# The Frenet–Serret formulas*

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## 1 The Frenet–Serret frame of a space curve

We will consider smooth curves given by a parametric equation in a three-dimensional space. That is, writing bold-face letters of vectors in three dimension, a curve is described as $\mathbf{r} = \mathbf{F}(t)$, where $\mathbf{F}'$ is continuous in some interval $I$; here the prime indicates derivative. The length of such a curve between parameter values $t_0 \in I$ and $t_1 \in I$ can be described as

$$\sigma(t_1) = \int_{t_0}^{t_1} |\mathbf{F}'(t)|\,dt = \int_{t_0}^{t_1} \left|\frac{d\mathbf{r}}{dt}\right| dt \tag{1}$$

where, for a vector $\mathbf{u}$ we denote by $|\mathbf{u}|$ its length; here we assume $t_0$ is fixed and $t_1$ is variable, so we only indicated the dependence of the arc length on $t_1$. Clearly, $\sigma$ is an increasing continuous function, so it has an inverse $\sigma^{-1}$; it is customary to write $s = \sigma(t)$. The equation

$$\mathbf{r} = \mathbf{F}(\sigma^{-1}(s)) \qquad s \in J \overset{def}{=} \{\sigma(t) : t \in I\} \tag{2}$$

is called the re-parametrization of the curve $\mathbf{r} = \mathbf{F}(t)$ $(t \in I)$ with respect to arc length. It is clear that $J$ is an interval. To simplify the description, we will always assume that $\mathbf{r} = \mathbf{F}(t)$ and $s = \sigma(t)$,

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so we will just use the variables $\mathbf{r}$, $t$, and $s$ instead of using function notation. We will use prime to indicate the differentiation $d/dt$, while the differentiation $d/ds$ will not be abbreviated. We will assume that $\mathbf{r}' \neq 0$ for any $t \in I$; then $\mathbf{r}'$ is a tangent vector to the curve corresponding to the given parameter value $t$.[1.1]

By the fundamental Theorem of Calculus, equation (1) implies

$$s' = \frac{ds}{dt} = \frac{d\sigma(t)}{dt} = |\mathbf{F}'(t)| = |\mathbf{r}'|. \tag{3}$$

Hence, by the chain rule of differentiation we have

$$\frac{d}{ds} = \frac{dt}{ds}\frac{d}{dt} = \left(\frac{1}{ds/dt}\right)\frac{d}{dt} = \frac{1}{|\mathbf{r}'|}\frac{d}{dt}, \tag{4}$$

where the second equation follows from equation (1) by the Fundamental Theorem of calculus. When using differential operator notation as in $d/ds$, everything after the differential operator up to the next $+$ or $-$ sign needs to be differentiated; the expression preceding the differential operator is not to be differentiated. The unit tangent vector $\mathbf{T}$ is defined as[1.2]

$$\mathbf{T} \overset{def}{=} \frac{1}{|\mathbf{r}'|}\mathbf{r}' = \frac{d\mathbf{r}}{ds}; \tag{5}$$

the second equation follows from equations (4). The curvature is defined as

$$\kappa \overset{def}{=} \left|\frac{d\mathbf{T}}{ds}\right|. \tag{6}$$

We will assume that $\kappa \neq 0$.[1.3] The unit normal vector is defined as

$$\mathbf{N} \overset{def}{=} \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}. \tag{7}$$

Note that $|\mathbf{T}|^2 = \mathbf{T} \cdot \mathbf{T}$, so by the product rule of differentiation,

$$\frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = \frac{1}{2}\left(\frac{d\mathbf{T}}{ds} \cdot \mathbf{T} + \mathbf{T} \cdot \frac{d\mathbf{T}}{ds}\right) = 0; \tag{8}$$

hence $\mathbf{T}$ is perpendicular to $d\mathbf{T}/ds$, and so $\mathbf{N}$ is perpendicular to $\mathbf{T}$. The unit binormal vector is defined as

$$\mathbf{B} \overset{def}{=} \mathbf{T} \times \mathbf{N}. \tag{9}$$

The vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ form the basic unit vectors of a coordinate system especially useful for describing the the local properties of the curve at the given point. These three vectors form what is called the *Frenet–Serret frame*. Equation (9) implies that the vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ form a right-handed system of pairwise perpendicular unit vectors. Any cyclic permutation of these vectors also form a right-handed system of pairwise perpendicular unit vectors; therefore we have

$$\begin{aligned} \mathbf{T} \times \mathbf{N} &= \mathbf{B}, \\ \mathbf{N} \times \mathbf{B} &= \mathbf{T}, \\ \mathbf{B} \times \mathbf{T} &= \mathbf{N}. \end{aligned} \tag{10}$$

[1.1] If the equation $\mathbf{r} = \mathbf{F}(t)$ describes a moving point, where $t$ is time, then $\mathbf{r}$ is the velocity vector of the moving point at time $t$. That is, the length of $\mathbf{r}'$ is its speed, while the direction of $r'$ is its direction of its movement. If $\mathbf{r}' = 0$, then the point stopped moving at the given time, and then it may resume its movement in a different direction. This means that even though the function $\mathbf{F}$ is differentiable, the tangent line to the curve described by the function may not be defined at this point.

[1.2] A unit vector is a vector of length 1.

[1.3] We will comment later on what happens when $\kappa \neq 0$.

## 2 The Frenet–Serret formulas

As $|\mathbf{N}| = 1$, we have $|\mathbf{N}|^2 = \mathbf{N} \cdot \mathbf{N} = 1$, and so, similarly to equation (8), we have

$$\frac{d\mathbf{N}}{ds} \cdot \mathbf{N} = 0.$$

That is $d\mathbf{N}/ds$ is perpendicular to $\mathbf{N}$, so we have

$$\frac{d\mathbf{N}}{ds} = \alpha\mathbf{T} + \tau\mathbf{B} \tag{11}$$

for some numbers $\alpha$ and $\tau$ (depending on $t$).[2.1] Here $\tau$ is called the torsion of the curve at the point; the value of $\alpha$ will be determined below. Using this equation, equations (10), and the product rule of differentiation for vector products, we have

$$\begin{aligned} \frac{d\mathbf{B}}{ds} &= \frac{d(\mathbf{T} \times \mathbf{N})}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} \\ &= \kappa\mathbf{N} \times \mathbf{N} + \mathbf{T} \times (\alpha\mathbf{T} + \tau\mathbf{B}) = \tau\mathbf{T} \times \mathbf{B} = -\tau\mathbf{N}; \end{aligned} \tag{12}$$

the third equation follows by equations (6) and (11). The first term of the third member[2.2] is zero; so is first term after distributing the cross product in the second term in the third member. Thus the fourth equation follows; to obtain the last equation we used equation (10). Hence, using equation (10), we have

$$\begin{aligned} \frac{d\mathbf{N}}{ds} &= \frac{d(\mathbf{B} \times \mathbf{T})}{ds} = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} \\ &= -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times (\kappa\mathbf{N}) = \tau\mathbf{B} + \kappa\mathbf{B} \times \mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}; \end{aligned} \tag{13}$$

This equation shows that $\alpha$ in equation (11) equals $-\kappa$; however, this fact and equation (11) itself is no longer of any interest, since this equation is subsumed in the last equation. Equations (7), (12), and (13) are called the *Frenet–Serret formulas.* To summarize these formulas, we have

$$\begin{aligned} \frac{d\mathbf{T}}{ds} &= \kappa\mathbf{N}, \\ \frac{d\mathbf{N}}{ds} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \frac{d\mathbf{B}}{ds} &= -\tau\mathbf{N}. \end{aligned} \tag{14}$$

## 3 The first three derivatives of r

As we mentioned above, we will indicate derivation with respect to $t$ by prime. According to equations (5) and (3), we have

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = s'\mathbf{T}. \tag{15}$$

[2.1] Every vector can be expressed as a linear combination of the basic unit vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$; as $d\mathbf{N}/ds$ is perpendicular to $\mathbf{N}$, the coefficient of $\mathbf{N}$ in this linear combination is 0.

[2.2] The member between the third and fourth equations

We can do further differentiations with respect to $t$ by using equations (14), (4), and (3). We have

$$\mathbf{r}'' = s''\mathbf{T} + s'\mathbf{T}' = s''\mathbf{T} + (s')^2\frac{d\mathbf{T}}{ds} = s''\mathbf{T} + (s')^2\kappa\mathbf{N}. \tag{16}$$

Further, repeatedly using equations (15) and (14), we have

$$\begin{aligned}\mathbf{r}''' &= s'''\mathbf{T} + s''\mathbf{T}' + \big(2s's''\kappa + (s')^2\kappa'\big)\mathbf{N} + (s')^2\kappa\mathbf{N}' \\ &= s'''\mathbf{T} + s''s'\frac{d\mathbf{T}}{ds} + \big(2s's''\kappa + (s')^2\kappa'\big)\mathbf{N} + (s')^3\kappa\frac{d\mathbf{N}}{ds} \\ &= s'''\mathbf{T} + s''s'\kappa\mathbf{N} + \big(2s's''\kappa + (s')^2\kappa'\big)\mathbf{N} + (s')^3\kappa(-\kappa\mathbf{T} + \tau\mathbf{B}) \\ &= \big(s''' - (s')^3\kappa\big)\mathbf{T} + \big(3s''s'\kappa + (s')^2\kappa'\big)\mathbf{N} + (s')^3\kappa\tau\mathbf{B}.\end{aligned} \tag{17}$$

It is now easy to express $\kappa$ and $\tau$ in terms of derivatives with respect to $t$. Equations (15), (16), and (10) give

$$\mathbf{r}' \times \mathbf{r}'' = (s')^3\kappa\mathbf{B}, \tag{18}$$

so using equation (3), we obtain that

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \tag{19}$$

by noting that $|\mathbf{B}| = 1$. By equations (18) and (17) we have

$$(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = (s')^6\kappa^2\tau.$$

Hence, using equation (18) and noting that $|\mathbf{B}| = 1$, we obtain that

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}. \tag{20}$$

# 4 Examples and discussion

Since the curvature $\kappa$ and the torsion $\tau$ are defined in terms of the local coordinate frame $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ in arc-length parametrization, they only depend on the shape of the curve and not on the choice of the coordinate system $x, y, z$ and the choice of the parameter $t$. Hence, for a curve that we want to calculate the curvature or the torsion, we may set up the coordinate system $x, y, z$ and choose a parametrization that make these calculations especially simple.

## 4.1 The curvature of a circle

We consider the circle of radius $R > 0$ lying in the $x, y$ plane, and centered at the origin. This circle can be parametrized by the equation

$$\mathbf{r} = R(\mathbf{i}\cos t + \mathbf{j}\sin t),$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit coordinate vectors in the directions of the positive $x$, $y$, and $z$ axes, respectively. We have

$$\mathbf{r}' = R(-\mathbf{i}\sin t + \mathbf{j}\cos t)$$

and

$$\mathbf{r}'' = -R(\mathbf{i}\cos t + \mathbf{j}\sin t).$$

Hence

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin t & R\cos t & 0 \\ -R\cos t & -R\sin t & 0 \end{vmatrix} = R^2(\sin^2 t + \cos^2 t)\,\mathbf{k} = R^2\,\mathbf{k}.$$

We also have

$$|\mathbf{r}'| = R\sqrt{\cos^2 t + \sin^2 t} = R.$$

Hence equation (19) gives that

$$\kappa = \frac{R^2}{R^3} = \frac{1}{R}.$$

Thus, the curvature of a circle is the reciprocal of the radius. For this reason, given any curve, $1/\kappa$ is called the radius of curvature; it is the radius of the *osculating circle*: given an curve by the equation $\mathbf{r} = \mathbf{F}(t)$, the osculating circle at a point corresponding to the parameter value $t = t_0$ (i.e., at the point with position vector $\mathbf{r}_0 = \mathbf{F}(t_0)$) is a circle with equation $\mathbf{r} = \mathbf{G}(t)$ at the for which

$$\mathbf{G}(t_0) = \mathbf{F}(t_0), \quad \mathbf{G}'(t_0) = \mathbf{F}'(t_0), \quad \text{and} \quad \mathbf{G}''(t_0) = \mathbf{F}''(t_0). \tag{21}$$

For the existence of such a circle, one needs to assume that $\mathbf{F}'(t_0) \neq 0$ and $\mathbf{F}''(t_0) \neq 0$. The case that $\mathbf{F}'(t_0) = 0$ is a case of bad parametrization, when the curve may or may not have a tangent line and a curvature, but the equation is not suitable for determining the tangent line or the curvature. In this case, one needs to re-parametrize the curve in such a way that the derivative at the point with position vector $\mathbf{F}(t_0)$ is not zero.[4.1] If $\mathbf{F}'(t_0) \neq 0$ but $\mathbf{F}''(t_0) = 0$, the curvature is 0, and the osculating circle degenerates into a straight line; in fact, the tangent line can be considered the osculating "circle" in this case, and one may say that the corresponding radius of curvature is infinite.

## 4.2 The curvature and the torsion of a helix

A helix in the standard position can be described by the equation

$$\mathbf{r} = \mathbf{i}\,R\cos t + \mathbf{j}\,R\sin t + ct\,\mathbf{k} \qquad (R > 0).$$

We have

$$\begin{aligned} \mathbf{r}' &= -\mathbf{i}\,R\sin t + \mathbf{j}\,R\cos t + c\,\mathbf{k}, \\ \mathbf{r}'' &= -\mathbf{i}\,R\cos t - \mathbf{j}\,R\sin t, \\ \mathbf{r}''' &= \quad \mathbf{i}\,R\sin t - \mathbf{j}\,R\cos t. \end{aligned}$$

Therefore

$$|\mathbf{r}'| = \sqrt{R^2(\sin^2 t + \cos^2 t) + c^2} = \sqrt{R^2 + c^2}.$$

Furthermore,

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin t & R\cos t & c \\ -R\cos t & -R\sin t & 0 \end{vmatrix} &= \mathbf{i}\,cR\sin t - \mathbf{j}\,cR\cos t + R^2(\sin^2 t + \cos^2 t)\,\mathbf{k} \\ &= \mathbf{i}\,cR\sin t - \mathbf{j}\,cR\cos t + R^2\,\mathbf{k}. \end{aligned}$$

[4.1] Such a parametrization may not exist. If there is a suitable re-parametrization, the re-parametrization with respect to arc length will work. It is, however, possible, that the derivative of the arc-length parametrization at the given point does not exist – in which case there is no smooth re-parametrization at the point.

Thus

$$|\mathbf{r}' \times \mathbf{r}''| = \sqrt{c^2R^2(\cos^2 t + \sin^2 t) + R^4} = R\sqrt{c^2 + R^2}$$

and

$$(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = R\sqrt{c^2 + R^2} = cR^2 \sin^2 t + cR^2 \cos^2 t = cR^2.$$

Hence, using formula (19) we obtain

$$\kappa = \frac{R\sqrt{c^2 + R^2}}{(R^2 + c^2)^{3/2}} = \frac{R}{R^2 + c^2}.$$

Similarly, using formula (20)

$$\tau = \frac{cR^2}{R^2(c^2 + R^2)} = \frac{c}{R^2 + c^2}.$$

An osculating curve to a given curve $\mathbf{r} = \mathbf{F}(t)$ is a curve $\mathbf{r} = \mathbf{G}(t)$ satisfying equations (21); these equations can also be described by saying that the curves $\mathbf{r} = \mathbf{F}(t)$ and $\mathbf{r} = \mathbf{G}(t)$ have a second order contact at the given point. One can generalize this to an arbitrary integer $n$ by saying that the curve $\mathbf{r} = \mathbf{F}(t)$ and $\mathbf{r} = \mathbf{F}(t)$ have an order $n$ contact for a given parameter value $t = t_0$ if $\mathbf{F}'(t_0) \neq 0$ and[4.2]

$$\mathbf{G}^{(k)}(t_0) = \mathbf{F}^{(k)}(t_0) \qquad \text{for all } k \text{ with } 0 \leq k \leq n. \tag{22}$$

The osculating plane is the plane spanned by the vectors $\mathbf{T}$ and $\mathbf{N}$. The osculating circle lies in this place. The osculating plane has a second order contact with the curve at the point given by a parameter value $t = t_0$. More generally, if $\mathbf{F}'(t_0) \neq 0$ and $\mathbf{F}''(t_0) \neq 0$, then any plane curve that has a second order contact with the curve $\mathbf{r} = \mathbf{F}(t)$ at the parameter value $t = t_0$ lies entirely in the osculating plane. If $\mathbf{F}''(t_0) = 0$ then the osculating plane is not determined since $\kappa = 0$ in this case so the vector $\mathbf{N}$ is not determined (cf. equations (19) and (7) to see this). The torsion expresses the speed with which the osculating place turns as the arc-length parameter changes (indeed, this follows from the third equation in (14), since $\mathbf{B}$ is normal to the osculating plane).

The derivation of the Frenet–Serret formulas (14) shows the theoretical usefulness of arc-length parametrization. Re-parametrizing a curve with respect to arc-length is rarely done in practice, since the integrals involved cannot usually be evaluated, and a more useful procedure is to rewrite the formulas derived with arc-length parametrization in terms of the original parameter, as was done in formulas (19) and (20).

[4.2]This condition just mentioned may be dropped if one is only interested in the curve as a moving points. If $F'(t_0) = 0$ then this condition expresses only that the points near the time $t = t_0$ have a "higher order closeness," but since both points stop moving at this time, this has no implication for the geometry of the two curves.