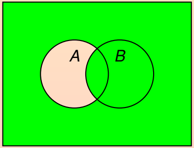
$\overline{B}$



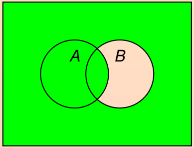
$\overline{A \cup B}$



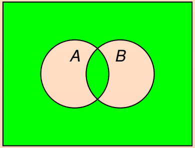
$\overline{A \cap B}$



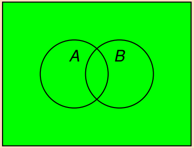
$\overline{A \setminus B}$



$\overline{B \setminus A}$

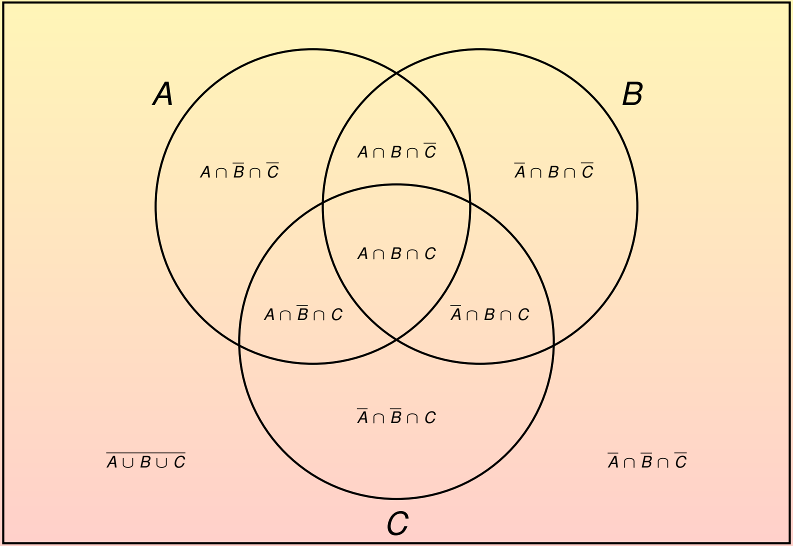


$\overline{A \triangle B}$



$\mathcal{U}$

Venn Diagrams for Three Intersecting Sets

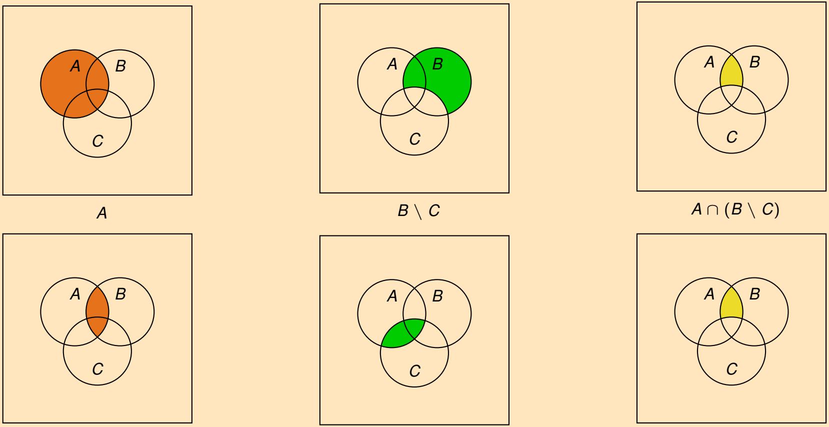


Using Venn Diagrams to Prove Identities

Theorem

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$$

Proof



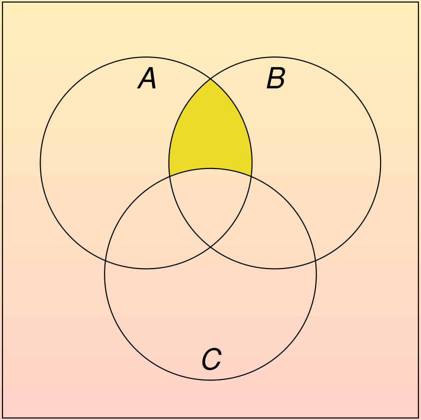
A | $B \setminus C$ | $A \cap (B \setminus C)$

$A \cap B$ | $A \cap C$ | $(A \cap B) \setminus (A \cap C)$

Using Venn Diagrams to Prove Identities

Theorem

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C) = A \cap B \cap \overline{C}$$



$A \cap B \cap \overline{C}$

The Membership Method to Prove Identities

Theorem

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$$

Proof

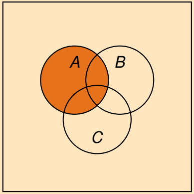
- Let $S = A \cap (B \setminus C)$ and let $T = (A \cap B) \setminus (A \cap C)$
- Assume $x \in S$:
  - ∗ As a result, $x \in A$ and $x \in B$ but $x \notin C$.
  - ∗ Thus, $x \in A \cap B$ and $x \notin (A \cap C)$.
  - ∗ Consequently, $x \in T$.
- Assume $x \in T$:
  - ∗ This implies that $x \in A \cap B$ as long as $x \notin C$.
  - ∗ Therefore, $x \in A$ but also $x \in (B \setminus C)$.
  - ∗ Hence, $x \in S$.
- It follows that $S = T$ because $x \in S$ **if and only if** $x \in T$.

Using Venn Diagrams to Show Non-Identities

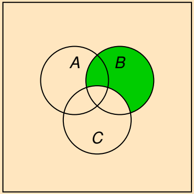
Theorem

$$A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$$

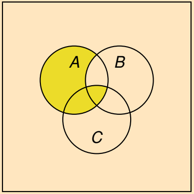
Proof



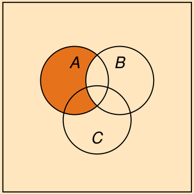
A



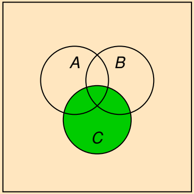
$B \setminus C$



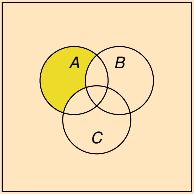
$A \setminus (B \setminus C)$



$A \setminus B$



C



$(A \setminus B) \setminus C$

Using Counter-Examples to Show Non-Identities

Theorem

$$A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$$

Counter-example: $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$

- $B \setminus C = \{3\}$
- $A \setminus (B \setminus C) = \{1, 2\}$
- $(A \setminus B) = \{2\}$
- $(A \setminus B) \setminus C = \{2\}$
- $A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$

Remarks

- Observe that $((A \setminus B) \setminus C) \subseteq (A \setminus (B \setminus C))$
- The inequality happens when $A \cap C$ is not empty
- In the counterexample, $A \cap C = \{1\}$

Venn Diagrams for More Than Three Sets

Setting for the k sets: $A_1, A_2, \ldots, A_k$

- Any object x may belong to a set or not for a total of 2^k membership options depending on x's membership in each set A_i.
  - ∗ In one of the options, x belongs to all of the k sets and therefore it belongs to the intersection of the k sets.
  - ∗ In another option, x does not belong to any of the sets and therefore it belongs to the intersection of all the complements of the k sets.
  - ∗ In general, an option represents the situation in which x belongs to some of the sets and to the complement of the other sets.
- A planar k-Venn Diagram allocates each A_i a **super-zone** such that any option out of the 2^k options is represented by a different **zone**.

Venn Diagrams for More Than Three Sets

Venn Diagrams for four and five sets

- https://www.youtube.com/watch?v=WBot4kqc3ro

Venn Diagrams for four sets with a sample proof

- https://www.youtube.com/watch?v=_i9x_vknLKI&t=2s

Venn Diagrams for seven sets

- https://i.stack.imgur.com/kImwq.png

The De Morgan's Laws

The two sets case

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

The three sets case

$$\neg(A \cup B \cup C) = (\neg A) \cap (\neg B) \cap (\neg C)$$
$$\neg(A \cap B \cap C) = (\neg A) \cup (\neg B) \cup (\neg C)$$

Online proofs with Venn Diagrams

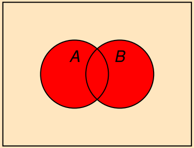
- https://www.youtube.com/watch?v=wFBFFIb2wa8
- https://www.youtube.com/watch?v=oOx7FSzSav4

The De Morgan's Laws for Two Sets

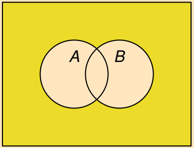
Complement of Union $\equiv$ Intersection of Complements

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$

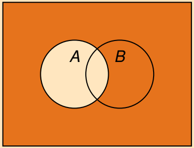
Proof



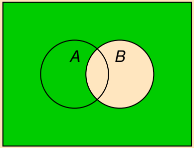
$A \cup B$



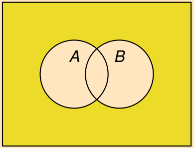
$\overline{A \cup B}$



$\overline{A}$



$\overline{B}$



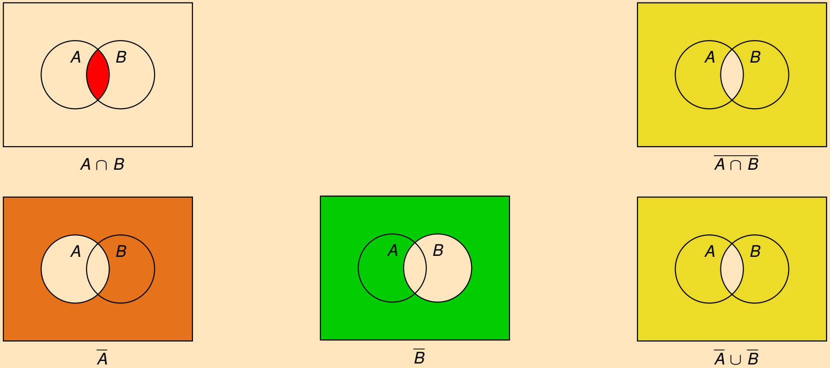
$\overline{A} \cap \overline{B}$

The De Morgan's Laws for Two Sets

Complement of Intersection $\equiv$ Union of Complements

- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof



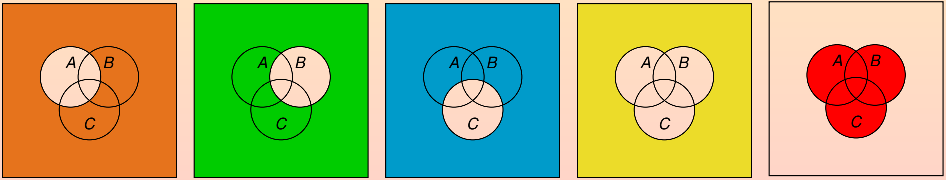
The De Morgan's Laws for Three Sets

Complement of Union ≡ Intersection of Complements

- $\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$

Union ≡ Complement of Intersection of Complements

- $A \cup B \cup C = \overline{\overline{A} \cap \overline{B} \cap \overline{C}}$



$\overline{A}$ $\overline{B}$ $\overline{C}$ $\overline{A} \cap \overline{B} \cap \overline{C}$ $\overline{\overline{A} \cap \overline{B} \cap \overline{C}}$

$A \cup B \cup C$

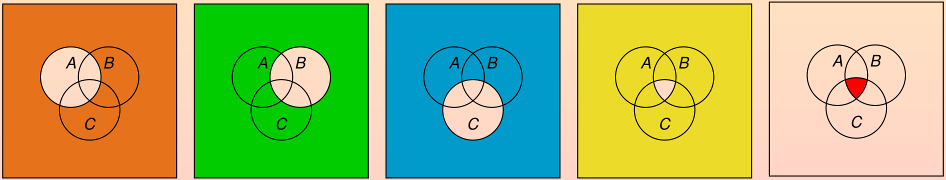
The De Morgan's Laws for Three Sets

Complement of Intersection $\equiv$ Union of Complements

- $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

Intersection $\equiv$ Complement of Union of Complements

- $A \cap B \cap C = \overline{\overline{A} \cup \overline{B} \cup \overline{C}}$



$\overline{A}$ | $\overline{B}$ | $\overline{C}$ | $\overline{A} \cup \overline{B} \cup \overline{C}$ | $\overline{\overline{A} \cup \overline{B} \cup \overline{C}}$

$A \cap B \cap C$

The General De Morgan's Laws

Complement of Unions $\equiv$ Intersection of Complements

$$\left(\bigcup_{i=1}^{n} A_i\right)' = (A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n' = \bigcap_{i=1}^{n} A_i'$$

Complement of Intersections $\equiv$ Union of Complements

$$\left(\bigcap_{i=1}^{n} A_i\right)' = (A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n' = \bigcup_{i=1}^{n} A_i'$$

The General De Morgan's Laws

Theorem

- Complement of Unions $\equiv$ Intersection of Complements

Proof

- Let $L = (A_1 \cup A_2 \cup \cdots \cup A_n)'$ and $R = A_1' \cap A_2' \cap \cdots \cap A_n'$
- Prove that $x \in L \Leftrightarrow x \in R$

$$
\begin{aligned}
x \in L \quad & \Longleftrightarrow \quad x \in (A_1 \cup A_2 \cup \cdots \cup A_n)' \\
& \Longleftrightarrow \quad x \notin (A_1 \cup A_2 \cup \cdots \cup A_n) \\
& \Longleftrightarrow \quad x \notin A_i \textbf{ for all } i \in \{1, \ldots, n\} \\
& \Longleftrightarrow \quad x \in A_i' \textbf{ for all } i \in \{1, \ldots, n\} \\
& \Longleftrightarrow \quad x \in (A_1' \cap A_2' \cap \cdots \cap A_n') \\
& \Longleftrightarrow \quad x \in R
\end{aligned}
$$

The General De Morgan's Laws

Theorem

- Complement of Intersections $\equiv$ Union of the Complements

Proof

- Let $L = (A_1 \cap A_2 \cap \cdots \cap A_n)'$ and $R = A_1' \cup A_2' \cup \cdots \cup A_n'$
- Prove that $x \in L \Leftrightarrow x \in R$

$$
\begin{aligned}
x \in L & \iff x \in (A_1 \cap A_2 \cap \cdots \cap A_n)' \\
& \iff x \notin (A_1 \cap A_2 \cap \cdots \cap A_n) \\
& \iff x \notin A_i \textbf{ for at least one } i \in \{1, \ldots, n\} \\
& \iff x \in A_i' \textbf{ for at least one } i \in \{1, \ldots, n\} \\
& \iff x \in (A_1' \cup A_2' \cup \cdots \cup A_n') \\
& \iff x \in R
\end{aligned}
$$

Online Resources

Introduction to Higher Math: A 30-minute introductory video

www.youtube.com/watch?v=HoWC0GQWGaA&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX&index=6&t=546s

TrevTutor lectures

- Introduction to Set Theory: https://www.youtube.com/watch?v=tyDKR4FG3Yw
- Subsets and Power Sets: https://www.youtube.com/watch?v=H5D6EAezsXQ
- Set Operations: https://www.youtube.com/watch?v=4TlCToZZ5gA
- All lectures and practice sessions (first 13 videos):

  https://www.youtube.com/playlist?list=PLDDGPdw7e6Ag1EIznZ-m-qXu4XX3A0cIz

On-line tutorials

- Short: www.tutorialspoint.com/discrete_mathematics/discrete_mathematics_sets.htm
- Long: www.math-only-math.com/set-theory.html

Online Resources

Basic Set Theory in 5 parts (approximately 22 minutes)

- Part I: Introduction to Sets and Set Notation
  https://www.youtube.com/watch?v=rRk0d6P4oUI
- Part II: Common Universal Sets
  https://www.youtube.com/watch?v=pL7m76igVFA
- Part III: Set Operations, Complements and Subsets
  https://www.youtube.com/watch?v=GOgNoYw9qYE
- Part IV: More About Elements and Subsets
  https://www.youtube.com/watch?v=ZapWv4v7ybQ
- Part V: Cardinality
  https://www.youtube.com/watch?v=CeW_FAp2v8U&t=3s

More videos

- Set theory basics (approximately 10 minutes):
  https://www.youtube.com/watch?v=IqX0emlxVEU
- Types of sets (approximately 8 minutes):
  https://www.youtube.com/watch?v=VBzlvKP-2yI

The Principle of Inclusion Exclusion

Setting

- Let S be a union of n sets: $S = S_1 \cup S_2 \cup \cdots \cup S_n = \bigcup_{i=1}^{n} S_i$

Goal

- Find the **cardinality** of S.

Proposition

- If all the n sets are **pairwise disjoint**: $S_i \cap S_j = \emptyset$ for any $1 \leq i, j \leq n$, then

$$|S| = |S_1| + |S_2| + \cdots + |S_n| = \sum_{i=1}^{i=n} |S_i|$$

The Principle of Inclusion Exclusion

An upper bound

- The size of S cannot exceed the sum of the sizes of all the n sets:

$$|S| \leq \sum_{i=1}^{i=n} |S_i|$$

A lower bound

- The size of S is at least the size of the largest set:

$$|S| \geq \max_{i=1}^{i=n} |S_i|$$

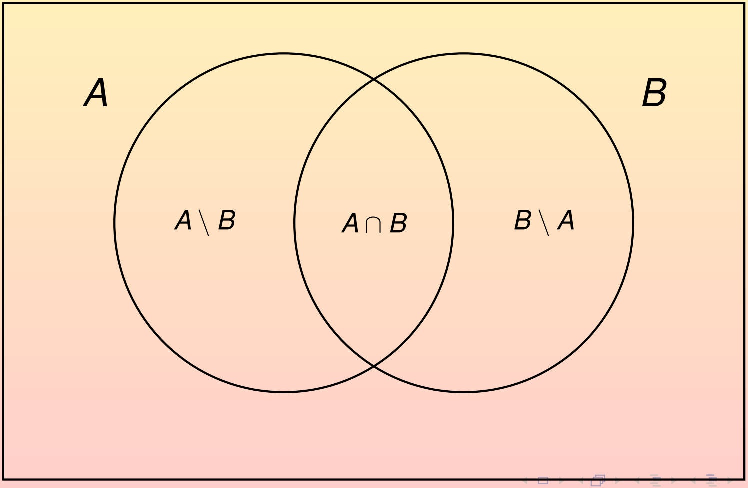
Challenge

- Find the **cardinality** of S when the n sets are not pairwise disjoint by avoiding **over-counting** or **under-counting**.

The Principle of Inclusion Exclusion for Two Sets

Theorem

$$|A \cup B| = |A| + |B| - |A \cap B|$$



The Principle of Inclusion Exclusion for Two Sets

Theorem

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof

- Any $x \in A \cup B$ is counted exactly once in the expression $|A| + |B| - |A \cap B|$.
- **Case $x \in A \setminus B$:**
  - x is counted only once in $|A|$.
- **Case $x \in B \setminus A$:**
  - x is counted only once in $|B|$.
- **Case $x \in A \cap B$:**
  - x is counted twice in $|A|$ and $|B|$.
  - but the count is decreased by one in $|A \cap B|$.

Corollary

$$|A \cup B| + |A \cap B| = |A| + |B|$$

The Principle of Inclusion Exclusion: Application

Definition

- Assume $n \geq 1$ is a multiple of 6 and $i \in \{2, 3, 6\}$.
- Denote by $M_i^n = \{i, 2i, 3i, \ldots, n\}$ the set of all the integers between 1 and n that are multiples of i.

Propositions

- $|M_i^n| = n/i$
- $M_6^n = M_2^n \cap M_3^n$

Example

- $M_2^{18} = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$ $\quad |M_2^{18}| = 18/2 = 9$
- $M_3^{18} = \{3, 6, 9, 12, 15, 18\}$ $\quad |M_3^{18}| = 18/3 = 6$
- $M_6^{18} = \{6, 12, 18\}$ $\quad |M_6^{18}| = 18/6 = 3$

The Principle of Inclusion Exclusion: Application

Problem

- Let n be a multiple of 6.
- Let $M(n) = M_2^n \cup M_3^n$ be the set of all integers between 1 and n that are multiples of either 2 or 3 or both.
- What is the size of $M(n)$?

Answer

- By the principle of inclusion exclusion:

$$\begin{aligned} |M(n)| &= |M_2^n| + |M_3^n| - |M_2^n \cap M_3^n| \\ &= |M_2^n| + |M_3^n| - |M_6^n| \\ &= \frac{n}{2} + \frac{n}{3} - \frac{n}{6} = \frac{3+2-1}{6} = \frac{4n}{6} = \frac{2n}{3} \end{aligned}$$

The Principle of Inclusion Exclusion: Application

Example: $n = 12$

- $M(12) = \{2, 3, 4, 6, 8, 9, 10, 12\}$
- $|M(12)| = \frac{2 \cdot 12}{3} = 8$

Example: $n = 18$

- $M(18) = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18\}$
- $|M(18)| = \frac{2 \cdot 18}{3} = 12$

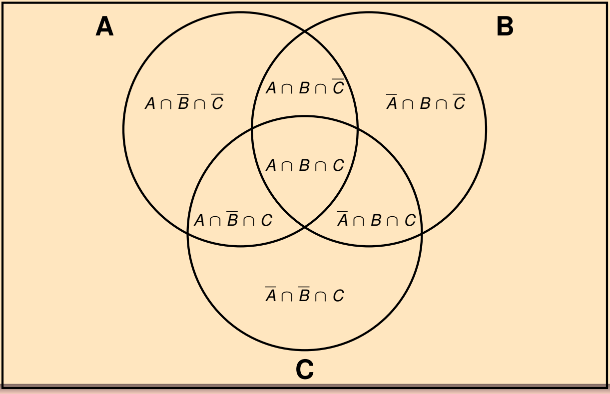
How many integers between 1 and 24 are not multiples of 2 or 3?

- $\overline{M(24)} = \{1, 5, 7, 11, 13, 17, 19, 23\}$
- $|\overline{M(24)}| = 24 - \frac{2 \cdot 24}{3} = 8$

The Principle of Inclusion Exclusion for Three sets

Theorem

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



An animated Proof

- https://www.youtube.com/watch?v=vVZwe3TCJT8

The Principle of Inclusion Exclusion for Three sets

Theorem

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof

- Let $R = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- Any $x \in A \cup B \cup C$ is counted exactly once in R depending on how many sets among A, B, and C contain x.
- **Case x belongs to exactly one of the sets A, B, and C:** ???
- **Case x belongs to exactly two of the sets A, B, and C:** ???
- **Case $x \in A \cap B \cap C$:** ???

The Principle of Inclusion Exclusion for Three sets

Theorem

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof

- Let $R = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- Any $x \in A \cup B \cup C$ is counted exactly once in R depending on how many sets among A, B, and C contain x.
- **Case x belongs to exactly one of the sets A, B, and C:**
  - x does not belong to the sets $A \cap B$, $A \cap C$, $B \cap C$, and $A \cap B \cap C$.
  - Therefore, x is counted exactly once in R.
- **Case x belongs to exactly two of the sets A, B, and C:** ???
- **Case $x \in A \cap B \cap C$:** ???

The Principle of Inclusion Exclusion for Three sets

Theorem

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof

- Let $R = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- Any $x \in A \cup B \cup C$ is counted exactly once in R depending on how many sets among A, B, and C contain x.
- **Case x belongs to exactly one of the sets A, B, and C:** $\checkmark\checkmark\checkmark$
- **Case x belongs to exactly two of the sets A, B, and C:**
  - x is counted twice in $|A|$, $|B|$, and $|C|$.
  - The count is decreased by one in one of $|A \cap B|$, $|A \cap C|$, $|B \cap C|$.
  - The count is not changed since x does not belong to $A \cap B \cap C$.
- **Case $x \in A \cap B \cap C$:** ???

The Principle of Inclusion Exclusion for Three sets

Theorem

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof

- Let $R = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- Any $x \in A \cup B \cup C$ is counted exactly once in R depending on how many sets among A, B, and C contain x.
- **Case x belongs to exactly one of the sets A, B, and C:** ✓✓✓
- **Case x belongs to exactly two of the sets A, B, and C:** ✓✓✓
- **Case $x \in A \cap B \cap C$:**
  - x is counted four times in $|A|$, $|B|$, $|C|$, and $A \cap B \cap C$.
  - The count is decreased three times in $|A \cap B|$, $|A \cap C|$, and $|B \cap C|$.

The Principle of Inclusion Exclusion for Three sets

Theorem

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof

- Let $R = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- Any $x \in A \cup B \cup C$ is counted exactly once in R depending on how many sets among A, B, and C contain x.
- **Case x belongs to exactly one of the sets A, B, and C:** $\checkmark\checkmark\checkmark$
- **Case x belongs to exactly two of the sets A, B, and C:** $\checkmark\checkmark\checkmark$
- **Case $x \in A \cap B \cap C$:** $\checkmark\checkmark\checkmark$

The Principle of Inclusion Exclusion: Application

Definition

- Assume $n \geq 1$ is a multiple of 30 and $i \in \{2, 3, 5, 6, 10, 15, 30\}$.
- Denote by $M_i^n = \{i, 2i, 3i, \ldots, n\}$ the set of all the integers between 1 and n that are multiples of i.

Propositions

- $|M_i^n| = n/i$
- $M_6^n = M_2^n \cap M_3^n$
- $M_{10}^n = M_2^n \cap M_5^n$
- $M_{15}^n = M_3^n \cap M_5^n$
- $M_{30}^n = M_2^n \cap M_3^n \cap M_5^n$

The Principle of Inclusion Exclusion: Application

Problem

- Let n be a multiple of 30.
- Let $M(n) = M_2^n \cup M_3^n \cup M_5^n$ be the set of all integers between 1 and n that are multiples of at least one of the integers from $\{2, 3, 5\}$.
- What is the size of $M(n)$?.

Answer

- By the principle of inclusion exclusion:

$$\begin{aligned}
|M(n)| &= |M_2^n| + |M_3^n| + |M_5^n| - |M_6^n| - |M_{10}^n| - |M_{15}^n| + |M_{30}^n| \\
&= \frac{n}{2} + \frac{n}{3} + \frac{n}{5} - \frac{n}{6} - \frac{n}{10} - \frac{n}{15} + \frac{n}{30} \\
&= \frac{(15 + 10 + 6 - 5 - 3 - 2 + 1)n}{30} \\
&= \frac{22n}{30} = \frac{11n}{15}
\end{aligned}$$

The Principle of Inclusion Exclusion: Application

Example: $n = 30$

- $M(30) = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30\}$
- $|M(30)| = \frac{11 \cdot 30}{15} = 22$

Example: $n = 60$

- $M(60) = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30$
  $32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 50, 51, 52, 54, 55, 56, 57, 58, 60\}$
- $|M(60)| = \frac{11 \cdot 60}{15} = 44$

How many integers between 1 and 90 are not multiples of 2, 3, 5?

- $\overline{M(90)} = \{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89\}$
- $|\overline{M(90)}| = 90 - \frac{11 \cdot 90}{15} = 24$

The Principle of Inclusion Exclusion for 2, 3, 4 Sets

Two Sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Three Sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Four Sets

$$\begin{aligned} |A \cup B \cup C \cup D| \quad = \quad & +|A| + |B| + |C| + |D| \\ & -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & -|A \cap B \cap C \cap D| \end{aligned}$$

The Principle of Inclusion Exclusion for Many Sets

General method

- Include (**add**) the cardinalities of the sets.
- Exclude (**subtract**) the cardinalities of the pair-wise intersections.
- Include (**add**) the cardinalities of the triple-wise intersections.
- Exclude (**subtract**) the cardinalities of the quadruple-wise intersections.
- Include (**add**) the cardinalities of the quintuple-wise intersections.
- Continue, until the cardinality of the n-tuple-wise intersection is
  - ∗ included (**added**) if n is odd
  - ∗ excluded (**subtracted**) if n even.

The Principle of Inclusion Exclusion: Resources

Venn Diagrams with Three Categories

- https://www.youtube.com/watch?v=LIzIhmKlYPk

The principle of Inclusion Exclusion for 4 sets

- https://drive.google.com/file/d/1YGINz8u_YIxhPvv2_OqeL2GlTGDivSSG/view?ts=6565076a

The Trev Tutor Course: Inclusion Exclusion Principle

- https://www.youtube.com/watch?v=GS7dIWA6Hpo&feature=youtu.be

A tutorial in 3 parts (approximately 38.5 minutes)

- The principle: https://www.youtube.com/watch?v=4qd9JkYVGBU
- The proof: https://www.youtube.com/watch?v=y9123Dx6cJA
- Basic example: https://www.youtube.com/watch?v=Xd2ZGvMqXsc

The Numbers

Natural numbers

- $\mathbb{N}$ – The set of all natural (counting) numbers $1, 2, 3, \ldots$

Integers

- $\mathbb{Z}$ – The set of all integers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

Rationals

- $\mathbb{Q}$ – The set of all fractions p/q where p (the numerator) and q (the denominator) are integers and $q \neq 0$

Reals

- $\mathbb{R}$ – The set of all numbers of the form $n.d_1 d_2 d_3 \ldots$ where n is an integer and $d_i \in \{0, 1, \ldots, 9\}$ for all $i = 1, 2, \ldots$

Observation

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Sets of the Same Cardinality (Size)

One-to-one mapping

- $f : S \to T$ from a set S to a set T is a **one-to-one mapping** if:
  - $*$ For all $s \in S$ there exists $t \in T$ such that $f(s) = t$.
  - $*$ $s_1 \neq s_2 \;\Rightarrow\; f(s_1) \neq f(s_2)$.
  - $*$ For all $t \in T$ there exists $s \in S$ such that $f(s) = t$.

Proposition

- A **one-to-one mapping** $f : S \to T$ implies a **one-to-one mapping** $g : T \to S$ in which $f(s) = t$ if and only if $g(t) = s$.

Cardinality

- Two sets S and T have the same **cardinality** ($|S| \approx |T|$) if and only if there is a **one-to-one mapping** $f : S \to T$ from S to T.

Remark

- For finite sets $|S| = |T|$ is **equivalent** to $|S| \approx |T|$.

Theorems

$|\mathbb{Z}| \approx |\mathbb{N}|$

- Although it **"seems"** like $|\mathbb{Z}|$ is twice as large as $|\mathbb{N}|$.

$|\mathbb{Q}| \approx |\mathbb{Z}|$

- Although there are **infinitely many** rational numbers between any two consecutive integers.

$|\mathbb{R}| \not\approx |\mathbb{Q}|$

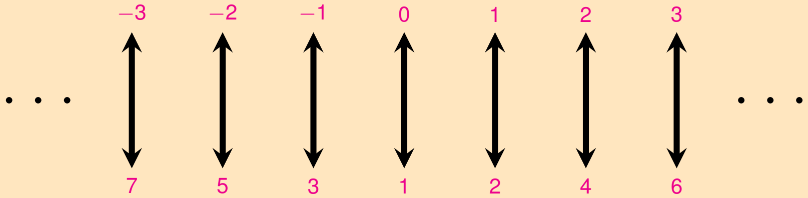
- Although there are **infinitely many** rational numbers between any two real numbers.
- Since $\mathbb{Q} \subset \mathbb{R}$ (every rational number is also a real number), it follows that "$\mathbb{Q} < \mathbb{R}$" from a cardinality point of view.

$|\mathbb{Z}| \approx |\mathbb{N}|$

The mapping $f : \mathbb{Z} \to \mathbb{N}$

$$f(n) = \begin{cases} 2n & \text{for a positive integer } n \\ -(2n-1) & \text{for a negative integer } n \\ 1 & \text{for } n = 0 \end{cases}$$

Proof without words



Infinity and Beyond

The infinite hotel

- https://www.youtube.com/watch?v=Uj3_KqkI9Zo

How big is infinity?

- https://www.youtube.com/watch?v=UPA3bwVVzGI&feature=youtu.be

How can infinity come in many sizes?

- https://www.quantamagazine.org/how-can-infinity-come-in-many-sizes-20260223/

The infinite gift and computing its measurements

- https://www.youtube.com/watch?v=YCCNWNXXSx4

Infinity is bigger than you think!

- https://www.youtube.com/watch?v=elvOZm0d4H0

$S = 1 - 1 + 1 - 1 + 1 - \cdots = ???$

$0 = 1$?

- $S = (1-1) + (1-1) + \cdots = 0 + 0 + \cdots = 0$
- $S = 1 + (-1+1) + (-1+1) + \cdots = 1 + 0 + 0 + \cdots = 1$

$S = n$ for any positive integer n?

$$\begin{aligned} S &= 1_1 - 1_2 + 1_3 - 1_4 + 1_5 - 1_6 + \cdots + 1_{2n-1} - 1_{2n} + \cdots \\ &= (1_1 + 1_3 + \cdots + 1_{2n-1}) + (1_{2n+1} - 1_2) + (1_{2n+3} - 1_4) + \cdots \\ &= n + 0 + 0 + \cdots = n \end{aligned}$$

$S = 1/2$?

$$\begin{aligned} S &= 1 - 1 + 1 - 1 + 1 + \cdots \\ 1 - S &= 1 - (1 - 1 + 1 - 1 + \cdots) \\ 1 - S &= 1 - 1 + 1 - 1 + 1 + \cdots \\ 1 - S &= S \\ S &= 1/2 \end{aligned}$$

The Barber and Russell's Paradoxes

The barber paradox

- In a small village all men are being shaved every day.
- The village has only one barber.
- This barber shaves only men who don't shave themselves.
- Does the barber shave himself?
- **No:** Then he must shave himself $\Longrightarrow$ **a contradiction!**
- **Yes:** Then he cannot shave himself! $\Longrightarrow$ **a contradiction!**

Russell's Paradox

- Let R be the set of all sets that are not members of themselves.
- If R is not a member of itself, then its definition dictates that R must contain itself: **a contradiction**
- If R contains itself, then R **contradicts** its own definition as the set of all sets that are not members of themselves.
- Symbolically: $(R = \{S | S \notin S\}) \Longrightarrow (R \in R \Leftrightarrow R \notin R)$

Online Resources for Both Paradoxes

The barber paradox:

- https://www.youtube.com/watch?v=nI-MMJiFpqo
- https://www.youtube.com/watch?v=qGjznKRLClA&feature=youtu.be
- https://www.youtube.com/watch?v=hDGjHYv9dmw

Russell's paradox:

- https://www.youtube.com/watch?v=83X1YWDXdDA

Both paradoxes:

- https://www.youtube.com/watch?v=8JDgNL5HsPo

The Zeno's Paradoxes

The dichotomy paradox

- https://www.youtube.com/watch?v=EfqVnj-sgcc&feature=youtu.be

Achilles and the Tortoise I

- https://www.youtube.com/watch?v=skM37PcZmWE

Achilles and the Tortoise II

- https://www.youtube.com/watch?v=3vNlf2zGLaE

Achilles and the tortoise and beyond

- https://www.youtube.com/watch?v=NCtw5f6XPF4

Relations and Functions

- Relations: https://www.youtube.com/watch?v=j5cj_4Mk9_M&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX&index=9&t=1346s
- Functions: https://www.youtube.com/watch?v=Tk5_B7w5fiY&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX&index=10&t=891s