Discrete Math

Counting and Combinatorics Practice Problems: Solutions

1. In all parts of this problem, the goal is to count permutations of the numbers $\{1, 2, \ldots, n\}$ (for $n = 4$, $n = 6$, and general n) that follow some restrictions.

   (a) $n = 4$: counting permutations of the numbers $\{1, 2, 3, 4\}$.

      i. There are $4! = 24$ permutations:

$$\begin{matrix} (1234) & (1243) & (1324) & (1342) & (1423) & (1432) \\ (2134) & (2143) & (2314) & (2341) & (2413) & (2431) \\ (3124) & (3142) & (3214) & (3241) & (3412) & (3421) \\ (4123) & (4132) & (4213) & (4231) & (4312) & (4321) \end{matrix}$$

      ii. There are $3! = 6$ permutations in which the first number is 1.

$$\begin{matrix} (1234) & (1243) & (1324) & (1342) & (1423) & (1432) \end{matrix}$$

      iii. There are $3! = 6$ permutations in which the last number is 4:

$$\begin{matrix} (1234) & (1324) & (2134) & (2314) & (3124) & (3214) \end{matrix}$$

      iv. There are $2! = 2$ permutations in which the first number is 1 and the last number is 4:

$$\begin{matrix} (1234) & (1324) \end{matrix}$$

      v. There are $4! - 3! = 3 \cdot 3! = 18$ permutations in which the first number is not 1:

$$\begin{matrix} (2134) & (2143) & (2314) & (2341) & (2413) & (2431) \\ (3124) & (3142) & (3214) & (3241) & (3412) & (3421) \\ (4123) & (4132) & (4213) & (4231) & (4312) & (4321) \end{matrix}$$

      vi. There are $4! - 3! = 3 \cdot 3! = 18$ permutations in which the last number is not 4.

$$\begin{matrix} (1243) & (1342) & (1423) & (1432) & (2143) & (2341) \\ (2413) & (2431) & (3142) & (3241) & (3412) & (3421) \\ (4123) & (4132) & (4213) & (4231) & (4312) & (4321) \end{matrix}$$

      vii. There are $4! - 2 \cdot 3! + 2! = 14$ permutations in which the first number is not 1 and the last number is not 4.

$$\begin{matrix} (2143) & (2341) & (2413) & (2431) & (3142) & (3241) & (3412) \\ (3421) & (4123) & (4132) & (4213) & (4231) & (4312) & (4321) \end{matrix}$$

(b) $n = 6$: counting permutations of the numbers $\{1, 2, 3, 4, 5, 6\}$.

i. There are $6! = 720$ permutations.

Proof: For any integer $n \geq 1$, there are $n!$ permutations.

ii. There are $5! = 120$ permutations in which the first number is 1.

Proof: After committing on 1 at the first position, there are 5! ways to arrange the numbers $2, 3, 4, 5, 6$ in the other 5 positions.

iii. There are $5! = 120$ permutations in which the last number is 6.

Proof: After committing on 6 at the last position, there are 5! ways to arrange the numbers $1, 2, 3, 4, 5$ in the other 5 positions.

iv. There are $4! = 24$ permutations in which the first number is 1 and the last number is 6.

Proof: After committing on 1 at the first position and 6 at the last position, there are 4! ways to arrange the numbers $2, 3, 4, 5$ in the remaining 4 positions.

v. There are $6! - 5! = 5 \cdot 5! = 600$ permutations in which the first number is not 1.

Proof I: The number of permutations in which the first number is not 1 is the total number of permutations 6! minus the number of permutations in which the first number is 1 which is 5! (the answer for part (ii)). It follows that there are $6! - 5! = 720 - 120 = 600$ permutations in which the first number is not 1.

Proof II: There are 5 options for the number at the first position in the permutation. Assume that this number is $j \in \{2, 3, 4, 5, 6\}$. At the other 5 positions, there are 5! ways to arrange the 4 numbers in $\{2, 3, 4, 5, 6\} \setminus \{j\}$ and the number 1. In total there are $5 \cdot 5! = 5 \cdot 120 = 600$ permutations in which the first number is not 1.

vi. There are $6! - 5! = 5 \cdot 5! = 600$ permutations in which the last number is not 6.

Proof I: The number of permutations in which the last number is not 6 is the total number of permutations 6! minus the number of permutations in which the last number is 6 which is 5! (the answer for part (iii)). It follows that there are $6! - 5! = 720 - 120 = 600$ permutations in which the last number is not 6.

Proof II: There are 5 options for the number at the last position in the permutation. Assume that this number is $j \in \{1, 2, 3, 4, 5\}$. At the other 5 positions, there are 5! ways to arrange the 4 numbers in $\{1, 2, 3, 4, 5\} \setminus \{j\}$ and the number 6. In total there are $5 \cdot 5! = 5 \cdot 120 = 600$ permutations in which the last number is not 6.

vii. There are $6! - 2 \cdot 5! + 4! = 504$ permutations in which the first number is not 1 and the last number is not 6.

Proof: By part (ii), there are 5! permutations in which the first number is 1 and by part (iii), there are 5! permutations in which the last number is 6. In both cases, the count includes the number of permutations in which the first number is 1 and the last number is 6 which is 4! by part (iv). As a result, there are $5! + 5! - 4!$ permutation in which either the first number is 1, or the last number is 6, or both the first number is 1 and the last number is 6. In any other permutation, the first number is not 1 and the last number is not 6. Therefore, by part (i), the number of permutations in which the first number is not 1 and the last number is not 6 is

$$6! - 5! - 5! + 4! = 720 - 120 - 120 + 24 = 504$$

(c) $n \geq 3$: counting permutations of the numbers $\{1, 2, \ldots, n\}$.

i. There are $n!$ permutations.
Proof: For any integer $n \geq 1$, there are $n!$ permutations.

ii. There are $(n-1)!$ permutations in which the first number is 1.
Proof: After committing on 1 at the first position, there are $(n-1)!$ ways to arrange the numbers $2, 3, \ldots, n$ in the other $n-1$ positions.

iii. There are $(n-1)!$ permutations in which the last number is n.
Proof: After committing on n at the last position, there are $(n-1)!$ ways to arrange the numbers $1, 2, \ldots, (n-1)$ in the other $n-1$ positions.

iv. There are $(n-2)!$ permutations in which the first number is 1 and the last number is n.
Proof: After committing on 1 at the first position and n at the last position, there are $(n-2)!$ ways to arrange the numbers $2, 3, \ldots, n-1$ in the remaining $n-2$ positions.

v. There are $n! - (n-1)! = (n-1)(n-1)!$ permutations in which the first number is not 1.

Proof I: The number of permutations in which the first number is not 1 is the total number of permutations $n!$ minus the number of permutations in which the first number is 1 which is $(n-1)!$ (the answer for part (ii)). It follows that there are $n! - (n-1)!$ permutations in which the first number is not 1.

Proof II: There are $n-1$ options for the number at the first position in the permutation. Assume that this number is $j \in \{2, 3, \ldots, n\}$. At the other $n-1$ positions, there are $(n-1)!$ ways to arrange the $n-2$ numbers in $\{2, 3, \ldots, n\} \setminus \{j\}$ and the number 1. In total there are $(n-1)(n-1)!$ permutations in which the first number is not 1.

vi. There are $n! - (n-1)! = (n-1)(n-1)!$ permutations in which the last number is not n.

Proof I: The number of permutations in which the last number is not n is the total number of permutations $n!$ minus the number of permutations in which the last number is n which is $(n-1)!$ (the answer for part (iii)). It follows that there are $n! - (n-1)!$ permutations in which the last number is not n.

Proof II: There are $n-1$ options for the number at the last position in the permutation. Assume that this number is $j \in \{1, 2, \ldots, n-1\}$. At the other $n-1$ positions, there are $(n-1)!$ ways to arrange the $n-2$ numbers in $\{1, 2, \ldots, n-1\} \setminus \{j\}$ and the number n. In total there are $(n-1)(n-1)!$ permutations in which the last number is not n.

vii. There are $n! - 2(n-1)! + (n-2)!$ permutations in which the first number is not 1 and the last number is not n.

Proof: Let A be the set of all the permutations, let B be the set of all the permutations that start with 1, and let C be the set of all the permutations that end with n. The goal is to find the number of permutations in $(A \setminus B) \cap (A \setminus C)$ which by the De Morgan Laws is the same as the number of permutations in $A \setminus (B \cup C)$. By the *Principle of Inclusion and Exclusion* $|B \cup C| = |B| + |C| - |B \cap C|$ and therefore the answers for parts (ii), (iii), and (iv) imply that

$$\begin{aligned}
|A \setminus (B \cup C)| &= |A| - (|B| + |C| - |B \cap C|) \\
&= |A| - |B| - |C| + |B \cap C| \\
&= n! - (n-1)! - (n-1)1 + (n-2)! \\
&= n! - 2(n-1)! + (n-2)!
\end{aligned}$$

2. The goal is to count the number of passwords with repetitions of length $k \geq 1$ on the digits $\{0, 1, \ldots, 9\}$. Unlike numbers, passwords may start with one or more 0.

   Examples: (80) is a password of length 2, (00289) is a password of length 5, (9876543210) is a password of length 10, and (777) is a password of length 3.

   (a) How many passwords of length 2 are there?

   Answer: $100 = 10 \cdot 10$.

   Proof: There are 10 options for the first digit and 10 options for the second digit.

   (b) How many passwords of length 2 are there that do not contain the digit 0?

   Answer: $81 = 9 \cdot 9$.

   Proof: There are 9 options for the first digit and 9 options for the second digit.

   (c) How many passwords of length 2 are there that contain the digit 0 at least once?

   Answer: $19 = 10^2 - 9^2 = 100 - 81$.

   Proof: The set of passwords that do not contain the digit 0, denoted by A, is the complement set of the set of passwords that contain the digit 0 at least once, denoted by B. Therefore, the sum of the sizes of A and B is the total number of passwords. The solutions to the previous two parts imply that the size of A is $19 = 100 - 81$.

   (d) How many passwords of length 2 are there that contain at least one 0 and one 1?

   Answer: 2.

   Proof: The only possible passwords are (01) and (10).

   (e) How many passwords of length 3 are there?

   Answer: $1000 = 10 \cdot 10 \cdot 10$.

   Proof: There are 10 options for the first digit, 10 options for the second digit, and 10 options for the third digit.

   (f) How many passwords of length 3 are there that do not contain the digit 0?

   Answer: $729 = 9 \cdot 9 \cdot 9$.

   Proof: There are 9 options for the first digit, 9 options for the second digit, and 9 options for the third digit.

   (g) How many passwords of length 3 are there that contain the digit 0 at least once?

   Answer: $271 = 10^3 - 9^3 = 1000 - 729$.

   Proof: The set of passwords that do not contain the digit 0, denoted by A, is the complement set of the set of passwords that contain the digit 0 at least once, denoted by B. Therefore, the sum of the sizes of A and B is the total number of passwords. The solutions to the previous two parts imply that the size of A is $271 = 1000 - 729$.

   (h) How many passwords of length 3 are there that contain at least one 0 and one 1?

   Answer: 54.

   Proof: There are two cases.

   - In the first case, the password contains only 0 and 1. There are 6 such passwords: (001), (010), (011), (100), (101), and (110).
   - In the second case, 0 appears once and 1 appears once and the third digit could be any of the remaining 8 digits $2, 3, \ldots, 9$. There are 3 options for the location of the third digit, denoted by d, and then there are 2 options to select the order between the 0 and 1. In total there are 6 options for the digits 0, 1 and d: (d01), (d10), (0d1), (1d0), (01d),and (10d). Since d has 8 options, it follows that the number of passwords of length 3 that has exactly one 0 and one 1 is $48 = 8 \cdot 6$.

   Therefore, in total there are $54 = 6 + 48$ passwords of length 3 that contain at least one 0 and one 1.

(i) How many passwords of length $k \geq 1$, as a function of k, are there?

Answer: 10^k.

Proof: There are 10 options for each one of the k digits.

(j) How many passwords of length $k \geq 1$, as a function of k, are there that do not contain the digit 0?

Answer: 9^k.

Proof: There are 9 options for each one of the k digits.

(k) How many passwords of length $k \geq 1$, as a function of k, are there that contain the digit 0 at least once?

Answer: $10^k - 9^k$.

Proof The set of passwords that do not contain the digit 0, denoted by A, is the complement set of the set of passwords that contain the digit 0 at least once, denoted by B. Therefore, the sum of the sizes of A and B is the total number of passwords. The solutions to the previous two parts imply that the size of A is $10^k - 9^k$.

(l) How many passwords of length $k \geq 2$, as a function of k, are there that contain at least one 0 and one 1?

Answer: $10^k - 2 \cdot 9^k + 8^k$.

Proof: Let A be the set of all the passwords that contain at least one 0 and let B be the set of all the passwords that contain at least one 1. The goal is to find the number of passwords in $A \cap B$. There are 9^k passwords that do not contain the digit 0, there are 9^k passwords that do not contain the digit 1, and there are 8^k passwords that do not contain both the digit 0 and the digit 1. Therefore, $|A| = |B| = 10^k - 9^k$ and $|A \cup B| = 10^k - 8^k$. By the *Principle of Inclusion and Exclusion*,

$$\begin{aligned} |A \cap B| &= |A| + |B| - |A \cup B| \\ &= (10^k - 9^k) + (10^k - 9^k) - (10^k - 8^k) \\ &= 10^k - 2 \cdot 9^k + 8^k \end{aligned}$$

For $k = 2$:

$$2 = 10^2 - 2 \cdot 9^2 + 8^2 = 100 - 2 \cdot 81 + 64$$

For $k = 3$:

$$54 = 10^3 - 2 \cdot 9^3 + 8^3 = 1000 - 2 \cdot 729 + 512$$

3. An ice cream shop sells eleven different flavors of ice cream.

   (a) How many different two-scoop cones are there if the order among the scoops does not matter? That is, a cone with a vanilla scoop on top of a chocolate scoop is considered the same as a cone with chocolate on top of vanilla.

   Answer: 66.

   Proof: There are $\binom{11}{2} = 55$ different pairs of flavors. Each pair corresponds to a possible cone with two different scoops when the order does not matter. There are an additional 11 one-flavor two-scoop cones. In total there are 66 ways to have cones with two scoops when the order of the scoops does not matter.

   (b) How many different two-scoop cones are there if the order of the scoops does matter? In this case, cones with two scoops of the same flavor, e.g., two scoops of vanilla, should only be counted once.

   Answer: 121.

   Proof: There are 11 options to select the bottom scoop and then there are 11 options to select the top scoop. The total numbers of ways to have a cone with two scoops when order matters is therefore $11 \cdot 11 = 121$.

4. Teams from twelve schools participate in a basketball tournament.

   (a) How many games are there in the tournament if each team must play each other team exactly once?

   Answer: 66.

   Proof I: The number of games is exactly the number of different pairs of teams. By definition it is

   $$\binom{12}{2} = \frac{12 \cdot 11}{2} = \frac{132}{2} = 66 \ .$$

   Proof II: Each team plays 11 other teams for a total of $12 \cdot 11 = 132$ games. However, each game is counted twice and therefore the answer is $\frac{132}{2} = 66$.

   (b) How many games are there in the tournament if each team plays exactly four other teams?

   Answer: 24.

   Proof: Each team plays 4 other teams for a total of $12 \cdot 4 = 48$ games. However, each game is counted twice and therefore the answer is $\frac{48}{2} = 24$.

5. A club has nine members.

   (a) In how many ways can the club select a president and a treasurer from among its members?

   Answer: 72.

   Proof: There are 9 ways to select the president. After the president is selected, there are 8 ways to select the treasurer because the president cannot be the treasurer. The total number of ways is $72 = 9 \cdot 8$. The same arguments work if the treasurer is selected first and then the president.

   (b) In how many ways can the club select a two-person executive committee from among its members?

   Answer: 36.

   Proof: A set of size 9 has $\binom{9}{2} = \frac{9 \cdot 8}{2} = 36$ subsets of size 2. The number of two-person committees is exactly this number of subsets.

   (c) In how many ways can the club select a president and a two-person executive advisory board from among its members (assuming that the president is not on the advisory board)?

   Answer: 252.

   Proof I: There are 9 ways to select the president and then there are $\binom{8}{2} = \frac{8 \cdot 7}{2} = 28$ ways to select the advisory board for a total of $252 = 9 \cdot 28$ ways.

   Proof II: There are $\binom{9}{2} = \frac{9 \cdot 8}{2} = 36$ ways to select the advisory board and then there are 7 ways to select the president for a total of $252 = 36 \cdot 7$ ways.

6. For an integer $n \geq 3$, the following formula is a mathematical identity:

$$n\binom{n-1}{2} = \binom{n}{2}(n-2)$$

Algebraic proof:

$$\begin{aligned} n\binom{n-1}{2} &= n\frac{(n-1)(n-2)}{2} \\ &= \frac{n(n-1)(n-2)}{2} \\ &= \frac{n(n-1)}{2}(n-2) \\ &= \binom{n}{2}(n-2) \end{aligned}$$

Combinatorial proof: Assume a club with n members that is selecting a president and a two-person executive advisory board from among its members where the president is not on the advisory board.

The following are two methods to count the number of ways to selct these three persons:

- There are n ways to select the president and then there are $\binom{n-1}{2}$ ways to select the advisory board for a total of $n\binom{n-1}{2}$ ways.
- There are $\binom{n}{2}$ ways to select the advisory board and then there are $n-2$ ways to select the president for a total of $\binom{n}{2}(n-2)$ ways.

The identity is true since both methods count the same number.

7. **Theorem:** For $n \geq 3$,

$$\binom{n}{3} + \binom{n+1}{3} + \binom{n+2}{3} = \frac{n(n^2+1)}{2}$$

Proof:

$$\begin{aligned}
\binom{n}{3} + \binom{n+1}{3} + \binom{n+2}{3} &= \frac{n(n-1)(n-2)}{6} + \frac{(n+1)n(n-1)}{6} + \frac{(n+2)(n+1)n}{6} \\
&= \frac{n((n-1)(n-2) + (n+1)(n-1) + (n+2)(n+1))}{6} \\
&= \frac{n((n^2-3n+2) + (n^2-1) + (n^2+3n+2))}{6} \\
&= \frac{n(3n^2+3)}{6} \\
&= \frac{3n(n^2+1)}{6} \\
&= \frac{n(n^2+1)}{2}
\end{aligned}$$

8. **Theorem:** For $0 \le k \le n$,

$$\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k}$$

Combinatorial proof:

- Let S be the set of all the subsets of $\{1, 2, \ldots, n+1\}$ of size $k+1$.
- For $k \le m \le n$, let S_m be the set of all the subsets of $\{1, 2, \ldots, n+1\}$ of size $k+1$ for which the maximum number is $m+1$.
- Since the maximum number in a subset of $\{1, 2, \ldots, n+1\}$ of size $k+1$ is at least $k+1$ and at most $n+1$, it follows that

$$|S| = \binom{n+1}{k+1} = \sum_{m=k}^{n} |S_m|$$

- For $k \le m \le n$, any subset of $\{1, 2, \ldots, n+1\}$ of size $k+1$ for which the maximum number is $m+1$ is the union of the singleton subset $\{m+1\}$ with a subset of $\{1, 2, \ldots, m\}$ of size k.
- Therefore, $|S_m| = \binom{m}{k}$ for $k \le m \le n$.
- The proof follows by replacing $|S_m|$ with $\binom{m}{k}$ in the above equation for $k \le m \le n$.

Algebraic proof: Apply the recursive identity

$$\binom{j}{h} = \binom{j-1}{h-1} + \binom{j-1}{h}$$

for terms $\binom{j}{h}$ for which $h = k+1$ and $j > h$. At the end replace $\binom{k+1}{k+1}$ with $\binom{k}{k}$ because both equal 1. Retain the terms $\binom{j}{h}$ for which $h = k$.

$$\begin{aligned}
\binom{n+1}{k+1} &= \binom{n}{k} + \binom{n}{k+1} \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-1}{k+1} \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \binom{n-2}{k+1} \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \binom{n-3}{k} + \binom{n-3}{k+1} \\
&\vdots \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \binom{n-3}{k} + \cdots + \binom{k+1}{k} + \binom{k+1}{k+1} \\
&= \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \binom{n-3}{k} + \cdots + \binom{k+1}{k} + \binom{k}{k} \\
&= \sum_{m=k}^{n} \binom{m}{k}
\end{aligned}$$