3. The Axiom of Completeness

A *cut* is a pair (A, B) such that A and B are nonempty subsets of the set $\mathbb{R}$ of real numbers with $A \cup B = \mathbb{R}$, and such that for every $x \in A$ and every $y \in B$ we have $x < y$. If (A, B) is a cut, it is clear that the sets A and B must be disjoint (since every element of A is less than every element of B). For a cut (A, B) and a real number t we say that the cut *determines* the number t if for every $x \in A$ we have $x \le t$ and for every $y \in B$ we have $t \le y$.

For example, a cut that determines the number 2 is the pair (A, B) with $A = \{t : t \le 2\}$ and $B = \{t : t > 2\}$. Another cut that determines the number 2 is the pair (C, D) with $C = \{t : t < 2\}$ and $D = \{t : t \ge 2\}$.

It is clear that a cut cannot be determine more than one number. Assume, on the contrary, that the cut (A, B) determines the numbers t_1 and t_2 and $t_1 \ne t_2$. Without loss of generality, we may assume that $t_1 < t_2$. Then the number $c = (t_1 + t_2)/2$ cannot belong to A, since we cannot have $t_1 < x$ for any element of A, whereas $t_1 < c$. Similarly $c = (t_1 + t_2)/2$ cannot be an element of B, since we cannot have $y < t_2$ for any element of B, whereas $c < t_2$.

Furthermore, the above example is of a general character, in that for every real number u there are exactly two cuts, (A, B) and (C, D) that determine u, where $A = \{t : t \le u\}$ and $B = \{t : t > u\}$. $C = \{t : t < u\}$ and $D = \{t : t \ge u\}$. The Axiom of Completeness is an important property of real numbers:

Axiom of Completeness. *Every cut determines a real number.*

Ordinarily, one does not expect to prove this statement, since axioms are basic statements that one does not prove. However, one can prove the Axiom of Completeness if one defines the real numbers as infinite decimals.[12]

Proof of the Axiom of Completeness. Let (A, B) be a cut. Assume first that A contains a positive number. Let n be a positive integer, and let $x^{(n)} = x_0^{(n)}.x_1^{(n)}x_2^{(n)}\dots x_n^{(n)}$ be the largest decimal fraction with n digits after the decimal point such that $x^{(n)}$ is less than or equal to every element of B; here $x_0^{(n)}$ is a nonnegative integer, and, for each i with $1 \le i \le n$, $x_i^{(n)}$ is a digit, i.e., one of the numbers 0, 1, …, 9.[13] There is such an $x^{(n)}$, since the set of decimal fractions with n digits after the decimal point that are nonnegative and less than or equal to every element of B is finite.[14]

It is easy to see that if m and n are positive integers and $m < n$ then for each i with $0 \le i \le m$ we have $x_i^{(m)} = x_i^{(n)}$. Indeed, we cannot have $x_i^{(m)} < x_i^{(n)}$. for any i with $0 \le i \le m$ since then the number $x' = x_0^{(n)}.x_1^{(n)}x_2^{(n)}\dots x_m^{(n)}$ would be an m-digit decimal greater than $x^{(m)}$ and less than or equal to every element of B. Furthermore, we cannot have $x_i^{(m)} > x_i^{(n)}$. for any i with $0 \le i \le m$, since then, with this i, the number $x'' = x_0^{(m)}.x_1^{(m)}x_2^{(n)}\dots x_i^{(m)}00\dots0$ ($n - i$ zeros at the end) would be an n-digit decimal greater than $x^{(n)}$ and less than or equal to every element of B.

Now, the infinite decimal $r = x_0^{(0)}.x_1^{(1)}x_2^{(2)}\dots x_n^{(n)}\dots$ is the number determined by cut (A, B). Indeed, $r \ge x$ for every $x \in A$. Assume, on the contrary, that $r < x$ and let n be such that $10^{-n} < x - r$. Then noting that $r - x^{(n)} < 10^{-n}$, the decimal number $x^{(n)} + 10^{-n} \le r + 10^{-n} < x$ would be less than every element of B (since x is less than

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[12] However, defining the reals numbers as infinite decimals is inelegant in that the decimal number system came about only by a historical accident. One could also define the reals, however, as infinite binary fractions; i.e., "decimal" fractions in the binary number system. The binary number system is somewhat more natural than the decimal system in that it is the simplest of all number systems.

[13] i.e., if $x^{(n)} = 345.84$ then $n = 2$, $x_0^{(n)} = 345$, $x_1^{(n)} = 8$, and $x_2^{(n)} = 4$.

[14] The set of these decimal fractions is nonempty, since it contains the number 0. It is finite, because if b is an arbitrary element of B, there are at most $10^n \cdot b + 1$ n-digit nonnegative decimal fractions less than or equal to b.

every element of B), contradiction the choice that $x^{(n)}$ was the largest such n-digit decimal fraction. Furthermore, $r \leq y$ for every $y \in B$. Assume, on the contrary that $r > y$ for some $y \in B$, and let n be such that $10^{-n} < r - y$. Then noting that $r - x^{(n)} < 10^{-n}$, we have $x^{(n)} > r - 10^{-n} > y$, contradiction the assumption that $x^{(n)} \leq y$ for every $y \in B$. The proof is complete in case the cut (A, B) is such that A contains a positive number.

If (A, B) is a cut such that B contains the negative number, writing $-A = \{-x : x \in A\}$ $-B = \{-y : y \in B\}$, the cut $(-B, -A)$ is such that $-B$ contains a positive number; hence, by the above argument, this cut determines a number r. Then the cut (A, B) determines the number $-r$.

Finally, if the cut (A, B) is such that A does not contain a positive number and B does not contain a negative number, then it is clear that the cut (A, B) determines the number 0. The proof is complete.

The Axiom of Completeness guarantees, for example, that the number $\sqrt{2}$ exists. Namely, the cut (A, B) with $A = \{x : x < 0 \text{ or } x^2 \leq 2\}$ and $B = \{x : x > 0 \text{ and } x^2 > 2\}$ and determines the number t such that $t^2 = 2$.

To show this, we will first show that for every $\epsilon > 0$ we have $|t^2 - 2| < \epsilon$. In order to show this, we may assume that $\epsilon < 1$. First note note that there is an $x \in A$ and a $y \in B$ with $x \geq 0$ and $y \geq 0$ (the latter holds for every $y \in B$) such that $y - x \leq \epsilon/6$. To see this, consider the set

$$S = \{n\epsilon/6 : n \geq 0 \text{ is an integer} \quad \text{and} \quad n\epsilon/6 \in A\}.$$

S is not empty, since $0 \in S$. Furthermore, it is clearly a finite set, and so it has a largest element. Now, choose x to be the largest element of this set and put $y = x + \epsilon/6$ (clearly, $y \in B$, since if we had $y \notin B$ then we would have $y \in A$, and so $y \in S$, and then x would not be the largest element of x.

Observe that $x < 2$ (because $x \in A$, and so $x^2 \leq 2$ unless $x < 0$ by the definition of A) and $y = x + \epsilon/6 \leq 2 + 1/6 < 3$ (this is where we used the assumption $\epsilon < 1$). Furthermore, $x \leq t \leq y$, since t is the number determined by the cut (A, B). We have

$$t^2 - 2 \leq t^2 - x^2 = (t - x)(t + x) \leq (y - x)(y + x) < \frac{\epsilon}{6} \cdot 5 < \epsilon;$$

the third inequality holds because we have $y - x \leq \epsilon/6$, $x < 2$, and $y < 3$. Similarly,

$$2 - t^2 \leq y^2 - t^2 = (y - t)(y + t) \leq (y - x)(y + y) < \frac{\epsilon}{6} \cdot 6 = \epsilon.$$

These two inequalities together show that

$$|t^2 - 2| < \epsilon.$$

as claimed.

Since $\epsilon > 0$ was arbitrary, this inequality holds for every $\epsilon > 0$. Now, assume that $t^2 \neq 2$. Then the number $|t^2 - 2|$ is positive. Choosing $\epsilon = |t^2 - 2|$, the above inequality cannot hold. This is a contradiction, showing that we must have $t^2 = 2$.

Let S be a nonempty set of reals. A number t such that $t \geq x$ for every $x \in S$ is called an *upper bound* of S. The set A is called *bounded* from above if it has an upper bound. The number c that is an upper bound of S such that $c \leq t$ for every upper bound t of S is called the *least upper bound* or *supremum* of S. The supremum of S is denoted as $\sup S$.

Lemma. *Let S be a nonempty set S of reals that is bounded from above. Then S has a supremum.*

Proof. Let B be the set of upper bounds of S, and let A be the set of those reals that are not upper bounds of S (i.e., $A = \mathbb{R} \setminus B$). Then it is clear that (A, B) is a cut. Indeed, if we have $x \geq y$ and $y \in B$ for the reals x and y, then x is also an upper bound of S (since it is at greater than or equal to another upper bound, namely y). This shows that for every $x \in A$ and $y \in B$ we must have $x < y$. A is not empty since S is not empty, so not every real is an upper bound of S; B is not empty, since S is bounded from above by assumption.

Let t be the real determined by the cut (A, B). Then t is an upper bound of S. Assume, on the contrary, that there is an $s \in S$ is such that $s > t$. Let $u = (s + t)/2$. Then $u > t$, so we must have $u \in B$ (since the cut (A, B) determines t). Yet $u < s$, so u is not an upper bound of S, contradicting the relation $u \in B$.

Furthermore, t is the least upper bound of S. Assume, on the contrary, that y is an upper bound of S such that $y < t$. Then $y \in B$ (since B contains every upper bound of S). But then we must have $y \geq t$ (since the cut (A, B) determines t). This contradicts the relation $y < t$. The proof is complete.

For the empty set, one usually writes that $\sup \emptyset = -\infty$, and for a set S that is not bounded from above, one writes that $\sup S = +\infty$. With this extension, the symbol $\sup S$ will be meaningful for any subset of S reals.

Let S be a nonempty set of reals. A number t such that $t \leq x$ for every $x \in S$ is called a *lower bound* of S. The set A is called *bounded* from below if it has an lower bound. The number c that is an lower bound of S such that $c \geq t$ for every lower bound t of S is called the *greatest lower bound* or *infimum* of S. The infimum of S is denoted as $\inf S$. Every nonempty set that is bounded from below has an infimum. The proof of this statement is similar to the proof of the Lemma above. Instead of carrying out this proof, one can argue more simply that $\inf S = -\sup(-S)$, where $-S \stackrel{def}{=} \{-s : s \in S\}$. One usually writes $\inf \emptyset = +\infty$, and if S is not bounded from below, then one writes $\inf S = -\infty$.