

Discrete Math

Counting and Combinatorics Practice Problems

Name and ID:

1. In all parts of this problem, the goal is to count permutations of the numbers $\{1, 2, \dots, n\}$ (for $n = 4$, $n = 6$, and general n) that follow some restrictions.

You **MUST** show how you came up with your answers.

(a) How many permutations of the numbers $\{1, 2, 3, 4\}$ are there?

In how many permutations of the numbers $\{1, 2, 3, 4\}$ the first number is 1?

In how many permutations of the numbers $\{1, 2, 3, 4\}$ the last number is 4?

In how many permutations of the numbers $\{1, 2, 3, 4\}$ the first number is 1 and the last number is 4?

In how many permutations of the numbers $\{1, 2, 3, 4\}$ the first number is not 1?

In how many permutations of the numbers $\{1, 2, 3, 4\}$ the last number is not 4?

In how many permutations of the numbers $\{1, 2, 3, 4\}$ the first number is not 1 and the last number is not 4?

(b) How many permutations of the numbers $\{1, 2, 3, 4, 5, 6\}$ are there?

In how many permutations of the numbers $\{1, 2, 3, 4, 5, 6\}$ the first number is 1?

In how many permutations of the numbers $\{1, 2, 3, 4, 5, 6\}$ the last number is 6?

In how many permutations of the numbers $\{1, 2, 3, 4, 5, 6\}$ the first number is 1 and the last number is 6?

In how many permutations of the numbers $\{1, 2, 3, 4, 5, 6\}$ the first number is not 1?

In how many permutations of the numbers $\{1, 2, 3, 4, 5, 6\}$ the last number is not 6?

In how many permutations of the numbers $\{1, 2, 3, 4, 5, 6\}$ the first number is not 1 and the last number is not 6?

(c) How many permutations of the numbers $\{1, 2, \dots, n\}$ are there?

In how many permutations of the numbers $\{1, 2, \dots, n\}$ the first number is 1?

In how many permutations of the numbers $\{1, 2, \dots, n\}$ the last number is n ?

In how many permutations of the numbers $\{1, 2, \dots, n\}$ the first number is 1 and the last number is n ?

In how many permutations of the numbers $\{1, 2, \dots, n\}$ the first number is not 1?

In how many permutations of the numbers $\{1, 2, \dots, n\}$ the last number is not n ?

In how many permutations of the numbers $\{1, 2, \dots, n\}$ the first number is not 1 and the last number is not n ?

2. The goal is to count the number of passwords with repetitions of length $k \geq 1$ on the digits $\{0, 1, \dots, 9\}$. Unlike numbers, passwords may start with one or more 0.

Examples: (80) is a password of length 2, (00289) is a password of length 5, (9876543210) is a password of length 10, and (777) is a password of length 3.

Solve the following 12 problems. Explain how you came up with your answers. You do not need to evaluate expressions for large numbers like n^i .

Remark: When you solve the general case of $k \geq 1$ in parts (i)–(l), check your answers with the trivial case of $k = 1$ and your answers for the cases $k = 2$ and $k = 3$ in parts (a)–(h).

(a) How many passwords of length 2 are there?

(b) How many passwords of length 2 are there that do not contain the digit 0?

(c) How many passwords of length 2 are there that contain the digit 0 at least once?

(d) How many passwords of length 2 are there that contain at least one 0 and one 1?

(e) How many passwords of length 3 are there?

(f) How many passwords of length 3 are there that do not contain the digit 0?

(g) How many passwords of length 3 are there that contain the digit 0 at least once?

(h) How many passwords of length 3 are there that contain at least one 0 and one 1?

(i) How many passwords of length $k \geq 1$, as a function of k , are there?

(j) How many passwords of length $k \geq 1$, as a function of k , are there that do not contain the digit 0?

(k) How many passwords of length $k \geq 1$, as a function of k , are there that contain the digit 0 at least once?

(l) How many passwords of length $k \geq 2$, as a function of k , are there that contain at least one 0 and one 1?

3. An ice cream shop sells eleven different flavors of ice cream.

Justify your answers for Part (a) and Part (b).


- (a) How many different two-scoop cones are there if the order among the scoops does not matter? That is, a cone with a vanilla scoop on top of a chocolate scoop is considered the same as a cone with chocolate on top of vanilla.

- (b) How many different two-scoop cones are there if the order of the scoops does matter? In this case, cones with two scoops of the same flavour, e.g., two scoops of vanilla, should only be counted once.

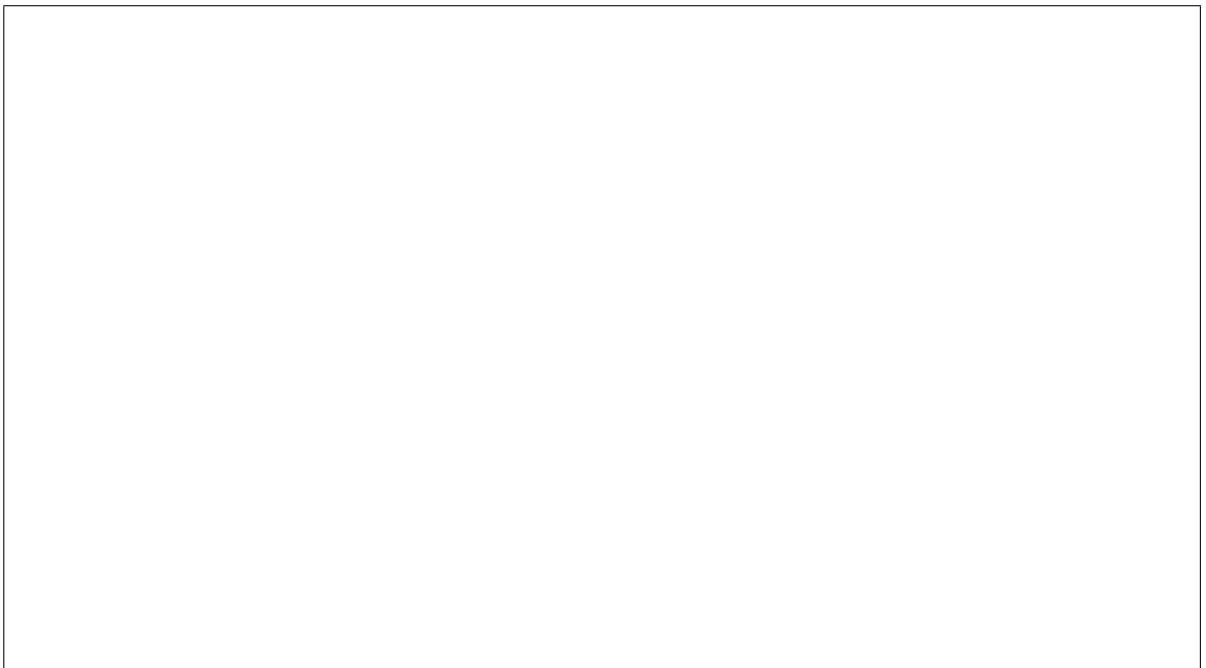
4. Teams from twelve schools participate in a basketball tournament.

Justify your answers for Part (a) and Part (b).

- (a) How many games are there in the tournament if each team must play each other team exactly once?



- (b) How many games are there in the tournament if each team plays exactly four other teams?



5. A club has nine members.

Justify your answers for Part (a), Part (b), and Part (c).

- (a) In how many ways can the club select a president and a treasurer from among its members?

- (b) In how many ways can the club select a two-person executive committee from among its members?

- (c) In how many ways can the club select a president and a two-person executive advisory board from among its members (assuming that the president is not on the advisory board)?

6. For an integer $n \geq 3$, the following formula is a mathematical identity:

$$n \binom{n-1}{2} = \binom{n}{2} (n-2).$$

(a) Prove that this identity is correct using the formula for $\binom{k}{2}$.

(b) Describe a combinatorial explanation of why the identity is true.

Hint: Think in terms of officers and committees in a club.

7. For $n \geq 3$, simplify

$$\binom{n}{3} + \binom{n+1}{3} + \binom{n+2}{3}$$



8. For $0 \leq k \leq n$, describe a combinatorial proof to the identity

$$\binom{n+1}{k+1} = \sum_{m=k}^n \binom{m}{k}$$

