How Hard is the Play of
the King-Rook-King-Knight Ending?
D.Kopec and T.Niblett

ABSTRACT
An exhaustive database of the KRKN ending revealed inadequacies in the published analyses. It was also used (see appendix 3) to demonstrate that the domain can be mastered by an endgame expert after special study. Experiments with Class ‘A’ players showed that the defence of the draw for the knight’s side is in general easier than demonstration of the win for the rook’s side, but that a special class of positions exist for which the correct defence runs counter to most players’ intuition. The nature of these positions is examined, and modifications are proposed to existing KRKN theory.

INTRODUCTION
The subject of the first part of our paper is a database for the KRKN endgame. The second part reports experiments with human players in the same endgame. The third discusses the main concepts for the domain.

USE OF A DATABASE TO INVESTIGATE THE KRKN DOMAIN
By ‘database’ we mean a complete computer-stored look-up table for optimal play. Such a database has already proven invaluable for the study of KPK (Clarke 1977). With its help Bramer (1977) has proved the optimality of a program to play this endgame. Using a database for KRK, Clarke (1977) has shown that White can force a win from any position in 16 moves or less — instead of 17 as was previously thought (Fine 1941). These endings however are ones for which Chess Masters have no difficulty in finding correct, if not optimal moves. We think this is due to the small number of ‘patterns’ needed to assess any position statically, despite the longest win being 19 moves for KPK, 16 for KRK. Very little lookahead seems to be needed for these endings.

KRKN is different: here we enter a domain that can be challenging even for a master. This is a subgame of chess which is possible to compute fully by machine (a database), but very hard for the unaided person to play correctly. The ending KQKR illustrated a similar case recently when two International Masters repeatedly failed to find a winning line in play against a database for this ending. We have had similar results with KRKN against master-strength players.

Other databases already constructed are those for KRKR and KQPKQ (Arlazarov and Futer 1979), the latter being notoriously difficult even for leading Grandmasters. Last year Grandmaster Bronstein, once challenger for the world championship, had a position with KQPKQ at the tail end of the world championship 1978. Unfortunately, his annotations were inadequate to enable a human annotator to determine the optimal moves. It is very likely, however, that a database could reveal the right moves in such a position, and this is the purpose of our paper.
adjournment and used the database to find a win he himself could not. Although KRKN is simpler than this, one of our later examples shows the breakdown of ‘pattern matching’ even for a Grandmaster. The major difference then between this endgame and KPK is the need for very much more tree-search, or lookahead by humans.

KRKN databases have a fairly long history. The first of which we are aware was completed in 1970 by Thomas Strohlein (1970) as part of a Ph.D. Dissertation on Graph Theory and Combinatorics. Recently Ken Thompson of Bell Labs computed databases for all the interesting four-piece endings including KRKN, and Gams (1978) has computed specifically the KRKN case and has partially validated Thompson’s work. Finally at the M.I.R.U. we have independently constructed one. With this we can check one database against another to ensure correctness. This is a major problem with databases: because of their size only machine checking is possible and even this is daunting. In the case of Grandmaster Bronstein’s consultation of the KQPKQ database, the actual line he was given would have resulted in a draw due to a database error at the very end!

Algorithm for Database Construction

Michael Clarke (1977) has already described a procedure for KPK. However, KRKN differs slightly, for breadth-first backing up is used. The procedure can be conveniently divided into three parts (see figure 1).

All positions for White where he can mate immediately or win the black knight in one move are defined as won at depth 1. The first step is to find all these without loss of the rook or stalemate. For each legal BTM position a counter is set for the number of illegal successors. This completes the initialisation.

Backing up from WTM positions won at depth N is done by finding all legal predecessors of the position. Since there are $8 + 8 = 16$ potential moves in all, the counter will show the value 15 at the moment that the set of legal moves is exhausted. If any hasn’t previously been marked as won, then if the counter has value 15, that position is marked lost depth N + 1; otherwise the counter is incremented by 1.

Backing up from BTM positions is simpler. Again all legal predecessors are found. Any not previously marked as won are marked as won depth N.

Results

The procedure described above was executed as a program on the DEC System-10 computer. The resulting database was used to partially check the correctness and completeness of Thompson’s KRKN database. The latter was the one actually used for most of the results now to be reported.

It is difficult to estimate the exact number of legal positions. Due to symmetry it is much less than $64^4$. Thompson (private communication) estimates 741 000 drawn positions and 651 000 positions won for White in the canonical state space.

Note that the backing-up method described entails that only won
A White-to-move (WTM) position is won depth = 1 if White can
mate or win the knight without stalemate or loss of the rook.

**Figure 1**

<table>
<thead>
<tr>
<th>Depth of loss (moves)</th>
<th>No. of positions</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>378,518</td>
</tr>
<tr>
<td>2</td>
<td>95,450</td>
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<td>3</td>
<td>46,269</td>
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<td>30,729</td>
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<td>15,071</td>
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<td>26</td>
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</tr>
<tr>
<td>27</td>
<td>2</td>
</tr>
</tbody>
</table>

*Figure 2. The distribution of losses with depth*
positions are assigned a depth in the database. Draws are recognised by lack of this marker. There is, incidentally, one position lost for White! There are therefore roughly 1 400 000 legal positions with White to move. This is about 15 times larger than KPK. The distribution of losses with depth, WTM, is shown in figure 2.

The two extreme positions lost in 27 moves with optimal play are:

- WK:d1 WR:h1 and WK:c1 WR:f8
- BK:b1 BN:g4 BK:a3 BN:e2

(For optimal winning sequence see appendix 1.) It is important to realise that these winning lines are far longer than needed for any position given in the literature. The deepest win we have been able to find as a published position is 16 moves with optimal play. The database can be used to generate (a) optimal play for White from any won starting position, (b) optimal play for Black from any lost starting position, (c) correct play for Black from any drawn position. Safe knight-capture is counted as ‘win’ without considering play of the residual KRK game. This section concludes with a discussion of KRKN as it is represented in the literature.

Analysis of Previously Published Treatments of KRKN

The history of KRKN is peculiarly long, peculiar in that the king, rook and knight move in the same way as in the ancient game of Chaturanga, which is otherwise very different from Chess. Several theoretical positions can be dated to the ninth century AD. In English the two most comprehensive treatments of the endgame are in Fine (1941) and Averbakh (1978). Fine's analysis is based on that of Berger whereas Averbakh relies on many more sources. Fine however gives a more detailed analysis for each position.

![Figure 3. Black to move](image)

To show the difficulty of the ending and its ‘opacity’ even in the face of extensive analysis we discuss the position shown in figure 3. This position is a variant of that given for Chaturanga by al-Adli in the ninth century. It was rediscovered in 1859 and (incorrectly) analysed in ‘The Chess-Players Chronicle’. Berger subjected the position to detailed analysis, and this was used by Fine in ‘Basic Chess Endings’ and continued by several analysts. The database shows that it can be won in 14 moves.

To show the difficulty of analysis we give Fine’s main line (his
punctuation) for the first few moves, comparing his move with that of the KRKN database:

1 ... Na5+ forced
2 Kb5 Nb7 all-optimal
3 Rf8! ... Rh5 is best, one move shorter.

Here Fine’s notes say ‘Not ... Rh7, because of ... Kb8, Kb6 drawn’. However Kc6 wins, and more quickly than Fine’s main line. This is one of two cases where Fine gives the wrong game-theoretic value for a position.

3 ... Nd6+ forced
4 Kc6 Nc4
5 Rd8! ... In spite of Fine’s exclamation mark, this move changes the depth of win from 11 to 17 moves in this position.

This demonstrates a typical phenomenon. The human master finds a plan which preserves the game-theoretic value of the position, but which is inefficient in the minimax sense.

We have analysed all the main lines given in Fine, and figure 4 illustrates the results. It is superficially surprising that there are so few non-optimal moves at large depths and so many at relatively shallow depths. However, if the win is very deep there is usually only one good winning move. At depths of 7-10 moves the master doesn’t try in his tree searching to find optimal moves, just good ones. At lower depth still he can see through to the end.

Using the database we have seen that optimal play is very hard to find, even for the analyst. There is a small subset of the position space, the longer wins (17 to 27 moves deep), which has never been explored. Masters have great difficulty in winning these positions. In our small-scale trials of human play against Thompson’s database (not with full tournament time allowance) we never saw it done. Subsequently, however, similar but more systematic tests were convincingly passed by A.J. Roycroft, not a Master but a specialist in the chess endgame study. His own account is reproduced as appendix 3.

**Experiments with Class ‘A’ Players**

Further information about the nature of the difficulties was obtained from experiments with Class ‘A’ players described below, in which the database did not figure.

Experiment I : Weak Class ‘A’ Players

**Objectives.** This experiment was to discover:

(a) How hard is it for an ‘A’ player to win a won position?
(b) How hard is it for an ‘A’ player to draw a drawn position?

Usually when humans are asked about this ending they will say either that they know nothing about it, or that it is drawn by keeping the king and knight close together and avoiding the edge of the board where mating threats can occur. Since Class ‘A’ players are better than average tournament
players, and are likely to know just the two above concepts, their performance on both the stronger and weaker side, under tournament-like conditions, was thought worth investigating.

Experimental Design. Three subjects, all university students averaging 1882 in ELO rating, were each assigned two positions, one drawn, which they were to defend with the knight’s side, and the other won which they were to win for the rook’s side. Drawn positions included starting positions from which testing was done on Ivan Bratko’s Advice Table (see Bratko and Michie, this volume). All these positions have in common the mutual proximity of the defending K and N. However in ‘Corncase’, (see figure 5), after 1 Rb2+ Ka1, 2 Rb8, we reach the only position which Bratko had to program as a special case, since only 2 ... Ne2, separating K and N as much as possible in the position, draws.

Won positions were selected as the longest (not necessarily main-line) variations given in Fine’s Basic Chess Endings, reportedly requiring 24, 17 and 15 moves to win (see figure 5, T4, T5, T6). Later checking of these positions with the Thompson database revealed that they actually require 14, 9 and 11 moves to win, with optimal play for both sides. Subjects were
assigned positions based on a predetermined rank-ordering of their difficulty. Proficiency was demonstrated by holding the game-theoretical value over 30 moves of play in one hour against a U.S. Master opponent (D.K.). Otherwise resignation, imminent checkmate, or a loss of material were the termination conditions.

Results and Conclusions. These Class 'A' human subjects had no difficulty defending typical drawn KNKR positions except for 'Corncase', where the subject later testified that he had played the correct 'separating' move without knowing why the other moves lost. The experimental design and results are summarized by figure 6. On the stronger side subjects failed in two of three cases to win. The one subject who did win, did so in 11 moves, 6 less than in the variation given by Fine, but Fine gives 17 moves for T5, i.e. 6 more than 11! A later check against the Thompson database showed that optimal play requires 9 moves (see appendix 2).

The database also showed that the experimenter's knight's-side defence in the experiment was suboptimal in the minimax sense.

Immediately following the experiment, a complete analysis was done to discover all game-theoretic value-changing moves (i.e. 'mistakes'). Since
Figure 5. The positions for Experiment I. T1, T2, T3 are drawn positions and T4, T5, T6 are won positions. (BCE stands for Basic Chess Endings by R.Fine.)

Subjects held the draw in all three cases, it was only a question of whether the human opponent had failed to exploit some error which they had made, but no errors were found. When subjects failed to win won positions, their errors occurred quite early in the sequences. The retrospective analysis of the continuation from position T6 proved to be most interesting. After: 1 Rb2+ Ka5 2 Kc6? (2 Re2 was correct) Ne4, is this position a draw or a win for White?
How Hard is the Play of the KRKN Ending?

Knight's Side

K  A1  A2  A3

K
T1  T2  T3
1  1  1
K = Kopec was opponent
A1, A2, A3 = three Class 'A' player subjects

Tn  1 or 0

where Tn represents the nth test position:
1 means successful task completion
0 means unsuccessful task completion

Figure 6. Design and results of Experiment 1

Figure 7. Continuation from T6 of Experiment 1 with subject P.Edwards.
Is this position a draw or a win for White?

At first 1 Rb3! was tried, when not 1 ... Ka4?, 2 Rd3! and Black is lost, but 1 ... Nd2, 2 Rd3 Nb1! etc. when the position is clearly drawn. Then 1 Kd5 was tried with some very surprising variations to follow, but we will let readers study the position longer before giving the variations which provide the answer. This retrospective analysis led to the discovery of three positions where counter-intuitive 'separating' moves were required to draw and the Bratko Advice Table erred. The failure of the Table for these positions was not really surprising, for it had been designed for positions where the BK and BN were relatively close together (i.e. < 3 king moves apart) with the belief that all positions where K and N are further separated are either lost or, if not, K and N must approach each other immediately. This is the view which had been presented in all the basic chess literature for this ending. Positions where K and N are reasonably separated were considered lost. Only recently, thanks to Mr A.J.Roycroft, have we learned of a Czech paper by Mandler (1970), which indicates that work has been done on this aspect. These positions, which at the time were
believed to be new to the theory of this ending, led to an entire benchmark of 16 positions where K and N are separated. Some of these positions will be discussed in the last part of this paper, but now we will discuss their importance in Experiment II.

Experiment II: strong Class ‘A’ players

Objectives. Subjects’ performance in the defence of KNKR in Experiment I pointed to the conclusion that it was not a difficult task for Class ‘A’ players. However the retrospective analysis brought new interest to the problems of correct play in drawn positions with the domain much richer than previously thought. Experiment II was seen as necessary to find out just how hard is the defence of KNKR when the domain is enriched with these ‘new’ positions, or more specifically: How well will strong Class ‘A’ human players defend KNKR positions where counter-intuitive ‘separating moves’ are required as well as other concepts?

Experimental Design. The 16 positions generated by the retrospective analysis of Experiment I and tests on the Bratko Table included 13 positions which the table did not handle correctly. For Experiment II nine of the hardest KNKR positions were selected, with highest ranking in difficulty going to positions where counter-intuitive ‘separating’ moves were required. Further ranking was based on the number and complexity of concepts considered necessary for correct play. Three Class ‘A’ players, with ratings of 1950, 1975 and 1915 were each allotted three positions in such a way as to approximately even out the mean ranking of difficulty and to include for each subject a reasonable diversity of concepts necessary for correct play (figure 9). The nine positions used are given in figure 8. Viewing them from left to right, top to bottom, we go from the position highest ranked in difficulty (D3) to the lowest ranked (D15). Note that D3, D2, and ‘Corncase’ all require counter-intuitive ‘separating’ moves, and thus are highest ranked. Since D3 includes D2 as a subset of correct play from it, but first requires a ‘natural’ knight move, it is higher ranked than D2. The method by which these nine positions were ranked is given in figure 9. Of these nine positions the Bratko Advice Table had failed to play the correct move in seven of them, handling ‘Corncase’ and D15 correctly.

Subjects and experimenter (D.K.) were allowed 40 minutes in which subjects were to play 20 moves which held the draw. The time had been slightly curtailed compared with Experiment I since once subjects succeeded in playing the first few moves correctly the task became easy and rather pointless. According to their post-mortem testimonies, two of the three subjects in this experiment knew that they should keep K and N together.

Results and Conclusions. As shown in figure 10, subjects failed in 6 of 9 trials. Failure in 4 of the 5 positions highest ranked in difficulty indicates that our a priori notion of what comprises a difficult KNKR position for the human was probably correct. However subjects played the first unforced move correctly in 5 of the 9 positions. In addition, we note that despite subjects’ correct handling of ‘Corncase’ in both Experiment I and
Experiment II, it is apparent from post-experimental questioning that subjects did not do the necessary calculation to see why the 'separating' move, 2 ... Ne2 was the only move which did not lose.

This experiment substantiated our notion that at least expert and perhaps master strength human play may be required for the correct defence of KNKR. We have also gained insight into specific concepts which may be necessary for correct play in positions where defending K and N are separated.

Positions were allotted to subjects in such a way as approximately to even out the mean ranking of difficulty, and at the same time to include for each subject a reasonable scatter. Symbols such as OKON, ONLOST2P, etc., are explained in the glossary below:

APPROKON: 'approach our king our knight'. Piece of advice needed to guide our knight or our king in separated positions.
<table>
<thead>
<tr>
<th>Position name</th>
<th>Rank</th>
<th>Concepts involved</th>
<th>Subject treatments</th>
<th>Average rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>D3</td>
<td>1</td>
<td>APPROKON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D2</td>
<td>2</td>
<td>?</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>Corncase</td>
<td>3</td>
<td>OKONDE4</td>
<td>(special case)</td>
<td></td>
</tr>
<tr>
<td>D4</td>
<td>3</td>
<td>APPROKON</td>
<td>COMMON GROUND</td>
<td></td>
</tr>
<tr>
<td>D10</td>
<td>5</td>
<td>ONLOST2P</td>
<td>OKONNDLE</td>
<td></td>
</tr>
<tr>
<td>D13</td>
<td>6</td>
<td>OKTRATT</td>
<td>ONLOST2P</td>
<td></td>
</tr>
<tr>
<td>D9</td>
<td>7</td>
<td>APPROKON</td>
<td>ONTRNDLE †</td>
<td></td>
</tr>
<tr>
<td>D6</td>
<td>8</td>
<td>OKTRATT</td>
<td>A₂:D₂,D₁₃,D₁₅</td>
<td>5.33</td>
</tr>
<tr>
<td>D₁₅</td>
<td>8</td>
<td>OKTRATT</td>
<td>A₃:D₄,Corncase,D₉</td>
<td>4.33</td>
</tr>
</tbody>
</table>

* These are positions which require "separating" moves whose underlying concepts we have been unable to specify.

† Note that during Experiment II it was discovered that if White plays Kc₄ after 1 ... Nc₆+ the addition of OKONNDLE would be necessary.

Figure 9. Ranking of positions for Experiment II (average rank 4.77)

<table>
<thead>
<tr>
<th>ELO RATING</th>
</tr>
</thead>
<tbody>
<tr>
<td>D₃</td>
</tr>
<tr>
<td>D₆</td>
</tr>
<tr>
<td>D₁₀</td>
</tr>
<tr>
<td>A₁</td>
</tr>
<tr>
<td>X</td>
</tr>
</tbody>
</table>

| D₂         | 0         |
| D₁₃        | 1         |
| D₁₅        | 1         |
| A₂         | 1975      |
| X          |           |

| D₄         | 0         |
| CORN-      | 1         |
| D₉         | 0         |
| A₃         | 1915      |
| X          |           |

D₁ = a drawn benchmark position
1 = task successfully completed
0 = task unsuccessfully completed (subject lost or resigned)
X = first unforced move was a losing one

Figure 10. Design and results of Experiment II. Kopec was experimenter in each case.
OKONDE4: 'our king our knight distance equals 4’
Common Ground: Potential meeting squares for K and N in separated positions.

ONLOST2P: 'our knight lost 2 ply'. Predicate which detects nearly all losses of our knight in two ply. Applied as 'Not ONLOST2P'.

OKONNDLE: 'our king our knight new distance less than or equal to'. Means that we are happy to maintain or decrease the distance (in king moves) between our king and our knight, but not increase it

ONTRNDLE: 'our knight their rook new distance less than or equal to’. Intended to prevent our knight from approaching their rook in separated positions

OKTRATT: 'attack their rook two ply'. Predicate for positions where our king threatens to capture their rook thereby allowing common ground or some other concept to be attained

CONCEPTS INVOLVED IN
A BENCHMARK OF 20 DRAWN POSITIONS
Counter-Intuitive ‘Separation’ and Common Ground
In this section we discuss some of the positions used in Experiment I and Experiment II, as well as a few others.

On the basis of the insight gained we augmented the set of 16 positions mentioned earlier with a further four with the aim of establishing a benchmark of 20 positions. These should specifically test all the features of difficulty of which we were aware. Returning to figure 7 from the experimental run with subject P. Edwards, after 1 Kd5 we reach the position labelled D2 (see figure 8). To answer our earlier question, ‘What is the value of the position in figure 7?’ we now need a 12-ply search, i.e. 1 ... Nc3+, 2 Kc4 Ne4, 3 Re2 Nd6+ (any other N move gets the knight stranded as well), 4 Kc5 Nb7+, 5 Kc6 Nd8+ (if 5 ... Ka6, 6 Ra2+ Na5+, 7 Kc5 and wins), 6 Kc7 Nf7, and now the knight is clearly stranded. Instead, in position D2 the only drawing move is 1 ... Nf6!! We shall give Black's responses to all of White's possible K moves from figure 11, in counter-clockwise order: if (a) 2 Ke6 Ne4 (b) 2 Kd6 Ne4+ (c) 2 Kc6 Ne4 returning to figure 7 (d) 2 Kc5 Nd7+ or Ne4+ (e) 2 Kc4 Nd7 (f) 2 Kd4 Nd7 (g) 2 Ke5 Nd7+. Returning to D2 (figure 8), after 1 ... Nc3+?, 2 Kc4 Ne4, if White plays 3 Kd4? instead of 3 Re2, we have D3 (figure 8), when only 3 ... Nd6 draws. Then if 4 Kc5 Ne4+, 5 Kd5, we return to D2, and so D3 is a superset of D2, whereby first the 'natural' Nd6+ is required, and later the 'counter-intuitive' Nf6+!! (D2). Considering D2, after 1 ... Nc3+?, 2 Kc4 Ne4, we try 3 Rb5+? Ka6, 4 Rd5 reaching D4. Normally White's last few moves comprise part of a winning technique which entails: (1) separate BK and BN, (2) drive N further away, (3) trap N and win it. However in D4, Black has the drawing move Kb7 threatening Nf6 ... Ne8, ... Nc7 or ... Nf6, ... Kc7, and Nd7. (Other threats are ... Kc6 or ... Kc7 and ... Nd6, or Nf6 and Nd7 to follow.) White's inability to prevent all these 'paths' by which BK and BN threaten to meet, ensures that Black can draw. We have labelled these meeting squares 'Common Ground'. Figure 12 depicts this concept for D4
Figure 11. Position arising from D2 after 1 ... Nf6+!! This is the first case from our experimental work where a counter-intuitive 'separating' is required.

Figure 12. Position after 1 ... Kb7 from D4. Squares which are 'Common Ground' or potential meeting squares for BK and BN are marked with an X. Paths to these squares are marked with arrows also indicating approximate direction to target square(s).

after 1 ... Kb7, 'paths' denoted by '★' and the Common Ground denoted by 'X'. Another heuristic indicating the move ... Kb7 is 'If K and N are 3 or more K moves apart, don't move your K onto a file or rank where N can move'. Otherwise that N move can be met by a R move forking K and N.

If in D4 Black tries 1 ... Kb6?, then 2 Rd4! wins, i.e. 2 ... Nc5 and 2 ... Nf6 both lose to 3 Rd6+ and after 2 ... Ng5, 3 Rg4 Nf7, 4 Rg6+ Ka5, 5 Kd5 Nd8, 6 Kg8 Nb7, 7 Kc6 and wins.

In 'Concase', the refutation to Black's other N moves after 1 Rb2+ Ka1, 2 Rb7 deserves more attention than the continuation after the correct move 2 ... Ne2. This might go: 2 ... Na2, 3 Kb3 Kb1 (if 3 ... Nc1+, 4 Kc2 wins), 4 Rb8!!.

In post-mortem discussion those two subjects who played the correct 2 ... Ne2 did not mention the passing move 4 Rb8 as the refutation of Na2. The idea of a 'passing' move is the only way White can win in the position. Then on 4 ... Nc1+, Kc3+ with 6 Kc2 to follow, wins. 2 ... Nd3, 3 Kb3 and then if 3 ... Nc5+, 3 ... Nb2, or 3 ... Nc1+, 4 Kc2 wins quickly.

'Guided Knight Tours'

Next we shall consider D9 and D10 (figure 8). In D10 Na4 and Ne4 are the candidate moves. Na4 is closer to the BK than Ne4 but on the edge, while the latter move is more centralized. In fact after 1 ... Na4, 2 Kb4 forces 2 ... Nb6 since Nc3 loses to 3 Rg2+ (ONLOST2P), and then Black's N is stranded with further separation from the BK to follow. Correct is 1 ... Ne4 when White can make no progress, though Black must still play very carefully, i.e. if 2 Kd3 Nc5+, 3 Kc4 repeats the position; or if 2 Kd4 Nd2 (not Nc3? Rg2+, etc.) 3 Rg2 Kc1 and we have the typical 'edge draw'. In D9 the candidate moves are obviously Nf5+ and Nc6+. Both moves bring the N a distance of five king moves from the BK; however Nf5+ offers
no further path back to the BK. After 1 Nf5+, 2 Ke5 Ne7, 3 Rh6 the N is stranded and soon will be lost. Even after the correct move, 1 ... Nc6+, 2 Kc4, the BN still needs careful guidance — is 2 ... Na5+ or Ne5+ correct? This decision and the one on Black’s first move can be guided by the important underlying heuristic ‘When N and K are separated, don’t move the N towards both the opposing K and R unless it is the only way back to your K’. Thus here the correct move is 2 ... Na5+ and after 3 Kb4 Nc6+ it is clear Black cannot be forced to further separate K and N. If 2 ... Ne5+, 3 Kd5 Nd7, 4 Rh6 and again the N is stranded.

OKTRATT for Common Ground

The last group of positions from Experiment II to be discussed is D13, D6 and D15 (figure 8). These all share the idea of OKTRATT (our king their rook attacks) in order to allow ‘Common Ground’ to be achieved.

In D13 there are two drawing moves. 1 ... Kc7 and 1 ... Nd2. 1 ... Kc7 is clearer joining K and N next move. If 1 ... Nd2 then 2 Rb8+ Kc6! draws.

Similarly, in D6 1 ... Kc8 allows ‘Common Ground’ to be achieved via either ... Nf6 or ... Nd6. This is the only move since 1 ... Nf6? is met by Rd6 forcing further separation. In D15 Black’s ‘normal’ move, 1 ... Ng3 is thwarted by 2 Re3 when 2 ... Nf5 is impossible due to the fork, 3 Re5+ and thus the knight is quickly lost. In order to avoid the fork and still allow a path for ‘Common Ground’ via 2 ... Ng3 or 2 ... Ne3, depending on how White plays, the ‘separation’ move 1 ... Ke6 draws. The more obscure 1 ... Nh2 also draws because White cannot cut of both the paths ... Nf3 and ... Ng4 to follow.

D8 (figure 13) was contrived to show that the distance between K and N is not as important as the availability of paths between them, especially for the N. Both the moves ... Nb5 and ... Nc6 bring the BN closer to the BK, though at the same time closer to the WR and WK. Yet these are the only moves to consider, for a K move would be too slow in joining BK and BN, i.e. 1 ... Ke2, 2 Kc4! Nc6, 3 Re8+ and the N is soon trapped. Comparing 1 ... Nc6 and 1 ... Nb5, we see that the former is more centralized, offering two clear paths through the centre (... Ne5 and Nd4) back to the BK. White cannot prevent both these ‘threats’ and thus Black draws easily with 1 ... Nc6. On the other hand, 1 ... Nb5 only offers ... Nd4+ as a direct path back to the BK, with ... Nd6 as an indirect route. The natural move, 2 Rd8, cuts off these two possibilities (see figure 13, D8.2). Yet ... Nc7 draws! After 1 ... Nb5? from D8 it was later found that 2 Kc4! Nd6+, 3 Kd3 wins. From D8.2 White is threatening Rd7 and then Kb4 winning the N. Black’s only way out of this is to play ... Nc7 immediately, before the N gets ‘fenced in’. The N wanders, persistently evading the R. If the WK tries to participate, the BK comes to the rescue of the N just in time, or the N escapes back to the BK. From D8.2 if we continue 1 ... Nc7, 2 Rd6 Kf2, 3 Rf6+! Kg3 (the only move — if 3 ... Ke3, 4 Kc4 and 5 Rc6 wins quickly) 4 Kc4, then the position D18 is reached. Here only 1 ... Kh4 draws. The explanation for why 1 ... Kg4 loses can be given as an extension of the heuristic given for D4, ‘If K and N are separated by a distance of 3
or more K moves, don’t move your K onto a file or rank which the N may
need to occupy in two moves, i.e. 1 ... Kg4?, 2 Re6 Ne8, 3 Re6 Ng7
(3 ... Nc7, 4 Re7 wins the N soon), 4 Rg6+ and wins.

CONCLUSIONS

How hard is the KRKN ending?

1. The use of an exhaustive database has shown that published treat-
ments of KRKN are marred by serious inaccuracy and incompleteness.
Hence the domain cannot be described as easy, even for the chess analyst.

2. The attacking side’s task in won positions is too hard for successful
play to be reliably demonstrated by Masters against optimal defence.
But a celebrated endgame study specialist was able to do this after intensive
preparation on a benchmark of six 24-movers, only three of which had been
made available to him for prior study (see appendix 3).

3. Play of the defence of king and knight against king and rook in
drawn positions is easier than play of the attacking side in won positions.
But even the defence proved to be too hard for strong Class ‘A’ players to
conduct reliably.

4. Most of the difficulty of the defender’s task is concentrated in a
relatively small class of positions which demand counter-intuitive moves
separating king and knight.

5. The key concept of separation between king and knight should not
be measured geometrically as has been customary. A revised definition
should be based on multiplicity as well as length of available paths between
the two pieces.

6. The KRKN ending is a great deal harder than has been assumed,
and its complete codification has eluded chess theorists. Improvement of
existing theory is here put forward. This was substantially aided by avail-
bility of the KRKN database.

ACKNOWLEDGEMENT

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REFERENCES


Appendix 1. Optimal Move Sequences
for the Two Longest Wins

Position: WK:d1, WR:h1, BK:b1, BN:g4

1  Rh4  Ne5  2  Re4  Nf7  3  Rb4+ Ka2
4  Kc2  Ka3  5  Kc3  Nd6  6  Rb6  Ne4+
7  Kd3  Nf2  8  Kc4  Nd1  9  Rb3+ Ka4
10  Rf3  Nb2+  11  Kc3  Ka3  12  Rg3  Na4+
13  Kc4  Ka2  14  Kb4  Nb2  15  Rg4  Nd3+
16  Kc3  Nc5  17  Rc4  Ne6  18  Ra4+ Kb1
19  Ra5  Ng7  20  Re5  Ka2  21  Kd4  Kb3
22  Kd5  Kc3  23  Kc6  Kd4  24  Kd6  Kd3
25  Ke7  Kd4  26  Rg5  etc.

Position: WK:c1, WR:f8, BK:a3, BN:e2

1  Kd2  Nd4  2  Kc3  Nb5+  3  Kc4  Nd6
4  Kc5  Nb7  5  Kb6  Nd6  6  Rf4  Kb3
7  Kc5  Nb7+  8  Kc6  Nd8+  9  Kb5  Ne6
10  Rf3+ Kc2  11  Kc4  Kd2  12  Rf5  Kc2
13  Rf2+ Kd1  14  Kd3+ Nc5+  15  Kd4  Nb3+
16  Kc3  Ke1  17  Rb2  Nc5  18  Kd4  Ne6
19  Ke3  Kf1  20  Rb6  Nc7  21  Ke4  Kf2
22  Ke5  Ke3  23  Rb7  Na6  24  Kd6  Kd4
25  Rb6  Ne5  26  Rb4  etc.

Many positions, for either side, have more than one optimal continuation.
The above two lines should therefore be regarded as specimen paths excerpted arbitrarily from two optimal-strategy trees.
Appendix 2

When given the position WK:c6, WR:d8, BK:a7, BN:e3(#499, BCE) (see figure 5, T5) for the stronger side in Experiment 1, the subject, Pat Coleman (1965 rating), improved on the variation given by Fine by 6 moves. What follows is the experimental record:

<table>
<thead>
<tr>
<th>W: P. Coleman</th>
<th>Minimax-Optimal Value (Moves)</th>
<th>Optimal move(s) and their minimax-optimal path-lengths</th>
</tr>
</thead>
<tbody>
<tr>
<td>B: D. Kopec</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Rd4 (13)*</td>
<td>9</td>
<td>Kb8(9)</td>
</tr>
<tr>
<td>1. ... Nf5</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>2. Ra4+(15)</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>2. ... Kb8</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3. Re4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3. ... Ka7</td>
<td>3</td>
<td>Ng3(5)</td>
</tr>
<tr>
<td>4. Kd5</td>
<td>8</td>
<td>Kc7(3)</td>
</tr>
<tr>
<td>4. ... Kb6</td>
<td>8</td>
<td>Nb3, Kb7, Kb6</td>
</tr>
<tr>
<td>5. Ke6 (21)</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5. ... Ng3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>6. Re1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>6. ... Kc5</td>
<td>5</td>
<td>Kc7, Kc6(6)</td>
</tr>
<tr>
<td>7. Ke5 (22)</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>7. ... Kc4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>8. Kf4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>8. ... Nh5+</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>9. Kg5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>9. ... Ng3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>10. Kg4 (25)</td>
<td>and captures N</td>
<td></td>
</tr>
</tbody>
</table>

* Figures in brackets after White’s moves in this column are the total time from the allotted hour, consumed by subject.
Appendix 3

The following notes are contributed by A.J. Roycroft, author of Test-tube Chess and editor of EG, the international endgame study magazine.

The task was simply this. I was presented with three positions known to be won for the side with the rook in a battle against a lone knight, and I was to win them. The diagrams R1, R2 and R3 show the positions, and this is how the play went from each.

\[
\begin{array}{ccc}
\text{R1} & \text{R2} & \text{R3} \\
1. & \text{Rb3} & \text{Ka5} & 1. & \text{Ke5} & \text{Na4} & 1. & \text{Ke7} & \text{Kg7} \\
2. & \text{Kc7} & \text{Nf2} & 2. & \text{Rh7+} & \text{Ke8} & 2. & \text{Ra3} & \text{Kg6} \\
3. & \text{Kc6} & \text{Na4} & 3. & \text{Kd6} & \text{Nb6} & 3. & \text{Ke6} & \text{Nc2} \\
4. & \text{Rf3} & \text{Nd1} & 4. & \text{Rh8+} & \text{Kf7} & 4. & \text{Rg3+} & \text{Kh5} \\
5. & \text{Kc5} & \text{Nb2} & 5. & \text{Rh4} & \text{Nc8+} & 5. & \text{Kf5} & \text{Kh4} \\
6. & \text{Rh3} & \text{Nd1} & 6. & \text{Kd7} & \text{Nb6+} & 6. & \text{Rc3} & \text{Ne1} \\
7. & \text{Kc4} & \text{Nb2+} & 7. & \text{Kc6} & \text{Nc8} & 7. & \text{Kf4} & \text{Ng2+} \\
8. & \text{Kc3} & \text{Ka3} & 8. & \text{Rh7+} & \text{Kf6} & 8. & \text{Kf3} & \text{Kh3} \\
9. & \text{Rg3} & \text{Na4+} & 9. & \text{Rh6+} & \text{Kg7} & 9. & \text{Ra3} & \text{Nh4+} \\
10. & \text{Kc4+} & \text{Ka2} & 10. & \text{Re6} & \text{Na7+} & 10. & \text{Kf4+} & \text{Kh2} \\
11. & \text{Rg5} & \text{Nb2+} & 11. & \text{Kd6} & \text{Kf8} & 11. & \text{Ra5} & \text{Ng2+} \\
12. & \text{Kc3} & \text{Nd1+} & 12. & \text{Kd7} & \text{Nb5} & 12. & \text{Kf3} & \text{Ne1+} \\
13. & \text{Kc2} & \text{Ne3+} & 13. & \text{Re3} & \text{Nd4} & 13. & \text{Kf2} & \text{Nd3+} \\
14. & \text{Kd2} & \text{Nc4+} & 14. & \text{Re4} & \text{Nb3} & 14. & \text{Ke2} & \text{Nf4+} \\
15. & \text{Kc3} & \text{Ne3} & 15. & \text{Kd6} & \text{Kg7} & 15. & \text{Kf3} & \text{Nd3} \\
16. & \text{Re5} & \text{Ng4} & 16. & \text{Kd5} & \text{Nd2} & 16. & \text{Rd5} & \text{Nb4} \\
17. & \text{Re6} & \text{Kb1} & 17. & \text{Rf4} & \text{Nb1} & 17. & \text{Rd6} & \text{Kg1} \\
18. & \text{Kd2} & \text{Nf2} & 18. & \text{Rf3} & \text{Kg6} & 18. & \text{Ke2} & \text{Nc2} \\
19. & \text{Rb6+} & \text{Ka2} & 19. & \text{Rd3} & \text{Kf5} & 19. & \text{Rg6+} & \text{Kh2} \\
20. & \text{Rb4} & \text{Nh1} & 20. & \text{Kc5} & \text{Ke4} & 20. & \text{Rg4} & \text{Na1} \\
21. & \text{Rg4} & \text{Ka3} & 21. & \text{Kc4} & \text{Ke5} & 21. & \text{Rb4} & \text{Kh3} \\
22. & \text{Ke3} & \text{Kb3} & 22. & \text{Kb4} & \text{Ke6} & 22. & \text{Kd3} & \text{Kh2} \\
23. & \text{Rg1} & \text{‘resigns’} & 23. & \text{Rd1} & \text{‘resigns’} & 23. & \text{Rb1} & \text{‘resigns’} \\
\end{array}
\]

I played with a clock (at the time-limit of 16 moves per hour) and I did not have any moves back. On the other hand, I had had over a week’s notice of the positions, and naturally I prepared for the contest. In chess terms it was more like having an adjourned game.

I was already familiar, without being move-perfect, with what the books give on this ending. Also, I had learned a lot from playing against
Three positions, all White to play. In all cases White captures the black knight on his 24th move.

the same database at the April two-day conference in Edinburgh (CW, May 25). Thirdly, I asked for, and was given in advance, four positions (and the single-line optimal play) with the same solution length as the contest positions, so that I could have some training that was relevant (I refer to these later as the Edinburgh lines).

The solutions were all 24 moves long, chosen at random from the 178 such positions, with the idea that the near-maximum length (no position has an optimal solution exceeding 27 moves) and the number of such positions would provide the toughest and fairest contest conditions.

About five minutes was spent on each task. My preparation was pretty good, and most of the difficult moves I had already written down. Not only did I have notes, but an auxiliary board on which I moved the pieces around when I wanted to before choosing the move to be played on the primary board. This was all agreed in advance. But I used the auxiliary board seriously only twice. The contest was between a chess analyst and a database, not between an over-the-board master and a database, so the conditions were deliberately made to ensure the analyst, myself, could not complain.

Coming to the play, the solution to R3 is very similar to that to R1, a mirror image, or almost, from move 5 onwards. We were all surprised at the lack of variety, certainly the apparent lack of variety, in the core of the solutions. Even R2 was like one of the positions given to me in advance, and the R1 and R3 lines were not new to me. But R1 and R2 are quite distinct from each other.

The program might have diverged, by selecting an alternative equal depth move at many possible branch-points, though it did not. But certainly, most of the work was in the preparation, and a great deal of the play was accidentally (not deliberately, anyway) to be found in the practice positions.

Speaking as an analyst, R2 is quite beautiful. And it will not be found in the textbooks. Take the position after Black’s third move (see R2.1). The black knight and black king are striving to meet, to set up a standard drawing set-up with both men next to each other on the edge. White has to play very precisely to prevent this. In fact, I find the next 10 moves a sheer delight.

Consider the position after 8 Rh7+ (see R2.2). Where can Black play
his king? If he goes to e8 or to f8, the knight is immediately lost to a rook check on h8. If he goes to g8, however, the rook plays to c7, winning the knight by domination, since it has no safe squares. The same applies to the move of Black’s king to g6. So, in spite of the apparently wide choice, Black really has only e6 and f6 to choose from. But even e6 fails to Rh6+ and the variety of ways in which the knight is now caught in just a few more moves is quite delightful. So, really, only f6 will do for Black in R2.2. The effect, in the mind, of comprehending each of these variations individually, and all of the variations as a group, is strongly aesthetic.

After R1, R2 and R3 had been disposed of, I asked for three more positions to be given to me without preparation. Here I would try to play fast, but with up to three moves back — a kind of compromise between over-the-board competitive play and how the analyst works. The result was exactly the same. I made no mistakes (to everyone’s surprise, including my own) and it was remarkably easy. I assumed that the positions would transpose into solutions that I already knew — indeed, into the essentially different R1 and R2 lines or into one of the Edinburgh lines. And they did. An example is R4. It was not too difficult to find 1 Ke7 Kh6, 2 Kf6 Ne3, 3 Rg3 Nd5+, 4 Ke6 Nc7, 5 Kf5 Ne8, 6 Rg6+ Kh5, 7 Rc6 and we are in R1, as near as makes no difference. I had no moves back in these three ‘unseen’ positions. It turned out that all the moves I made were optimal. That is, no moves I made prolonged the solution unnecessarily.

I tackled the contest by looking for patterns. For instance, the play in all these long-solution positions seems to fall naturally into three phases. In the first phase the chessmen are relatively dispersed, but Black must get his king and knight together, or perpetually threaten to. At the end of this we enter a second phase, the kernel of the struggle, where a sequence of a dozen or so moves decides it, because eventually the knight is forced away from its king. Phase three mops up the knight. Of course, mate threats operate in any phase of the struggle, even in all phases.
Appendix 4. Benchmark of Twenty Positions

The following pages contain our benchmark of 20 positions which illustrate all the features of special difficulty for the knight's side of the ending of which we are aware. All positions are drawn with correct play.
D5.2 ... Nb1+

D6 ... Kc8

D8 ... Nc6

D8.2 ... Nc7

D8.4 ... Kg3

D9 ... Nc6+

D10 ... Ne4

D12 ... Nb6
or Kc2 or Kd2 or Kb2
How Hard is the Play of the KRKN Ending?

D13 ... Kc7 or Nd2

D15 ... Kd6

D16 ... Kb6

D17 ... Kg3

D18 ... Kh4

'Corncase' Rb2+ ...