Discrete Structures

Recursion Practice Problems

Name and ID: ....

1. Solve the following recurrences and prove that your solutions are correct.

$$T(n) = \begin{cases} 2 & \text{for } n = 1 \\ T(n-1) + 7 & \text{for } n \geq 2 \end{cases}$$

$$T(n) = \begin{cases} 3 & \text{for } n = 1 \\ 2T(n-1) & \text{for } n \geq 2 \end{cases}$$

$$T(n) = \begin{cases} 2 & \text{for } n = 1 \\ (n+1)T(n-1) & \text{for } n \geq 2 \end{cases}$$

2. Solve the following recurrence and prove that your solution is correct.

$$P(n) = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n = 1 \\ 5P(n-1) - 6P(n-2) & \text{for } n \geq 2 \end{cases}$$

Guide: Do the bottom-up evaluation, guess the solution, and prove by induction the correctness of your guess.

3. Some facts about the Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

$$F_n = \begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1 \\ F_{n-1} + F_{n-2} & \text{for } n \geq 2 \end{cases}$$

What is the smallest n for which $F_n > 100$? What is the smallest n for which $F_n > 1000$?

Let $A_n = (F_1 + F_2 + \cdots + F_n)/n$ be the average of the first n Fibonacci numbers. What is the smallest n for which $A_n > 10$?

Find all n for which $F_n = n$. Explain why these are the only cases.

Find all n for which $F_n = n^2$. Explain why these are the only cases.

Find at least three interesting and fascinating facts or properties about the Fibonacci numbers and/or the Golden Ratio that do not appear in the class presentation.

You do not need to provide proofs.

Support your findings with pointers to their resources.

4. Prove the following identity for $n \geq 2$:

$$F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2}$$

Hint: There exists a simple proof without induction that is based on the recursive definition of the Fibonacci numbers.

5. Define the following (*almost Fibonacci*) recurrence

$$G_n = \begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1 \\ G_{n-1} + G_{n-2} + 1 & \text{for } n \geq 2 \end{cases}$$

Find the values of $G_0, G_1, \ldots, G_{10}$.

Express G_n as a function of Fibonacci numbers.

Prove that your expression for G_n is correct for all $n \geq 0$.

6. Prove the following identity for $n \geq 1$:

$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

7. For $n \geq 1$, in how many out of the $n!$ permutations $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ of the numbers $\{1, 2, \ldots, n\}$ the value of $\pi(i)$ is either $i - 1$, or i, or $i + 1$ for all $1 \leq i \leq n$?

   Example: The permutation (21354) follows the rules while the permutation (21534) does not because $\pi(3) = 5$.

   Hint: Find the answer for small n by checking all the permutations and then find the recursive formula depending on the possible values for $\pi(n)$.