

Hyperbolic functions*

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Contents

1	Hyperbolic functions	1
2	Connection with the unit hyperbola	2
2.1	Geometric meaning of the parameter	3
3	Complex exponentiation: connection with trigonometric functions	3
3.1	Trigonometric representation of complex numbers	4
3.2	Extending the exponential function to complex numbers.	5

1 Hyperbolic functions

The hyperbolic functions are defined as follows

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad \dots$$

The following formulas are easy to establish:

$$(1) \quad \cosh^2 x - \sinh^2 x = 1, \quad \operatorname{sech}^2 x = 1 - \tanh^2 x,$$
$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y;$$

the analogy with trigonometry is clear. The differentiation formulas also show a lot of similarity:

$$(\sinh x)' = \cosh x, \quad (\cosh x)' = \sinh x,$$
$$(\tanh x)' = \operatorname{sech}^2 x = 1 - \tanh^2 x, \quad (\operatorname{sech} x)' = -\tanh x \operatorname{sech} x.$$

Hyperbolic functions can be used instead of trigonometric substitutions to evaluate integrals with quadratic expressions under the square root. For example, to evaluate the integral

$$\int \frac{\sqrt{x^2 - 1}}{x^2} dx$$

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for $x > 0$, we can use the substitution $x = \cosh t$ with $t \geq 0$.¹ We have $\sqrt{x^2 - 1} = \sinh t$ and $dt = \sinh x dx$, and so

$$\int \frac{\sqrt{x^2 - 1}}{x^2} dx = \int \frac{\sinh t}{\cosh^2 t} \sinh t dt = \int \tanh^2 t dt = \int (1 - \operatorname{sech}^2 t) dt = t - \tanh t + C.$$

Finally, we need to write the result in term of x . Denoting the inverse of the function \cosh by arcosh by the requirement that $t = \operatorname{arcosh} x$ if $x = \cosh t$ and $t \geq 0$,² we can find t by solving the equation $x = \cosh t$, i.e., $x = (e^t + e^{-t})/2$ for t . Multiplying the latter equation by e^t , we obtain a quadratic equation for e^t :

$$(e^t)^2 - 2x(e^t) + 1 = 0.$$

Solving this equation for e^t , we obtain

$$e^t = x \pm \sqrt{x^2 - 1}.$$

The product of the two solutions is the constant term of the equation, i.e., 1; so one solution must be greater than 1 and the other must be less than 1. Clearly, the solution given by the + sign will be the one that is greater than 1; this solution will give a value of $t > 0$; the other solution will give $t < 0$. We want the former solution; that is, we have

$$(2) \quad t = \operatorname{arcosh} x = \ln \left(x + \sqrt{x^2 - 1} \right).$$

Further,

$$\tanh t = \frac{\sinh t}{\cosh t} = \frac{\sqrt{x^2 - 1}}{x}.$$

Substituting these into the result obtained for the above integral, we obtain that

$$\int \frac{\sqrt{x^2 - 1}}{x^2} dx = \ln \left(x + \sqrt{x^2 - 1} \right) - \frac{\sqrt{x^2 - 1}}{x} + C.$$

2 Connection with the unit hyperbola

Consider the parametric equations

$$(3) \quad x = \cosh t, \quad y = \sinh t$$

What curve do these equations describe? According to the first equation in (1), we have

$$x^2 - y^2 = 1.$$

This equation represents a hyperbola, called the unit hyperbola, with center at the origin, and with a horizontal real axis. The parametric equations represent only the right branch of the hyperbola, since $\cosh t > 0$ for all t . Every point on this right branch is represented by the parametric equations, since, if (x, y) is a point on this right branch, then we obtain this point with $t = \operatorname{arcosh} x$ if $y \geq 0$ and $t = -\operatorname{arcosh} x$ if $y < 0$ (see (2)).

¹Note that $\cosh t > 0$ for all t , so this substitution works only in case $x > 0$ (in which case, we need to assume that $x \geq 1$ so as not to have a negative number under the square root). The case $x < 0$ can be handled by the substitution $-x = \cosh t$. The stipulation $t \geq 0$ is used to determine t uniquely, given x ; note that $\cosh t = \cosh(-t)$.

²Note we have $\cosh t = \cosh(-t)$; thus t is not uniquely determined by the stipulation that $x = \cosh t$. For this reason, we required $t \geq 0$ above, so that t be uniquely determined by x . The prefix ar in arcosh is an abbreviation of area – see the next section where the connection of hyperbolic functions with the unit hyperbola is discussed. This is in contrast with the inverse trigonometric functions where the prefix arc is used, and it literally stands for an arc of the unit circle.

2.1 Geometric meaning of the parameter

The meaning the parameter t in equations (3) is half the area that segment OP' sweeps with the point O being the origin or the coordinate system, and P' being the point with coordinates $x' = \cosh t'$ and $y' = \sinh t'$ as t' changes between 0 and t , the area being positive if t (or, which is, the same, y) is positive, and it being negative if t (or y) is negative.³ In fact, this is not hard to show.

To this end, assume for simplicity that $t_0 > 0$.⁴ Using polar coordinates, let T be the polar angle of the point (x_0, y_0) determined equations (3) for the given t_0 . We will convert equations (3) to polar coordinates by using

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{x}{y};$$

the latter equation will give the correct value of θ only if $x > 0$, which is the case for the right branch of the hyperbola:

$$(4) \quad r = \sqrt{\cosh^2 t + \sinh^2 t}, \quad \theta = \arctan \tanh t.$$

Using the area formula in polar coordinates, the area of the hyperbola segment is

$$A = \frac{1}{2} \int_0^T r^2 d\theta,$$

where $r = f(\theta)$ is the equation describing the right branch of the hyperbola in polar coordinates. Fortunately, there is no need to determine the function f , nor there is a need to determine the upper limit T in this integral, since we can change to the variable t described in equations (4). We then have

$$d\theta = \frac{1}{1 + \tanh^2 t} \cdot \operatorname{sech}^2 t dt = \frac{1}{\sinh^2 t + \cosh^2 t} dt.$$

Hence we obtain

$$A = \frac{1}{2} \int_0^T (\cosh^2 t + \sinh^2 t) \cdot \frac{1}{\sinh^2 t + \cosh^2 t} dt = \frac{1}{2} \int_0^T dt = \frac{T}{2},$$

as we wanted to show.

3 Complex exponentiation: connection with trigonometric functions

There is a close connection between hyperbolic functions and trigonometric functions. The formulas, to be explained later, stating these connection connection, are

$$(5) \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2},$$

where i is the imaginary unit, that is, $i = \sqrt{-1}$. To make sense of these formulas, one needs to know what is to be meant by e^z when z is a complex number.

The usual way of extending e^z to complex numbers proceeds by the use of infinite series.⁵ This approach is technically efficient but has no intuitive appeal. After seeing this approach, one is still mystified by equations (5). Here we present a different approach.

³Note the analogy with trigonometric functions, where, for example, $\cos t$ represents the x -coordinate corresponding to an arc t in the unit circle. An arc t in the unit circle is, however, associated with a circular sector of area $t/2$, completing the analogy with hyperbolic functions.

⁴The argument below works without change even if $t_0 \leq 0$, but it is easier to visualize in case $t_0 > 0$.

⁵By considering the Taylor series of e^z , $\sin z$, and $\cos z$, and considering them valid also for complex values of z .

3.1 Trigonometric representation of complex numbers

A complex number is a number of the form $a + bi$, where a and b are real numbers. It is customary to graph this number on the coordinate plane as the point (a, b) , and calling the coordinate plane used to graph complex numbers the complex plane.⁶ One defines the absolute value of a complex number as its distance from the origin of the coordinate system, that is,

$$(6) \quad |a + bi| = \sqrt{a^2 + b^2},$$

and its argument as the angle between the positive x axis and the ray from the origin toward the point (a, b) , angles corresponding to counter-clockwise rotation of the positive x axis to the latter ray are considered positive. The argument of a point is only determined upto an additive multiple of 2π except for the number $0 = (0, 0)$, which can have any argument. For the argument of a number (a, b) we have

$$(7) \quad \arg(a + bi) = \arctan \frac{b}{a} + 2k\pi \quad \text{or} \quad \arg(a + bi) = \arctan \frac{b}{a} + (2k + 1)\pi$$

for some integer (positive, negative, or zero) k ; the first equation is valid if $a > 0$, and the second one if $a < 0$; unfortunately, this formula does not work in case $b = 0$.⁷ Writing $\rho = |(a, b)|$ and $\theta = \arg(a, b)$, it is clear that $a = \rho \cos \theta$ and $b = \rho \sin \theta$, and so

$$(8) \quad a + bi = \rho(\cos \theta + i \sin \theta).$$

The right-hand side is called the trigonometric form of the complex number on the left.

Let

$$z_1 = \rho_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = \rho_2(\cos \theta_2 + i \sin \theta_2)$$

be two complex numbers. Then it follows from the addition formulas for sin and cos that

$$(9) \quad z_1 z_2 = \rho_1 \rho_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

This means that

$$|z_1 z_2| = |z_1| |z_2| \quad \text{and} \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2.$$

These equations can also be easily verified directly, the latter by using geometric arguments, and then these equations can be used to prove the addition formulas for sin and cos.

From (9) one can easily establish de Moivre's formula

$$(10) \quad (\rho(\cos \theta + i \sin \theta))^n = \rho^n (\cos n\theta + i \sin n\theta).$$

⁶In fact, the complex number $a + bi$ is interpreted as the ordered pair (a, b) of real numbers, so one need not wonder about what mysterious meaning $\sqrt{-1}$ may have. Then one defines operations on pairs of numbers; for example, one puts

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc),$$

which corresponds to the multiplication formula

$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$

Addition, subtraction, and division can be defined in a similar way.

⁷The fact that the formula does not always work is an inconvenience in numerical calculations. To remedy this, several programming languages provide the function `atan2()`; this function gives a value in the range $[-\pi, \pi]$, and `atan2(a, b)` returns a value for any values of `a` and `b` (for example, the C programming language distinguishes between $+0$ and -0 , and in C this function is defined even if `a` or `b` or both are equal to ± 0). See e.g.

<http://en.wikipedia.org/wiki/Atan2>

for details.

3.2 Extending the exponential function to complex numbers.

We are about to extend the natural exponential function e^x to complex exponents. Occasionally, it will be convenient to use $\exp x$ or $\exp(x)$ instead of e^x , especially when the exponent x is replaced by a complicated expression. Our starting point will be the equation

$$(11) \quad \exp x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n,$$

valid for every real x , where n is running over positive integers. The easiest way to prove this is to establish this limit for a continuous real variable t with the aid of l'Hospital's rule:

$$(12) \quad \begin{aligned} \lim_{t \rightarrow +\infty} \left(1 + \frac{x}{t}\right)^t &= \lim_{t \rightarrow +\infty} \exp \left(\ln \left(1 + \frac{x}{t}\right)^t \right) = \lim_{t \rightarrow +\infty} \exp \left(t \ln \left(1 + \frac{x}{t}\right) \right) \\ &= \exp \left(\lim_{t \rightarrow +\infty} t \ln \left(1 + \frac{x}{t}\right) \right) = \exp \left(\lim_{t \rightarrow +\infty} \frac{\ln(1 + x/t)}{1/t} \right) \\ &= \exp \left(\lim_{t \rightarrow +\infty} \frac{\frac{-x/t^2}{1+x/t}}{-1/t^2} \right) = \exp \left(\lim_{t \rightarrow +\infty} \frac{x}{1 + x/t} \right) = \exp x; \end{aligned}$$

the interchange of the limit and the exponential function after the third equation is justified by the continuity of the exponential function, and the fifth equation is valid in view of l'Hospital's rule. We will extend the exponential function to complex numbers by requiring that equation (11) remains valid for complex values of x , that is, by requiring that

$$(13) \quad \exp z = \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n,$$

valid for every complex z . Various properties of the exponential function can be established with the aid of this equation; we will confine ourselves to establishing the equation

$$(14) \quad e^{ix} = \cos x + i \sin x,$$

called Euler's formula.

Given a fixed real number x , we have

$$(15) \quad 1 + \frac{ix}{n} = \sqrt{1 + \frac{x^2}{n^2}} \left(\cos \arctan \frac{x}{n} + i \sin \arctan \frac{x}{n} \right)$$

according to (6), (7), and (8). Hence, by de Moivre's formula (10) we have

$$(16) \quad \left(1 + \frac{ix}{n}\right)^n = \left(1 + \frac{x^2}{n^2}\right)^{n/2} \left(\cos \left(n \arctan \frac{x}{n}\right) + i \sin \left(n \arctan \frac{x}{n}\right) \right).$$

The limits as $n \rightarrow \infty$ of the expressions on the right-hand side are easy to establish by first changing n into a continuous variable t and then using l'Hospital's rule. In establishing the first of these limits, we proceed similarly to (12), so we will give fewer details. We have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \left(1 + \frac{x^2}{t^2}\right)^{t/2} &= \exp \left(\lim_{t \rightarrow +\infty} \frac{\ln(1 + x/t^2)}{2/t} \right) = \exp \left(\lim_{t \rightarrow +\infty} \frac{\frac{-2x/t^3}{1+x/t^2}}{-2/t^2} \right) \\ &= \exp \left(\lim_{t \rightarrow +\infty} \frac{x}{t(1 + x/t^2)} \right) = \exp 0 = 1; \end{aligned}$$

the first equation used the continuity of the exponential function, and the second one used l'Hospital's rule. As for the second limit,

$$\lim_{t \rightarrow +\infty} t \arctan \frac{x}{t} = \lim_{t \rightarrow +\infty} \frac{\arctan(x/t)}{1/t} = \lim_{t \rightarrow +\infty} \frac{\frac{-x/t^2}{1+x^2/t^2}}{-1/t^2} = \lim_{t \rightarrow +\infty} \frac{x}{1+x^2/t^2} = x;$$

the second equation here follows from l'Hospital's rule. Using these two limits and making $n \rightarrow \infty$ in (16), we obtain

$$\lim_{n \rightarrow \infty} \left(1 + \frac{ix}{n}\right)^n = \cos x + i \sin x.$$

Thus (14) follows in view of (13).

Writing $-x$ instead of x in (14), we obtain

$$\exp(-ix) = \cos(-x) + i \sin(-x) = \cos x - i \sin x.$$

Solving the system of equations given by this equation and by (14) for $\sin x$ and $\cos x$, we obtain equations (5).