Problem 1. When 12 is divided into 75, the quotient is 6 and the remainder is 3. Write the equation this statement expresses.

Problem 2. What is the remainder when 11 is divided into $(1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 \times 13) + 1$. Do not do detailed calculations to give your answer.

Problem 3. Explain why the number $(1 \times 2 \times 3 \times 4 \times \ldots \times n) + 1$ cannot have a prime divisor less than or equal to n.

Problem 4. Is it possible for an extremely large prime to be expressed as a large integer raised to a very large power? Explain.

Problem 5. Let n be a positive integer, and let m be its smallest positive divisor that is different from 1. Show that m is a prime number.

Problem 6. Are there infinitely many natural numbers that are not prime? If so, explain why.

Problem 7. Explain why none of the numbers $(1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7) + 2$, $(1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7) + 3$, $(1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7) + 4$, $(1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7) + 5$, $(1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7) + 6$, $(1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7) + 7$ are prime, without doing detailed calculations.

Problem 8. Find all the prime numbers up to 200.

Problem 9. The clock now reads 7:00 o'clock. What time will it be 52 hours later (use a 12 hour clock, not a 24 hour clock).

Problem 10. Decide whether the number 0 41196 01012 1 is a correct Universal Product Number? (Note: if d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12} are the digits, from left to right, of a Universal Product Number, the congruence $3d_1 + d_2 + 3d_3 + d_4 + 3d_5 + d_6 + 3d_7 + d_8 + 3d_9 + d_{10} + 3d_{11} + d_{12} \equiv 0$ mod 10 must be satisfied.)

Problem 11. Describe when a positive integer p is called a prime.

Problem 12. Explain what a pair of twin primes are. Give three examples of pairs of twin primes.

Problem 13. Find 9^{74} mod 11.

Problem 14. Assume that
$$ac \equiv bc \mod p,$$
where a, b, c are integers, p is a prime and $p \nmid c$. Show that

$$a \equiv b \mod p.$$

(Cancelation Law for Congruences).

Problem 15. Explain, without actually doing the calculations, why the values 8×1 mod 11, 8×2 mod 11, 8×3 mod 11, 8×4 mod 11, 8×5 mod 11, 8×6 mod 11, 8×7 mod 11, 8×8 mod 11, 8×9 mod 11, 8×10 mod 11, are all different from one another? In particular, explain why we cannot have $8 \times 2 \equiv 8 \times 9$ mod 11. (Give a conceptual reason, and not one based on calculations; that is, saying that $16 \not\equiv 72$ mod 11 does not get at the real reason.)

Problem 16. Can you explain in a few sentences how the observation made in the preceding problem can be used to show that $8^{10} \equiv 1 \mod 11$.

Problem 17. Using the RSA Public Key Crypto-System, Bob's public key is the pair $(7, 143)$, and his secret key is $(103, 143)$. Answer the following questions: (i) Who is allowed to know the public key? (ii) Who is supposed to know the secret key? (iii) How does Alice send a message (represented by a positive integer W less than 143 that has no common factor with 143) to Bob? (iv) How does Bob read Alice's message? (v) How does Bob sign a message (represented by a positive integer W as in question (iii))? (vi) How does Alice know that the Bob's signature is valid (i.e., that the message was not sent by an impostor).

Problem 18. Show that if m is even then m^2 is divisible by 4.

Problem 19. Show that if we have $m^2 = 2n$ for integers m and n, then both m and n must be even.

Problem 20. Can the equation $m^2 = 2n^2$ hold for nonzero integers m and n?

Problem 21. Show that $\sqrt{2}$ is irrational.

Problem 22. Show that if m is divisible by 3 then m^2 is divisible by 9.

Problem 23. Show that if we have $m^2 = 3n$ for integers m and n, then both m and n must be divisible by 3.

Problem 24. Can the equation $m^2 = 3n^2$ hold for nonzero integers m and n?

Problem 25. Show that $\sqrt{3}$ is irrational.

Problem 26. Show that $\sqrt{6}$ is irrational.

Problem 27. Show that $\sqrt{2} + \sqrt{3}$ is irrational.

Problem 28. Let B be a number such that $3^B = 10$. Show that B is irrational.

Problem 29. Explain why the ratio of two integers must always be a repeating decimal. (We consider 3.2 a repeating decimal; namely $3.2 = 3.200000\ldots$, the digit 0 being repeated indefinitely.)

Problem 30. Explain why a repeating decimal must be rational. For example, find integers m and n such that $m/n = 2.124343434\overline{43}$ (the bar over the digits 43 means that the group of digits 43 is being repeated indefinitely; instead of $m/n = 2.124343434\overline{43}$ we could more simply have written $m/n = 2.12\overline{43}$).

Problem 31. Write the decimal expansion of an irrational number.

Problem 32. Explain why $0.99999\ldots$ (the 9 being repeated indefinitely) is equal to 1.

Problem 33. Describe a one-to-one correspondence between the set of all positive integers $\{1, 2, 3, 4, \ldots\}$ and the set $\{2, 3, 4, 5 \ldots\}$ of integers greater than 1.

Problem 34. Describe a one-to-one correspondence between the set of all positive integers $\{1, 2, 3, 4, \ldots\}$ and the set of all integers (positive, negative, and zero).

Problem 35. Describe a one-to-one correspondence between the set of all positive integers $\{1, 2, 3, 4, \ldots\}$ and the set of all rational numbers.

Problem 36. Assume f represents a one-to-one mapping from the integers to the real numbers as follows:

$$
\begin{aligned}
f(1) &= 268.3724962007\ldots \\
f(2) &= .9208426003\ldots \\
f(3) &= -37.3204270099\ldots \\
f(4) &= 0.0044287114\ldots \\
f(5) &= 2348.1222834920\ldots \\
f(6) &= -18.0855728420\ldots \\
f(7) &= -22.4223507242\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Explain why this mapping cannot list all real numbers by writing down a number that differs from these numbers on the diagonal.

Problem 37. Assume f represents a one-to-one mapping from the integers to infinite sequences of 0s and 1s as follows:

$$f(1) = 100110101101\ldots$$
$$f(2) = 110100110100\ldots$$
$$f(3) = 000110010110\ldots$$
$$f(4) = 011010001110\ldots$$
$$f(5) = 000111011010\ldots$$
$$f(6) = 010101101010\ldots$$
$$f(7) = 110010011101\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Explain why this mapping cannot list all sequences of 0s and 1s by writing down a sequence that differs from these sequences on the diagonal.

Problem 38. Assume f represents a one-to-one mapping from the integers to sets of integers as follows:

$$f(1) = \{\ 1,\ \ 3,\ \ 5, 11, 13, \ldots\}$$
$$f(2) = \{\ 1,\ \ 4,\ \ 5, 14, 22, \ldots\}$$
$$f(3) = \{\ 2,\ \ 3,\ \ 9, 10, 12, \ldots\}$$
$$f(4) = \{\ 1,\ \ 3,\ \ 6,\ \ 7, 12, \ldots\}$$
$$f(5) = \{\ 5,\ \ 6,\ \ 9, 10, 14, \ldots\}$$
$$f(6) = \{11, 22, 23, 25, 27, \ldots\}$$
$$f(7) = \{\ 3,\ \ 5,\ \ 6,\ \ 8, 12, \ldots\}$$
$$f(8) = \{\ 3,\ \ 6,\ \ 7,\ \ 8, 22, \ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Explain why this mapping cannot list all sets of integers by writing down a set that differs from these sets "on the diagonal."

Problem 39. Assume f represents a one-to-one mapping from the set $S = \{1, 2, 3, 4, 5\}$ to the set $\mathcal{P}(S)$ of all subsets of S as follows:

$$f(1) = \{1, 4, 5\}$$
$$f(2) = \{2, 3, 5\}$$
$$f(3) = \{1, 4\}$$
$$f(4) = \emptyset$$
$$f(5) = \{1, 2, 3, 4, 5\}$$

Explain why this mapping cannot list all subsets of S by writing down a set that differs from these sets "on the diagonal."

Problem 40. Let S be an arbitrary set. Show that the cardinality of the power set $\mathcal{P}(S)$ of S is larger than the cardinality of S.

Problem 41. Explain why talking about the set of all sets causes problems.

Problem 42. Describe Russell's paradox.

Problem 43. Explain how to map a line segment onto a line segment of a different length in a one-to-one way.

Problem 44. Explain how a finite line segment can be mapped onto the whole (infinite) line in a one-to-one way.

Problem 45. Explain how to map a square onto a line segment in a one-to-one way.

Problem 46. When trying to map the points of the square to the points of a line segment, the points of the square are represented by pairs (x, y), and the points of the line are represented by single numbers t. Explain why one would not get a one-to-one mapping of the a square onto a line segment by alternately taking a digit from x and y, and using these digits to build up the number t.

Problem 47. Prove the Pythagorean Theorem.

Problem 48. Explain how you find a spanning arc when starting out to triangulate a polygon (representing an art gallery).

Problem 49. Assume that an art gallery has n vertices.[2] Show that it is possible to place at most $n/3$ video cameras at some vertices of the art gallery so that every point of the gallery will be in the view of at least one camera.

Problem 50. Describe the five Platonic solids.

Problem 51. Describe how to obtain the dual of a Platonic solid. Describe the dual of each of the five Platonic solids.

Problem 52. How do you determine the number of edges of a Platonic solid.

Problem 53. Show that there can be no more than 5 regular solids.

Problem 54. Describe what a graph is; when a graph is called connected; and when a graph is called planar.

Problem 55. Describe Euler's formula connecting the number of vertices, faces, and edges in a connected planar graph.

Problem 56. Explain why Euler's formula connecting the number of vertices, faces, and edges in a connected planar graph is valid.

Problem 57. Explain why Euler's relation, $V + F - E = 2$ holds with V denoting the number of vertices in a Platonic solid, and F denoting the number of faces, and E denoting the number of edges in the same solid.

Problem 58. Describe what a pinwheel triangle is.

Problem 59. Draw a pinwheel tiling forming one supertile.

Problem 60. Explain why an interior triangle in a pinwheel tiling cannot be used as any other tile.

Problem 61. Explain what the pigeonhole principle is.

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[2]The plural of vertex.