Discrete Math

Logic Practice Problems: Solutions

1. (a) and (b) below are the only two truth assignments to the 4 variables x, y, z, w (out of the possible 16 assignments) that satisfy the formula:

$$(x \vee y) \wedge (\neg y \vee z) \wedge (\neg z \vee w) \wedge (\neg w \vee \neg x)$$

   (a) $(x, y, z, w) \equiv (T, F, F, F)$

   (b) $(x, y, z, w) \equiv (F, T, T, T)$

Proof: Let $A = (x \vee y)$, $B = (\neg y \vee z)$, $C = (\neg z \vee w)$, and $D = (\neg w \vee \neg x)$. Observe, that any assignment that satisfies the formula must satisfy A, B, C, and D.

   (a) Assume $x = T$ which satisfies A. Then $w = F$ in order to satisfy D. Then $z = F$ in order to satisfy C. Finally, $y = F$ in order to satisfy B. As a result, the only assignment in which $x = T$ that satisfies the formula is $(x, y, z, w) \equiv (T, F, F, F)$.

   (b) Assume $x = F$ which satisfies D. Then $y = T$ in order to satisfy A. Then $z = T$ in order to satisfy B. Finally, $w = T$ in order to satisfy C. As a result, the only assignment in which $x = F$ that satisfies the formula is $(x, y, z, w) \equiv (F, T, T, T)$.

Proof using a truth table:

x	y	z	w	$(x \vee y)$	$(\neg y \vee z)$	$(\neg z \vee w)$	$(\neg w \vee \neg x)$	formula
T	T	T	T	T	T	T	F	F
T	T	T	F	T	T	F	T	F
T	T	F	T	T	F	T	F	F
T	T	F	F	T	F	T	T	F
T	F	T	T	T	T	T	F	F
T	F	T	F	T	T	F	T	F
T	F	F	T	T	T	T	F	F
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	T	F	T	T	F	T	F
F	T	F	T	T	F	T	T	F
F	T	F	F	T	F	T	T	F
F	F	T	T	F	T	T	T	F
F	F	T	F	F	T	F	T	F
F	F	F	T	F	T	T	T	F
F	F	F	F	F	T	T	T	F

2. Tautologies and Contradictions.

   (a) Prove that $(x \wedge y) \rightarrow (x \vee y)$ is a tautology.

   Proof: The entries in the last column in the truth table are all T.

x	y	$x \wedge y$	$x \vee y$	$(x \wedge y) \rightarrow (x \vee y)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

   (b) Prove that $(x \oplus y) \wedge (x \equiv y)$ is a contradiction.

   Proof: The entries in the last column in the truth table are all F.

x	y	$x \oplus y$	$x \equiv y$	$(x \oplus y) \wedge (x \equiv y)$
T	T	F	T	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

3. Prove the two logic distributive laws.

   (a) $(x \wedge (y \vee z)) \equiv ((x \wedge y) \vee (x \wedge z))$.

   Proof: The last columns in both truth tables are identical.

x	y	z	$y \vee z$	$x \wedge (y \vee z)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

x	y	z	$x \wedge y$	$x \wedge z$	$(x \wedge y) \vee (x \wedge z)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

   (b) $x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$.

   Proof: The last columns in both truth tables are identical.

x	y	z	$y \wedge z$	$x \vee (y \wedge z)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

x	y	z	$x \vee y$	$x \vee z$	$(x \vee y) \wedge (x \vee z)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

4. Prove the two De Morgan's laws for four variables.

(a) $\neg(x \wedge y \wedge z \wedge w) \equiv (\neg x) \vee (\neg y) \vee (\neg z) \vee (\neg w)$.

Proof I: Let $L = \neg(x \wedge y \wedge z \wedge w)$ and $R = (\neg x) \vee (\neg y) \vee (\neg z) \vee (\neg w)$. Show that $L \equiv R$ by proving that if L is TRUE then R is TRUE and if L is FALSE then R is FALSE.

- If L is TRUE then $(x \wedge y \wedge z \wedge w)$ is FALSE. Therefore, at least one variable in $\{x, y, z, w\}$ is FALSE which implies that at least one negated variable in $\{\neg x, \neg y, \neg z, \neg w\}$ is TRUE. Consequently, R is TRUE.
- If L is FALSE then $(x \wedge y \wedge z \wedge w)$ is TRUE. Therefore, all the variables in $\{x, y, z, w\}$ are TRUE which implies that all the negated variables in $\{\neg x, \neg y, \neg z, \neg w\}$ are FALSE. Consequently, R is FALSE.

Proof II: The following equalities are based on one of the De Morgan's laws for two variables.

$$\begin{aligned}
\neg(x \wedge y \wedge z \wedge w) &= \neg((x \wedge y) \wedge (z \wedge w)) \\
&= \neg(x \wedge y) \vee \neg(z \wedge w) \\
&= (\neg x \vee \neg y) \vee (\neg z \vee \neg w) \\
&= \neg x \vee \neg y \vee \neg z \vee \neg w
\end{aligned}$$

Truth tables:

x	y	z	w	$x \wedge y \wedge z \wedge w$	$\neg(x \wedge y \wedge z \wedge w)$
T	T	T	T	T	F
T	T	T	F	F	T
T	T	F	T	F	T
T	T	F	F	F	T
T	F	T	T	F	T
T	F	T	F	F	T
T	F	F	T	F	T
T	F	F	F	F	T
F	T	T	T	F	T
F	T	T	F	F	T
F	T	F	T	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	T	F	F	T
F	F	F	T	F	T
F	F	F	F	F	T

x	y	z	w	$\neg x$	$\neg y$	$\neg z$	$\neg w$	$\neg x \vee \neg y \vee \neg z \vee \neg w$
T	T	T	T	F	F	F	F	F
T	T	T	F	F	F	F	T	T
T	T	F	T	F	F	T	F	T
T	T	F	F	F	F	T	T	T
T	F	T	T	F	T	F	F	T
T	F	T	F	F	T	F	T	T
T	F	F	T	F	T	T	F	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	F	F	F	T
F	T	T	F	T	F	F	T	T
F	T	F	T	T	F	T	F	T
F	T	F	F	T	F	T	T	T
F	F	T	T	T	T	F	F	T
F	F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	F	T
F	F	F	F	T	T	T	T	T

(b) $\neg(x \vee y \vee z \vee w) \equiv (\neg x) \wedge (\neg y) \wedge (\neg z) \wedge (\neg w)$.

Proof I: Let $L = \neg(x \vee y \vee z \vee w)$ and $R = (\neg x) \wedge (\neg y) \wedge (\neg z) \wedge (\neg w)$. Show that $L \equiv R$ by proving that if L is TRUE then R is TRUE and if L is FALSE then R is FALSE.

- If L is TRUE then $(x \vee y \vee z \vee w)$ is FALSE. Therefore, all the variables in $\{x, y, z, w\}$ are FALSE which implies that all the negated variables in $\{\neg x, \neg y, \neg z, \neg w\}$ are TRUE. Consequently, R is TRUE.
- If L is FALSE then $(x \vee y \vee z \vee w)$ is TRUE. Therefore, at least one variable in $\{x, y, z, w\}$ is TRUE which implies that at least one negated variable in $\{\neg x, \neg y, \neg z, \neg w\}$ is FALSE. Consequently, R is FALSE.

Proof II: The following equalities are based on one of the De Morgan's laws for two variables.

$$\begin{aligned} \neg(x \vee y \vee z \vee w) &= \neg((x \vee y) \vee (z \vee w)) \\ &= \neg(x \vee y) \wedge \neg(z \vee w) \\ &= (\neg x \wedge \neg y) \wedge (\neg z \wedge \neg w) \\ &= \neg x \wedge \neg y \wedge \neg z \wedge \neg w \end{aligned}$$

Truth tables:

x	y	z	w	$x \vee y \vee z \vee w$	$\neg(x \vee y \vee z \vee w)$
T	T	T	T	T	F
T	T	T	F	T	F
T	T	F	T	T	F
T	T	F	F	T	F
T	F	T	T	T	F
T	F	T	F	T	F
T	F	F	T	T	F
T	F	F	F	T	F
F	T	T	T	T	F
F	T	T	F	T	F
F	T	F	T	T	F
F	T	F	F	T	F
F	F	T	T	T	F
F	F	T	F	T	F
F	F	F	T	T	F
F	F	F	F	F	T

x	y	z	w	$\neg x$	$\neg y$	$\neg z$	$\neg w$	$\neg x \wedge \neg y \wedge \neg z \wedge \neg w$
T	T	T	T	F	F	F	F	F
T	T	T	F	F	F	F	T	F
T	T	F	T	F	F	T	F	F
T	T	F	F	F	F	T	T	F
T	F	T	T	F	T	F	F	F
T	F	T	F	F	T	F	T	F
T	F	F	T	F	T	T	F	F
T	F	F	F	F	T	T	T	F
F	T	T	T	T	F	F	F	F
F	T	T	F	T	F	F	T	F
F	T	F	T	T	F	T	F	F
F	T	F	F	T	F	T	T	F
F	F	T	T	T	T	F	F	F
F	F	T	F	T	T	F	T	F
F	F	F	T	T	T	T	F	F
F	F	F	F	T	T	T	T	T

5. Expressing $\mathcal{NOR}$ ($\downarrow$) with $\mathcal{NAND}$ ($\uparrow$) and $\mathcal{NAND}$ ($\uparrow$) with $\mathcal{NOR}$ ($\downarrow$).

(a) **Observation:** $P \uparrow P \equiv \neg P$ for any expression P.

Proof: By the definition of $\mathcal{NAND}$, if $P = T$ then

$$P \uparrow P \equiv T \uparrow T \equiv F \equiv \neg P$$

and if $P = F$ then

$$P \uparrow P \equiv F \uparrow F \equiv T \equiv \neg P$$

Proposition: $(x \downarrow y) \equiv ((x \uparrow x) \uparrow (y \uparrow y)) \uparrow ((x \uparrow x) \uparrow (y \uparrow y))$

Proof: Define

$$\begin{aligned} R &\equiv ((x \uparrow x) \uparrow (y \uparrow y)) \uparrow ((x \uparrow x) \uparrow (y \uparrow y)) \\ X &\equiv (x \uparrow x) \\ Y &\equiv (y \uparrow y) \\ Z &\equiv X \uparrow Y \end{aligned}$$

The above observation implies that $X \equiv \neg x$ and $Y \equiv \neg y$. Hence, $Z \equiv (\neg x \uparrow \neg y)$. Since $R \equiv (Z \uparrow Z)$, the above observation implies that $R \equiv \neg Z \equiv \neg(\neg x \uparrow \neg y)$.

The following truth table show the values for $\neg(\neg x \uparrow \neg y)$.

x	y	$\neg x$	$\neg y$	$\neg x \uparrow \neg y$	$\neg(\neg x \uparrow \neg y)$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	F	T

The proposition is correct since the last column of the above truth table is $\mathcal{NOR}$.

(b) **Observation:** $P \downarrow P \equiv \neg P$ for any expression P.

Proof: By the definition of $\mathcal{NOR}$, if $P = T$ then

$$P \downarrow P \equiv T \downarrow T \equiv F \equiv \neg P$$

and if $P = F$ then

$$P \downarrow P \equiv F \downarrow F \equiv T \equiv \neg P$$

Proposition: $(x \uparrow y) \equiv ((x \downarrow x) \downarrow (y \downarrow y)) \downarrow ((x \downarrow x) \downarrow (y \downarrow y))$

Proof: Define

$$\begin{aligned} R &\equiv ((x \downarrow x) \downarrow (y \downarrow y)) \downarrow ((x \downarrow x) \downarrow (y \downarrow y)) \\ X &\equiv (x \downarrow x) \\ Y &\equiv (y \downarrow y) \\ Z &\equiv X \downarrow Y \end{aligned}$$

The above observation implies that $X \equiv \neg x$ and $Y \equiv \neg y$. Hence, $Z \equiv (\neg x \downarrow \neg y)$. Since $R \equiv (Z \downarrow Z)$, the above observation implies that $R \equiv \neg Z \equiv \neg(\neg x \downarrow \neg y)$.

The following truth table show the values for $\neg(\neg x \downarrow \neg y)$.

x	y	$\neg x$	$\neg y$	$\neg x \downarrow \neg y$	$\neg(\neg x \downarrow \neg y)$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	F	T

The proposition is correct since the last column of the above truth table is $\mathcal{NAND}$.

6. Prove that if $p \to q$ and $q \to r$ then $p \to r$.

Proof I: The following is the truth table for $((p \to q) \wedge (q \to r)) \to (p \to r)$.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The table shows that $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology. Equivalently, if $p \to q$ and $q \to r$ then $p \to r$.

Proof II: There are two cases: q is either TRUE ($q = T$) or FALSE ($q = F$).

(a) Assume $q = T$. Then since $(p \to q) = T$ it follows that $p = T$ or $p = F$ and since $(q \to r) = T$ it follows that $r = T$. The claim is correct because both $T \to T$ and $F \to T$ are True.

(b) Assume $q = F$. Then since $(p \to q) = T$ it follows that $p = F$ and since $(q \to r) = T$ it follows that $r = T$ or $r = F$. The claim is correct because both $F \to T$ and $F \to F$ are True.

Proof III: There are two cases: p is either TRUE ($p = T$) or FALSE ($p = F$).

(a) Assume $p = T$. Then since $(p \to q) = T$ it follows that $q = T$ and since $(q \to r) = T$ it follows that $r = T$. The claim is correct because in this case $p \to r$ is $T \to T$ which is True.

(b) Assume $p = F$. Then always $(p \to r) = T$ because both $F \to T$ and $F \to F$ are True. So it does not matter if q and r are T or F.

7. Assume that $U = \{1, 2, 3\}$.

(a) Let $x = 1$. Then $x^2 = 1$ is smaller than $y + 1$ for $y = 1$, $y = 2$, and $y = 3$. Therefore, the statement $\exists_{x \in U} \forall_{y \in U} (x^2 < y + 1)$ is TRUE for $x = 1$.

(b) Let $x = y = 3$. Then $x^2 + y^2 = 18$. Therefore, the statement $\forall_{x \in U} \forall_{y \in U} (x^2 + y^2 < 18)$ is FALSE.

(c) Let $y = 1$. Then $x^2 + 1 < 11$ for $x = 1$, $x = 2$, and $x = 3$. Therefore, the statement $\forall_{x \in U} \exists_{y \in U} (x^2 + y^2 < 11)$ is TRUE for $y = 1$.

8. Consider the following three sets

$$\begin{aligned} A &= \{2, 4, 6, 8\} \\ B &= \{1, 3, 5, 7\} \\ C &= \{1, 2, 7, 8\} \end{aligned}$$

and the following four statements about a set S:

$$\begin{aligned} (p_1) &\equiv \forall_{x \in S} \{x \text{ is even}\} \\ (p_2) &\equiv \exists_{x \in S} \{x \text{ is even}\} \\ (p_3) &\equiv \forall_{x \in S} \{x \text{ is odd}\} \\ (p_4) &\equiv \exists_{x \in S} \{x \text{ is odd}\} \end{aligned}$$

(a) For each of the three sets A, B, and C, the following table states which of the four statements (p_1), (p_2), (p_3), and (p_4) is TRUE and which is FALSE.

Set	(p_1)	(p_2)	(p_3)	(p_4)	a short explanation
$A = \{2, 4, 6, 8\}$	TRUE	TRUE	FALSE	FALSE	all integers in A are even
$B = \{1, 3, 5, 7\}$	FALSE	FALSE	TRUE	TRUE	all integers in B are odd
$C = \{1, 2, 7, 8\}$	FALSE	TRUE	FALSE	TRUE	C contains 2 odd and 2 even integers

(b) **Claim:** There is no set of integers for which at least three of the four statements (p_1), (p_2), (p_3), and (p_4) are TRUE.

Proof I: Let S be a set of integers. If S has at least one even number and one odd numbver then (p_1) and (p_3) are FALSE. If all the integers in S are even then (p_3) and (p_4) are FALSE. If all the integers in S are odd then (p_1) and (p_2) are FALSE. The claim follows because these are the only three options.

Proof II: Assume toward a contradiction that there exists a set S for which at least three of the statements (p_1), (p_2), (p_3), and (p_4), are TRUE. It must be the case that either (p_1) is TRUE or (p_3) is TRUE. But if (p_1) is TRUE then both (p_3) and (p_4) must be FALSE and and if (p_3) is TRUE then both (p_1) and (p_2) must be FALSE. A contradiction.

9. Three boxes are presented to you. One contains gold while the other two are empty. Each box has a clue imprinted on it as to its contents. The clues are:

   - Box 1: "The gold is not here"
   - Box 2: "The gold is not here"
   - Box 3: "The gold is in Box 2"

   Only one clue is TRUE while the other two clues are FALSE. Which box has the gold?

   Answer I: The clues imprinted on box 2 and on box 3 contradict each other. Therefore, one of them must be TRUE and one of them must be FALSE. Since there are two FALSE clues, the one imprinted on box 1 is FALSE. Hence the gold is in Box 1.

   Answer II: For $1 \leq i, j \leq 3$, define the value $v(i,j)$ in row i and column j of the 3×3 matrix below as follows. If the gold is in box i, then $v(i,j) \equiv TRUE$ if the clue imprinted on box j is TRUE and $v(i,j) \equiv FALSE$ if the clue imprinted on box j is FALSE.

Box with the gold	Box 1 clue	Box 2 clue	Box 3 clue
Box 1	FALSE	TRUE	FALSE
Box 2	TRUE	FALSE	TRUE
Box 3	TRUE	TRUE	FALSE

   Since only one clue is TRUE, it follows that only rows with one TRUE value and two FALSE values could be the correct rows. Therefore, the gold is in Box 1.

10. There are three boxes. One box has two white balls; one box has two black balls; and one box has one white ball and one black ball. Three labels are attached to the boxes: WW (White and White); BB (Black and Black); and BW (Black and White). Unfortunately, each box is labeled with an **incorrect** label.

You are allowed to open only one box, pick one ball at random, obseve its color, and put it back into the box without seeing the color of the other ball.

Which box should you open to fix the labeling of the three boxes?

Answer I : Open the box with the label BW. There are two cases.

- If you observe a black ball, since the label is incorrect, you know that this box must be the box with the two black balls. As a result, the box with the label WW must contain one black ball and one white ball in order to have an incorrect label. This leaves the box with the BB label to contain two white balls.
- If you observe a white ball, since the label is incorrect, you know that this box must be the box with the two white balls. As a result, the box with the label BB must contain one white ball and one black ball in order to have an incorrect label. This leaves the box with the WW label to contain two black balls.

Remark: After learning the content of the box with the BW label, switch this label with the correct label. Next switch the labels of the other two boxes to get a correct labeling.

Answer II: There are only two ways to relabel the boxes assuming that each box is labeled incorrectly:

Current label	BW	BB	WW
Option I	BB	WW	BW
Option II	WW	BW	BB

The only way to distinguish between the two options is to open the box with the label BW. The color of the selected ball would determine which option is the correct labeling.