Charpit's method to find the complete integral*

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1 Description of the method

Consider a first order partial differential equation with two independent variables

$$F(x, y, u, p, q) = 0, \tag{1}$$

where $p = \partial u/\partial x$ and $q = \partial u/\partial y$, and we assume that $F_p^2 + F_q^2 \neq 0$. The equation for the characteristic strips for this equation are

$$\frac{dx}{ds} = F_p, \quad \frac{dy}{ds} = F_q, \quad \frac{dp}{ds} = -F_x - pF_u, \quad \frac{dp}{ds} = -F_y - qF_u, \quad \frac{du}{ds} = pF_p + qF_q$$

(see [3, p. 2]). Eliminating the parameter s from these equations, one often writes them as

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{dp}{-F_x - pF_u} = \frac{dq}{-F_y - qF_u}. \tag{2}$$

These equations are called Lagrange–Charpit equations. In interpreting these equations, it is convenient to allow zero denominators. For example, if $F_p = 0$, these equations require that $dx = 0$; that is, the denominator being zero just means that the numerator is also zero. Assume that from equations (1) and (2) one can derive a new equation

$$\phi(x, y, u, p, q) = a, \tag{3}$$

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where a is an arbitrary constant. Assume that the Jacobian determinant of F and ϕ with respect to p and q does not vanish, i.e., that

$$J = \det \frac{\partial(F,\phi)}{\partial(p,q)} = \begin{vmatrix} F_p & F_q \\ \phi_p & \phi_q \end{vmatrix} = F_p\phi_q - \phi_p F_q \neq 0; \tag{4}$$

this assumption implies that equations (1) and (3) can be solved locally for p and q. Let this solution be

$$p = p(x,y,u,a) \quad \text{and} \quad q = q(x,y,u,a). \tag{5}$$

To simplify the notation, we consider a fixed value of a, and we write $p(x,y,u,a) = p(x,y,u)$ and $q(x,y,u,a) = q(x,y,u)$ for now. Then, to get $u = u(x,y)$, we need to solve the system

$$u_x = p(x,y,u), \quad u_y = q(x,y,u) \tag{6}$$

of partial differential equations (for each value of the parameter a).

Several questions arise at this point: does this really simplify the problem, and in any case, are these equations solvable. The second question arises because we must have $u_{xy} = u_{yx}$ under reasonable assumptions.[1] Differentiating the first equation in (6) with respect to y and taking second equation into account, we have

$$u_{xy} = p_y + p_u u_y = p_y + p_u q.$$

Similarly, differentiating the second equation with respect to x, we obtain

$$u_{yx} = q_x + q_u u_x = q_x + q_u p.$$

The equality of these mixed derivatives[2] leads to the requirement

$$p_y + p_u q = q_x + q_u p. \tag{7}$$

Therefore, to make the above approach viable, we need to do two things: we need to show that condition (7) is satisfied, and then to show that this condition is sufficient to solve the system of equations given in (6). Solving the above system is described as *integrating* the system, and condition (7) is called *integrability condition*, and a system satisfying this condition is an *integrable* system.

The system of equations given in (6) can also be written as

$$du = p(x,y,u)\,dx + q(x,y,u)\,dy. \tag{8}$$

Here du denotes the total differential of $u = u(x,y)$, and since $du = u_x\,dx + u_y\,dy$, the equality meaning the equality of the coefficients in the differential expression, this equation is just another way of writing the above system of equations.

Assuming that condition (7) is satisfied, and a function $u = u(x,y)$ satisfying (6) exists (locally, i.e., on an on open set containing the point (x_0,y_0)), given the value of u at a point (x_0,y_0), such a u can be determined as follows. Let $x = x(t)$, $y = y(t)$ be differentiable functions such that $x(0) = x_0$ and $y(0) = y_0$, and write $U(t) = U(x(t),y(t))$. Then it follows from (6) (or, equivalently, from(8)) with the aid of the chain rule that

$$U'(t) = p(x(t),y(t),U(t))\,x'(t) + q(x(t),y(t),U(t))\,y'(t) \tag{9}$$

holds. Solving this differential equation for U with the initial condition $U(0) = u(x_0,y_0)$, we can determine $u(x(t),y(t)) = U(t)$ (locally, i.e., for small values of t, where equation (9) is solvable).

Charpit's method is described in [2, §10-10, pp. 242–244] and in [1].

[1] For example, this is the case if u has continuous second derivatives.

[2] See [4] for a discussion of various conditions ensuring the equality of mixed partial derivatives.

2 An example

Before discussing the theoretical issues raised at the end of the previous section, we will describe an example of how the method can be used. Consider the equation

$$p^2u + q^2 - 4 = 0. \tag{10}$$

This equation is given in [2, Problem 7, p. 244]; the solution is given in [2, p. 287], but it is not shown how to arrive at this solution. The Lagrange–Charpit equations (see (2)) for the above equation can be written as

$$\frac{dx}{2pu} = \frac{dy}{2q} = \frac{du}{2p^2u + 2q^2} = \frac{dp}{-p^3} = \frac{dq}{-p^2q}.$$

The fourth equation here can be written as $dp/p = dq/q$, i.e., $\log|p| = \log|q| + C$, that is, $q = ap$ with $a = \pm e^{-C}$. Substituting this into equation (10), we obtain $p^2u + a^2p^2 - 4 = 0$, i.e.

$$p = \pm\frac{2}{\sqrt{u + a^2}}.$$

For the sake of simplicity, we will consider only the + sign in the $\pm$ here. With this equation, together with the equation $q = ap$ obtained just before, equation (8) can be written as

$$du = \frac{2}{\sqrt{u+a^2}}\,dx + \frac{2a}{\sqrt{u+a^2}}\,dy.$$

Taking $x = t$ and $y = \lambda t$ with λ an arbitrary constant (cf. (9)), we obtain

$$du = \frac{2}{\sqrt{u+a^2}}\,dt + \frac{2a}{\sqrt{u+a^2}}\lambda\,dt,$$

i.e.,

$$(u + a^2)^{1/2}\,du = 2(1 + a\lambda)\,dt.$$

Integrating both sides, we obtain

$$\frac{2}{3}(u + a^2)^{3/2} = 2(1 + a\lambda)t + C' = 2x + 2ay + C',$$

where C' is an arbitrary constant; the second equation follows since we have $x = t$ and $y = \lambda t$. Solving this for u, we obtain

$$u = (3x + 3ay + b)^{2/3} - a^2,$$

where a and b are arbitrary constants ($b = (3/2)C'$). This is a complete integral of (10).[3]

[3]This deals with the case $x \neq 0$, since the equations $x = t$, $y = \lambda t$ does not allow the possibility of $x = 0$ and $y \neq 0$. It would have been easy to deal with this case as well, by taking $x = \xi t$ and $y = \eta t$ with ξ and η being arbitrary constants; the case $x = 0$ can also be dealt with by making $x \to 0$ in the above solution, since the solution of the given equation is continuous. This shows that the solution we obtained is also valid in case $x = 0$.

If we take the $-$ sign in $\pm$ above instead of the + sign, we arrive at the same solution, since in one of the steps we squared both sides of an equation.

3 Integrability

Condition (7) ensures that equations (6) or, equivalently, (8) are solvable. This is expressed by the following

Theorem 1. *Let $(x_0, y_0, u_0) \in \mathbb{R}^3$ be a point, and assume that the functions p and q have continuous partial derivatives in a neighborhood of (x_0, y_0, u_0). Assume, further, that*

$$p_y + p_u q = q_x + q_u p \tag{11}$$

in this neighborhood. Then there is a unique function u in a neighborhood of (x_0, y_0) in $\mathbb{R}^2$ such that $u(x_0, y_0) = u_0$ and

$$du = p(x, y, u)\,dx + q(x, y, u)\,dy \tag{12}$$

in a neighborhood of (x_0, y_0).

This theorem has a far-reaching generalization for systems of first-order partial differential equations, called Frobenius's theorem, usually formulated in terms of vector fields or differential 1-forms, that plays an important role in differential geometry.

Proof. Consider the differential equations

$$u_x(x, y_0) = p(x, y_0, u(x, y_0)), \quad u_y(x, y) = q(x, y, u(x, y)), \quad u_y(x_0, y_0) = u_0. \tag{13}$$

It is clear that any function u satisfying equation (12) must also satisfy these equations. These latter equations can be thought of ordinary differential equations: the first equation together with the initial condition given by the third equation is to describe $u(x, x_0)$, and the second equation for a fixed value of x given by the value of $u(x, y_0)$ determined by the first and third equations as initial condition is to determine $u(x, y)$ for arbitrary (x, y). It follows from the theory of ordinary differential equations that these equations uniquely determine $u(x, y)$ in a neighborhood of (x_0, y_0); we need to show that the function so determined satisfies the requirements of the theorem.

The Picard–Lindelöf theorem states the following: *Let (x_0, y_0) be a point and let f be a continuous function on the set*

$$S = \{(x, y) \in \mathbb{R}^2 : |x - x_0| \le a,\ |y - y_0| \le b\},$$

where a and b are positive real numbers. Assume that there is a real number L such that

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2|$$

(such a condition is called a *Lipschitz* condition) *for all points (x, y_1) and (x, y_2) in S. Let*

$$M = \max\{|f(x, y)| : (x, y) \in S\}.$$

Then the differential equation

$$\frac{dg(x)}{dx} = f(x, g(x))$$

with initial condition $g(x_0) = y_0$ has a unique solution for g in the interval $(x_0 - h, x_0 + h)$, where $h = \min(a, b/M)$.

This theorem ensures the uniqueness and the existence of a function $u(x, y)$ satisfying equations (13) in a neighborhood of the point (x_0, y_0). The theorem is discussed in many standard books on ordinary differential equations.

To complete the proof of the theorem, we need to show that the function $u(x,y)$ so defined satisfies the first equation in (6); the second equation there is satisfied, since that equation is required in (13) used to define $u(x,y)$. Integrating the equations in (13), we obtain

$$u(x,y) = u(x_0,y_0) + \int_{x_0}^{x} p(\xi, y_0, u(\xi, x_0))\,d\xi + \int_{y_0}^{y} q(x,\eta,u(x,\eta))\,d\eta.$$

Differentiating this with respect to x, we obtain, by the Fundamental Theorem of Calculus and by the interchangeability of differentiation and integration, that

$$\begin{aligned} u_x(x,y) &= p(x,y_0,u(x,y_0)) + \int_{y_0}^{y} (q_x(x,\eta,u(x,\eta)) + u_x(x,\eta)q_u(x,\eta,u(x,\eta))\,d\eta \\ &= p(x,y_0,u(x,y_0)) + \int_{y_0}^{y} (q_x(x,\eta,u(x,\eta)) + p(x,\eta,u(x,\eta)q_u(x,\eta,u(x,\eta))\,d\eta \\ &\quad + \int_{y_0}^{y} (u_x(x,\eta) - p(x,\eta,u(x,\eta))q_u(x,\eta,u(x,\eta))\,d\eta \\ &= p(x,y_0,u(x,y_0)) + \int_{y_0}^{y} (p_y(x,\eta,u(x,\eta)) + q(x,\eta,u(x,\eta))p_u(x,\eta,u(x,\eta))\,d\eta \\ &\quad + \int_{y_0}^{y} (u_x(x,\eta) - p(x,\eta,u(x,\eta))q_u(x,\eta,u(x,\eta))\,d\eta, \end{aligned}$$

where the last equation was obtained by (11). Using the second equation in (13), we can further write

$$\begin{aligned} u_x(x,y) &= p(x,y_0,u(x,y_0)) + \int_{y_0}^{y} (p_y(x,\eta,u(x,\eta)) + u_y(x,\eta)p_u(x,\eta,u(x,\eta))\,d\eta \\ &\quad + \int_{y_0}^{y} (u_x(x,\eta) - p(x,\eta,u(x,\eta))q_u(x,\eta,u(x,\eta))\,d\eta \\ &= p(x_0,y_0,u(x,y_0)) + p(x,y,u(x,y)) - p(x,y_0,u(x,y_0)) \\ &\quad + \int_{y_0}^{y} (u_x(x,\eta) - p(x,\eta,u(x,\eta))q_u(x,\eta,u(x,\eta))\,d\eta \\ &= p(x,y,u(x,y)) + \int_{y_0}^{y} (u_x(x,\eta) - p(x,\eta,u(x,\eta))q_u(x,\eta,u(x,\eta))\,d\eta, \end{aligned}$$

where the second equation used the Newton-Leibniz formula to evaluate a definite integral. Therefore, we have

$$(14) \qquad |u_x(x,y) - p(x,y,u(x,y))| \le \int_{y_0}^{y} |u_x(x,\eta) - p(x,\eta,u(x,\eta)||q_u(x,\eta,u(x,\eta)|\,d\eta.$$

Let U be a neighborhood of (x_0,y_0) in which the function $u(x,y)$ described in (13) is defined and in which the functions $u_x(x,y)$, $p(x,y,u(x,y))$, and $p_u(x,y,u(x,y))$ are bounded. Let $M > 0$ be such that

$$M \ge |p_u(x,y,u(x,y)| \quad \text{for} \quad (x,y) \in U.$$

Let δ with $0 < \delta \le 1/(2M)$ be such that

$$\{(x,y)| : |x - x_0| < \delta,\ |y - y_0| < \delta\} \subset U.$$

Let

$$A = \sup\{|u_x(x,y) - p(x,y,u(x,y))| : |x - x_0| < \delta,\ |y - y_0| < \delta\}. \tag{15}$$

Using (14), for (x, y) with $|x - x_0| < \delta$ and $|y - y_0| < \delta$ we obtain

$$|u_x(x,y) - p(x,y,u(x,y))| \le AM|y - y_0| \le AM\delta \le A/2.$$

This implies that

$$\sup\{|u_x(x,y) - p(x,y,u(x,y))| : |x - x_0| < \delta,\ |y - y_0| < \delta\} \le A/2,$$

contradicting (15) unless $A = 0$. This shows the first equation in (6) is satisfied in a neighborhood of (x_0, y_0), completing the proof. □

4 Integrability of Charpit's equations

In order to show that the integrability condition (7) to solve equations (6) (or, equivalently, (8)) is satisfied, we need the following

Lemma 1. *The function ϕ given in (3) satisfies the equation*

$$F_p\phi_x + F_q\phi_y + (pF_p + qF_q)\phi_u - (F_x + pF_u)\phi_p - (F_y + qF_u)\phi_q = 0. \tag{16}$$

The above equation can also be written as

$$(F_p\phi_x - \phi_pF_x) + (F_q\phi_y - \phi_qF_y) + p(F_p\phi_u - \phi_pF_u) + q(F_q\phi_u - \phi_qF_u) = 0, \tag{17}$$

and so it is clear that this equation is satisfied if the F and ϕ are the same functions. On the other hand, equation (4) ensures that F and ϕ are not the same.

Proof. This equation means that the gradient vector $(\phi_x, \phi_y, \phi_u, \phi_p, \phi_p, \phi_q)$ is orthogonal to the tangent vector $(F_p, F_q, pF_p + qF_q, -F_x - pF_u, -Fy - qF_u)$ of the characteristic curves described by equations (2).[4] As equation (3) is a consequence of equations (2) and (1), it must be compatible with characteristic curves.[5] The gradient of the surface $\phi(x, y, u, p, q) = a$ must be orthogonal to any curve contained in this surface; hence, *a fortiori*,[6] it must be orthogonal to the characteristic curves contained in it. □

Now, consider equations (1) and (3) with p and q as in (5). That is, suppressing the parameter a, we have

$$F(x,y,u,p(x,y,u),q(x,y,u)) = 0 \quad \text{and} \quad \phi(x,y,u,p(x,y,u),q(x,y,u)) = 0.$$

Differentiating these with respect to x we obtain

$$F_x + F_pp_x + F_qq_x = 0 \quad \text{and} \quad \phi_x + \phi_pp_x + \phi_qq_x = 0.$$

[4] The characteristic curves are in $\mathbb{R}^5$. If one talks about these "curves" as objects in $= \mathbb{R}^3$, it is better to call them characteristic strips.

[5] That is, if the surface $\phi(x, y, u, p, q) = a$ contains a point of a characteristic curve, it must contain the whole characteristic curve (or, at least, a local piece of it, since these arguments only work locally). This is because equations (2) describe the characteristic curves, and equation (1) must also be compatible with these characteristic curves.

[6] Even more so. The phrase "a fortiori" is often used in mathematical writing.

Using the notation $J = F_p\phi_q - F_q\phi_p$ (cf. (4)), multiplying the first equation by ϕ_p, the second one by F_p, and subtracting the equations, we obtain

$$F_x\phi_p - \phi_x F_p - Jq_x = 0. \tag{18}$$

Similarly, differentiating the above equations with respect to y, we obtain

$$F_y + F_p p_y + F_q q_y = 0 \quad \text{and} \quad \phi_y + \phi_p p_y + \phi_q q_y = 0.$$

Multiplying the first equation by ϕ_q and the second one by F_q, we obtain

$$F_y\phi_q - \phi_y F_q + Jp_y = 0. \tag{19}$$

Next, differentiating the above equations with respect to u, we obtain

$$F_u + F_p p_u + F_q q_u = 0 \quad \text{and} \quad \phi_u + \phi_p p_u + \phi_q q_u = 0.$$

Multiplying the first equation by ϕ_p and the second one by F_p, we obtain

$$F_u\phi_p - \phi_u F_p - Jq_u = 0. \tag{20}$$

Similarly, multiplying the first equation by ϕ_q and the second one by F_q, we obtain

$$F_u\phi_q - \phi_u F_q + Jp_u = 0. \tag{21}$$

Adding equations (18), (19), p times (20), and q times (21) to the equation (17), we obtain

$$(-q_x + p_y - pq_u + qp_u)J = 0.$$

Noting that $J \neq 0$ according to (4), equation (7) follows.

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