Inner products*

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1 Inner product spaces

Let V be a vector space over F, where F is either the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. For a complex number α, $\bar{\alpha}$ will denote its conjugate.

Definition 1.1. An inner product is a mapping $\langle \cdot, \cdot \rangle : V \times V \to F$ such that

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(a) For all $x \in V$, $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ only if $x = 0$,

(b) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in V$,

(c) $\alpha\langle x, y\rangle = \langle x, \alpha y\rangle$ for all $\alpha \in F$ and $x, y \in V$,

(d) $\langle x, y\rangle + \langle x, z\rangle = \langle x, y + z\rangle$ for all $x, y, z \in V$.

A vector space with an inner product is called an *inner product space.*

In Clause (a), $\langle x, x\rangle \geq 0$ means that the complex number $\langle x, x\rangle$ is actually a nonnegative real. According to Clauses (b) and (c), we have $\alpha\langle x, y\rangle = \langle \bar{\alpha} x, y\rangle$. If $F = \mathbb{R}$, the complex conjugation has no effect. Inner product spaces are discussed in [1, Chapter 7, starting on p. 304]. Inner products are defined in [1, (7.1.5) Definition].

1.1 Hermitian transpose

Given an $m \times n$ matrix A, one obtains its *Hermitian transpose* A^*, named after the French mathematician Charles Hermite, by first taking its transpose A^T, then taking the complex conjugate of each entry of A^T. That is, if A is the matrix $A = (a_{ij})_{1\leq i\leq m,\ 1\leq j\leq n}$, then its Hermitian transpose is $A^* = (\bar{a}_{ij})_{1\leq j\leq n,\ 1\leq i\leq m}$; here the notation indicates that in A, i refers to rows and j, to columns, while in A^*, i refers to columns and j, to rows.[1.1] It is easy to see that if A and B are matrices such that the product AB is meaningful the

$$(AB)^* = B^* A^*, \tag{1.1}$$

similarly to the equation $(AB)^T = B^T A^T$. Further, note that if α is a scalar, then, considering α identical to the 1×1 matrix (α), we have $\alpha^* = \bar{\alpha}$. The Hermitian transpose is described in [1, (7.1.1) Definition, p. 304].

A matrix A is called *Hermitian* if $A^* = A$ (cf. [1, (7.2.2) Definition, p. 309]). Clearly, for this to happen, A must be a square matrix with real entries in its main diagonal. If B is a matrix with real entries, then B^* is its transpose B^T, since conjugation has no effect. A Hermitian matrix with real entries is called symmetric.

1.2 Inner products of column vectors

Given an integer $n > 0$, one can define inner products on the space of column vectors over $\mathbb{R}$ or $\mathbb{C}$ with n entries (i.e., $n \times 1$ matrices). The Hermitian transpose of such a column vector is a row vector with n entries. Denoting by $F_{m,n}$ the set of $m \times n$ matrices over the field F where F is $\mathbb{R}$ or $\mathbb{C}$, with $\mathbf{x}$ and $\mathbf{y}$ running over elements of $F_{n,1}$, it is easy to verify that $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^*\mathbf{y}$ is an inner product on the space $F_{n,1}$. if $\mathbf{x} = (x_1, x_2, \ldots x_n)^T$ and $\mathbf{y} = (y_1, y_2, \ldots y_n)^T$, then

$$\mathbf{x}^*\mathbf{y} = \sum_{k=1}^{n} \bar{x}_k y_k. \tag{1.2}$$

This inner product is called the canonical inner product on the space $F_{n,1}$ – cf. [1, (7.1.2) Definition]. We are going to show that, indeed, $\mathbf{x}^*\mathbf{y}$ satisfies Clause (a) of Definition 1.1. Indeed, writing

[1.1] As usual, the letter i denotes the imaginary unit $\sqrt{-1}$. However, the letter i will also be freely used in other cases when the context clearly indicates that i does not refer to the imaginary unit.

$\bar{x} = (x_1, x_2, \dots x_n)^T$, we have

$$\mathbf{x}^*\mathbf{x} = \sum_{k=1}^{n} \bar{x}_k x_k = \sum_{k=1}^{n} |x_k|^2 > 0 \qquad \text{unless} \quad \mathbf{x} = 0. \tag{1.3}$$

We will show that the other properties of this inner product by considering a more general case.

Let H be a Hermitian matrix, and let $\mathbf{x}, \mathbf{y} \in F_{n,1}$. Then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^* H \mathbf{y} \tag{1.4}$$

satisfies Clauses (b)–(d) in the definition of an inner product on $F_{n,1}$, but it does not necessarily satisfy Clause (a). The canonical inner product is the special case of this with $H = I$, the identity matrix. Indeed, to verify Clause (b) of Definition 1.1, note

$$\overline{\mathbf{y}^* H \mathbf{x}} = (\mathbf{y}^* H \mathbf{x})^* = \mathbf{x}^* H^* \mathbf{y}^{**} = \mathbf{x}^* H \mathbf{y};$$

here, the first equation holds since $\mathbf{y}^* H \mathbf{x}$ is a scalar, and, as remarked above, for a scalar α we have $\alpha^* = \bar{\alpha}$. Clauses (c) and (d) In order to ensure that the product defined in (1.4) also satisfy Clause (a) need another property of the matrix H:

Definition 1.2. Let $n > 0$ be an integer, and let $H \in \mathbb{C}_{n,n}$ be a Hermitian matrix. H is called *positive definite* if $\mathbf{x}^* H \mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{C}_{n,1}$. If only the inequality $\mathbf{x}^* H \mathbf{x} \geq 0$ is satisfied then H us called *positive semi-definite.*

Note that $\mathbf{x}^* H \mathbf{x}$ is always real in view of the fact that the product defined in (1.4) satisfies Clause (b) of Definition 1.1. If H is also positive definite, it also satisfies Clause (a), and so it it an inner product. Since the identity matrix I is positive definite according to (1.3), it follows that the canonical inner product defined in (1.2) is indeed an inner product. Positve definite Hermitian matrices are described in [1, (7.2.2) Definition, p. 309].

We will show that the inner product described in (1.4) for a positive definite Hermitian matrix H describe all inner products on the space $\mathbb{C}_{n,1}$ of column vectors. Indeed, let $\langle \cdot, \cdot \rangle$ be an arbitrary this space. Write $\mathbf{e}_j$ for the jth column vector on this space $(1 \leq j \leq n)$. That is $\mathbf{e}_j$ is the column vector all whose entries are 0 except its jth entry is one. Write

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

δ_{ij} is called Kronecker's delta. We have

$$\mathbf{e}_j = (\delta_{1j}, \delta_{2j}, \dots, \delta_{nj})^T. \tag{1.5}$$

Let H be the $n \times n$ matrix whose entry at place (i, j) is $\langle \mathbf{e}_i, \mathbf{e}_j \rangle$; it is easy to see that H is Hermitian. Indeed, we have $\langle \mathbf{e}_j, \mathbf{e}_i \rangle = \overline{\langle \mathbf{e}_i, \mathbf{e}_j \rangle}$ at place (j, i) in this matrix.

We will show that equation (1.4) is satisfied. Indeed, the entry in the (i, j) place of an arbitrary $n \times n$ matrix A is $\mathbf{e}_i^* A \mathbf{e}_j$.[1.2] Hence, we have

$$\mathbf{e}_i^* H \mathbf{e}_j = \langle \mathbf{e}_i, \mathbf{e}_j \rangle \tag{1.6}$$

[1.2] Indeed, noting that the entries of $\mathbf{e}_i$ are real, we have $\mathbf{e}_i^T = \mathbf{e}_i^T$. Furthermore, $\mathbf{e}_i^T A \mathbf{e}_j$ is the product of size $1 \times n$, $n \times n$, and $n \times 1$ matrices, and, writing $A = (a_{ij})$, its only entry is

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \delta_{ik} a_{kl} \delta_{l_j} = a_{ij};$$

the equality holds since the only nonzero term in the sum is obtained when $k = i$ and $l = j$.

by the definition of H. Writing I for the the $n \times n$ identity matrix and noting that

$$\sum_{i=1}^{n} \mathbf{e}_i \mathbf{e}_i^* = I, \tag{1.7}$$

we have

$$\mathbf{x}^* H \mathbf{y} = \mathbf{x}^* I H I \mathbf{y} = \mathbf{x}^* \Big(\sum_{i=1}^{n} \mathbf{e}_i \mathbf{e}_i^* \Big) H \Big(\sum_{j=1}^{n} \mathbf{e}_j \mathbf{e}_j^* \Big) \mathbf{y} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{x}^* \mathbf{e}_i (\mathbf{e}_i^* H \mathbf{e}_j) \mathbf{e}_j^* \mathbf{y},$$

where the last equation made use of the associativity of matrix product. Making use of equation (1.6) and noting that $\mathbf{x}^* \mathbf{e}_i$ and $\mathbf{e}_j^* \mathbf{y}$ are scalars, we obtain that the right-hand side equals

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{x}^* \mathbf{e}_i \langle \mathbf{e}_i, \mathbf{e}_j \rangle \mathbf{e}_j^* \mathbf{y} = \sum_{i=1}^{n} \sum_{j=1}^{n} \big\langle \mathbf{e}_i\, \overline{\mathbf{x}^* \mathbf{e}_i}, \mathbf{e}_j (\mathbf{e}_j^* \mathbf{y}) \big\rangle.$$

On the right-hand side we wrote the scalar on the of the vector in the both members of the inner product, since this will make the ensuing calculations simpler. We have $\overline{\mathbf{x}^* \mathbf{e}_i} = (\mathbf{x}^* \mathbf{e}_i)^* = \mathbf{e}_i^* \mathbf{x}$; hence, making use of the associativity of the matrix product again, we find that the right-hand side equals

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \mathbf{e}_i \mathbf{e}_i^* \mathbf{x}, \mathbf{e}_j \mathbf{e}_j^* \mathbf{y} \rangle &= \left\langle \sum_{i=1}^{n} \mathbf{e}_i \mathbf{e}_i^* \mathbf{x}, \sum_{j=1}^{n} \mathbf{e}_j \mathbf{e}_j^* \mathbf{y} \right\rangle \\ &= \left\langle \Big(\sum_{i=1}^{n} \mathbf{e}_i \mathbf{e}_i^* \Big) \mathbf{x}, \Big(\sum_{j=1}^{n} \mathbf{e}_j \mathbf{e}_j^* \Big) \mathbf{y} \right\rangle = \langle I \mathbf{x}, I \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle; \end{aligned}$$

the third equation here follows from (1.7). This shows that equation (1.4) also holds for the matrix H defined so as to satisfy (1.6). The fact that this matrix H is positive definite follows because the given inner product $\langle \cdot, \cdot \rangle$ satisfies Clause (a) of Definition 1.1.

1.3 Inner products over finite dimensional spaces

Let $n \geq 1$ be an integer, let F be the field $\mathbb{C}$ or $\mathbb{R}$, and U and V be n-dimensional vector spaces over F. Let $T : U \to V$ is a linear transformation from U onto V, and let $\langle \cdot, \cdot \rangle_U$ be an inner product on U. Then, it is easy to see that $\langle \cdot, \cdot \rangle_T$ defined as

$$\langle Tx, Ty \rangle_T \stackrel{def}{=} \langle x, y \rangle_U \qquad (x, y \in U) \tag{1.8}$$

is an inner product on V. In showing this, the only complication is to verify Clause (a) of Definition 1.1. For this, one needs to note that U and V having the same finite dimension, the assumption that T is onto implies that T is also one-to-one in view of the Rank-Nullity Theorem.

Considering a special case of this situation, let $n \geq 1$ be an integer, let F be the field $\mathbb{C}$ or $\mathbb{R}$, and V be an n-dimensional vector space over F, and let $\mathcal{X} = (x_1, x_2, \dots, x_n)$ be a basis of V. Given a vector $v \in V$, we have $v = \sum_{i=1}^{n} x_i \alpha_i$ for some $\alpha_i \in F$, where we deliberately wrote the scalars after the basis vectors. Thinking of $\mathcal{X}$ as a row vector and writing $\mathbf{a} = (\alpha_1, \alpha_2, \dots, \alpha_n)^T \in F_{n,1}$, it is natural to write $v = \mathcal{X}\mathbf{a}$. In this case, one also writes $\mathbf{a} = \mathcal{R}_{\mathcal{X}} v$. $\mathbf{a}$ is called the representation of v in the basis $\mathcal{X}$. It is easy to verify that $\mathcal{R}_{\mathcal{X}}$ is a linear transformation of V onto F. Indeed,

$\mathcal{R}_\mathcal{X}$ is the inverse of the transformation $\mathbf{a} \mapsto \mathcal{X}\mathbf{a}$, which is clearly linear, and the inverse of a linear transformation is also linear. Given an inner product $\langle \cdot, \cdot \rangle$ on V, the quantity $\langle \cdot, \cdot \rangle_{\mathcal{R}_\mathcal{X}}$ defined as

$$\langle \mathcal{R}_\mathcal{X} u, \mathcal{R}_\mathcal{X} v \rangle_{\mathcal{R}_\mathcal{X}} \overset{def}{=} \langle u, v \rangle \qquad (x, y \in V) \tag{1.9}$$

is an inner product on $F_{n,1}$ according to what we said on account of equation (1.8). As $\mathbf{e}_i = \mathcal{R}_\mathcal{X} x_i$, we have

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle_{\mathcal{R}_\mathcal{X}} = \langle x_i, x_j \rangle.$$

Thus, defining the matrix $H \in F_{n,n}$ as the matrix with entry $\langle x_i, x_j \rangle$ in the (i, j) place, we have

$$\langle u, v \rangle = (\mathcal{R}_\mathcal{X} u)^* H (\mathcal{R}_\mathcal{X} v), \tag{1.10}$$

since, as we showed above, given an inner product on $F_{n,1}$, for a matrix H satisfying (1.6), equation (1.4) holds. The matrix H satisfying equation (1.10) is called the representation of the inner product $\langle \cdot, \cdot \rangle$ in the basis $\mathcal{X}$, and we write

$$H = \mathcal{P}_\mathcal{X} \langle \cdot, \cdot \rangle. \tag{1.11}$$

Representation of inner products are described in [1, (7.2.8) Definition, p. 312].

1.4 Schwarz's inequality

Schwarz's inequality is given by the following

Theorem 1.1 (Schwarz's inequality). *Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$, and write $\langle \cdot, \cdot \rangle$ for the inner product; let $x, y \in V$. Then we have*

$$|\langle x, y \rangle|^2 \le \langle x, x \rangle \cdot \langle y, y \rangle. \tag{1.12}$$

We will first prove this result for real vector spaces, and then for complex vector spaces; of course, the proof for complex vector spaces handles also the real case, but treating the real case is simpler, and it gives the main insight of the proof. Finally, we will give an "efficient" proof that avoids any intuitive indication as to why it works, but it has the advantage of being short. These proofs are closely related to each other.

Proof for real inner products. Assume V is a vector space over $\mathbb{R}$. We may assume that $x \ne 0$, since otherwise $\langle x, y \rangle = 0$, so the inequality to be proved clearly holds. Then, we have $\langle x, x \rangle > 0$ according to Clause (a) of Definition 1.1. Let λ be an arbitrary real number. Then, again by the same Clause, the equation

$$\langle \lambda x + y, \lambda x + y \rangle) = 0$$

can hold only if $\lambda x + y = 0$. As $x \ne 0$, this equation can only hold for a single value of λ. Indeed, if $\lambda_1 x + y = \lambda_2 x + y = 0$. then $(\lambda_1 - \lambda_2)x = 0$; since $x \ne 0$, we then must have $\lambda_1 - \lambda_2 = 0$.

Now,

$$\langle \lambda x + y, \lambda x + y \rangle = \lambda^2 \langle x, x \rangle + 2\langle \lambda x, y \rangle + \langle y, y \rangle.$$

Considering

$$\lambda^2 \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle = 0$$

as a quadratic equation for λ with the various inner products as coefficients,[1.3] this equation has at most one real solution, Hence its discriminant cannot be positive. That is,

$$\big(2\langle x,y\rangle\big)^2 - 4\langle x,x\rangle\langle y,y\rangle \le 0.$$

Rearranging this, we obtain the inequality to be proved. $\square$

Proof for complex inner products. We will only consider the case when V is an inner product space over $\mathbb{C}$, since the proof for that case also works when V is an inner product space over $\mathbb{R}$, except that in this latter case complex conjugation has no effect. The proof is similar to the one given in the real case, except that taking complex inner products causes minor additional complications.

We may assume that $\langle x,y\rangle \ne 0$, since otherwise the inequality to be proved clearly holds; then we also have $x \ne 0$. Let λ be a complex number. Then, by Clause (a) of Definition 1.1 of inner product, we have

$$\langle \lambda x + y, \lambda x + y\rangle \ge 0,$$

and equation here holds only if $\lambda x + y = 0$. Since we assumed that $x \ne 0$, this equation can only hold for a single value of λ if at all. Hence

$$\begin{aligned} 0 \le \langle \lambda x + y, \lambda x + y\rangle &= \langle \lambda x, \lambda x\rangle + \langle \lambda x, y\rangle + \langle y, \lambda x\rangle + \langle y,y\rangle \\ &= \bar\lambda\lambda\langle x,x\rangle + \bar\lambda\langle x,y\rangle + \lambda\langle y,x\rangle + \langle y,y\rangle \\ &= |\lambda|^2\langle x,x\rangle + 2\Re\big(\lambda\overline{\langle x,y\rangle}\big) + \langle y,y\rangle; \end{aligned} \tag{1.13}$$

the third equation holds since $\bar\lambda\lambda = |\lambda|^2$, and, with $z = \lambda\langle y,x\rangle = \overline{\lambda\langle x,y\rangle}$, we have $\bar z = \bar\lambda\langle x,y\rangle$ according to Clause (b) of Definition 1.1, and $\bar z + z = 2\Re z$, where the $\Re z$ denotes the real part of z. Let

$$\lambda_0 = \frac{|\langle x,y\rangle|}{\overline{\langle x,y\rangle}},$$

and put $\lambda = \rho\lambda_0$, where ρ is an arbitrary real (recall that we assumed that $\langle x,y\rangle \ne 0$). Then $|\lambda_0| = 1$ and so $|\lambda|^2 = \rho^2$. Further, the expression

$$\overline{\lambda\langle x,y\rangle} = \rho\lambda_0\overline{\langle x,y\rangle} = \rho|\langle x,y\rangle|$$

is real, and so $\Re\big(\lambda\overline{\langle x,y\rangle}\big) = \rho|\langle x,y\rangle|$. Thus, inequality (1.13) becomes

$$\rho^2\langle x,x\rangle + 2\rho|\langle x,y\rangle| + \langle y,y\rangle \ge 0. \tag{1.14}$$

According to what we said about the former inequality, we have equality here for at most one real value of ρ.[1.4] Hence the equation

$$\rho^2\langle x,x\rangle + 2\rho|\langle x,y\rangle| + \langle y,y\rangle = 0.$$

is a quadratic equation for ρ with real coefficients (recall that $\langle x,x\rangle \ne 0$ by Clause (a) of Definition 1.1 of inner product, since $x \ne 0$). that has at most one real solution. Hence its discriminant cannot be positive. That is,

$$\big(2\langle x,y\rangle\big)^2 - 4\langle x,x\rangle\langle y,y\rangle \le 0.$$

Rearranging this, we obtain the inequality to be proved. $\square$

[1.3] This equation is a genuine quadratic equation, since $\langle x,x\rangle \ne 0$, that is, the coefficient of λ^2 is not zero, according to what we said above.

[1.4] Saying that ρ is real is important here, since this inequality does not even have to hold if ρ is not real. This inequality is a consequence of inequality (1.13) only for real ρ. This is because the equation $\Re\lambda\overline{\langle x,y\rangle} = \rho|\langle x,y\rangle|$ holds only for real ρ.

This solution can be greatly shortened by taking

$$\lambda = -\frac{|\langle x, y\rangle|^2}{\langle x, x\rangle\overline{\langle x, y\rangle}} = -\frac{\langle x, y\rangle\overline{\langle x, y\rangle}}{\langle x, x\rangle\overline{\langle x, y\rangle}} = -\frac{\langle x, y\rangle}{\langle x, x\rangle} \tag{1.15}$$

in inequality (1.13). Indeed, this choice corresponds to the choice

$$\rho = -\frac{|\langle x, y\rangle|}{\langle x, x\rangle},$$

which is the value of ρ for which the left-hand side of inequality (1.14) assumes its minimum. Such a shortening is, however, no real simplification, since it is achieved by skipping the explanation why this choice of λ is taken. This is the proof next, except that the numerator and the denominator of λ given in equation (1.15) is distributed between x and y to avoid the use of fractions.

Direct proof, no motivation. Let V be an inner product space over $\mathbb{C}$ or $\mathbb{R}$; in the latter case, complex conjugation has no effect. Let x, y be vectors in V; we may assume that $x \neq 0$, since otherwise inequality (1.12) clearly holds. By Clause (a) of Definition 1.1 we have

$$\begin{aligned} 0 &\leq \big\langle \langle x, y\rangle x - \langle x, x\rangle y, \langle x, y\rangle x - \langle x, x\rangle y \big\rangle \\ &= \overline{\langle x, y\rangle}\langle x, y\rangle\langle x, x\rangle - \overline{\langle x, y\rangle}\langle x, x\rangle\langle x, y\rangle - \langle x, x\rangle\langle x, y\rangle\langle y, x\rangle + \langle x, x\rangle^2\langle y, y\rangle \\ &= -|\langle x, y\rangle|^2\langle x, x\rangle + \langle x, x\rangle^2\langle y, y\rangle; \end{aligned}$$

we are about to explain why these equations hold. To obtain the first equation, first note that $\langle x, x\rangle$ is real according to Clause (a) of Definition 1.1. In taking the scalar factors out of the inner product, we used Clause (c) We also used the equation $\langle \alpha u, v\rangle = \bar{\alpha}\langle u, v\rangle$, which, as we explained after Definition 1.1, holds in view of Clauses (b) and (c). To obtain the second equation, we used Clause (b) and the observation that $\bar{\alpha}\alpha = |\alpha|^2$ for any complex number α; note that this resulted in some calcelation among the first three terms of the expression on the left of the second equality. Given that $\langle x, x\rangle > 0$ by Clause (a) in view of our assumption that $x \neq 0$, we can divide through by $\langle x, x\rangle$ the extreme sides of the last displayed formula to obtain inequality (1.12) (note that we are dividing an inequality by a positive number). □

1.5 Normed vector spaces

On a vector space V over F (with $F = \mathbb{C}$ or $\mathbb{R}$) one often defines a norm:

Definition 1.3. A *norm* is a mapping $\|\cdot\| : V \to \mathbb{R}$ such that

(a) $\|x\| \geq 0$ for all $x \in V$, and $\|x\| = 0$ only if $x = 0$,

(b) $\|\alpha x\| = |\alpha|\,\|x\|$ for all $\alpha \in F$ and for all $x \in V$,

(c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

A vector space with a norm is called a *normed vector space* or, more shortly, a *normed space.*

Clause (c) is called Minkowski's inequality. With an inner product $\langle \cdot, \cdot\rangle$ one can define the induced norm as

$$\|x\| = \sqrt{\langle x, x\rangle}. \tag{1.16}$$

If the norm is induced by an inner product, Minkowski's inequality can be proved by Schwarz's inequality;

Proof of Minkowski's inequality for induced norms. Let V be an inner product space over $\mathbb{C}$ or $\mathbb{R}$; if V is over $\mathbb{R}$, complex conjugation will have no effect. We have

$$\begin{aligned}(\|x\|+\|y\|)^2 &= \|x\|^2+2\|x\|\|y\|+\|y\|^2 \geq \langle x,x\rangle+2|\langle x,y\rangle|+\langle y,y\rangle\\ &\geq \langle x,x\rangle+2\Re(\langle x,y\rangle)+\langle y,y\rangle=\langle x,x\rangle+\langle x,y\rangle+\overline{\langle x,y\rangle}+\langle y,y\rangle\\ &= \langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle=\langle x+y,x+y\rangle=\|x+y\|^2;\end{aligned}$$

here the first inequality follows from Schwarz's inequality, given in Theorem 1.1, $\square$

In an inner product space, by the norm we will always mean the induced norm unless otherwise mentioned.

Two vectors x and y are called *orthogonal* if $\langle x,y\rangle=0$. Let x_1, x_2, ..., x_n be a system of vectors such that x_i and x_j are orthogonal whenever $1\leq i<j\leq n$. Then

$$\left\|\sum_{k=1}^{n}x_k\right\|^2=\sum_{k=1}^{n}\|x_k\|^2. \tag{1.17}$$

Indeed, we have

$$\left\|\sum_{k=1}^{n}x_k\right\|^2=\left\langle\sum_{k=1}^{n}x_k,\sum_{l=1}^{n}x_l\right\rangle=\sum_{k=1}^{n}\sum_{l=1}^{n}\langle x_k,x_l\rangle=\sum_{k=1}^{n}\|x_k\|^2;$$

the last equation holds since $\langle x_k,x_l\rangle=0$ unless $k=l$. The equation we just established can be considered an analog of the Pythagorean theorem.

2 Diagonalization of Hermitian matrices

2.1 Eigenvalues of a Hermitian matrix are real

Hermitian matrices were defined in Subsection 1.1. Inner products of column vectors were described in Subsection 1.2; the canonical inner product was defined in (1.2), and inner products associated with a Hermitian matrix H were given in (1.4). One important property of Hermitian matrices is that all its eigenvalues are real. In fact, if H is Hermitian and λ is one of its eigenvalues with eigenvector $\mathbf{v}$, then we have

$$\overline{\mathbf{v}^*H\mathbf{v}}=(\mathbf{v}^*H\mathbf{v})^*=\mathbf{v}^*H^*\mathbf{v}^{**}=\mathbf{v}^*H\mathbf{v}=\mathbf{v}^*\lambda\mathbf{v}=\lambda\mathbf{v}^*\mathbf{v}.$$

The first two equations shows that $\mathbf{v}^*H\mathbf{v}$ is real. The product $\mathbf{v}^*\mathbf{v}$ is a nonzero real according to equation (1.3). Hence it follows that λ is real. If H is also a real matrix (a Hermitian matrix with real entries is called a *symmetric matrix*), then the eigenvector $\mathbf{v}$ can also be chosen to be real. This is because the matrix $H-\lambda I$ is then singular, and so its columns are linearly dependent over the field of reals; so there is a real nonzero vector $\mathbf{v}$ such that $(H-\lambda I)\mathbf{v}=\mathbf{0}$ (the column vector on the left is a linear combination of the columns of the matrix $H-\lambda I$, the entries of $\mathbf{v}$ being the coefficients in this linear combination).

2.2 Unitary matrices

A square matrix over the complex numbers is called unitary if $U^*U = I$. If so, U^* is a left inverse of U; since a left inverse of a matrix is also its right inverse, we have also $UU^* = I$. Hence, if U is unitary, then U is invertible, and $U^{-1} = U^*$ is also unitary. Furthermore, If U and V are unitary matrices, then UV is also a unitary matrix. Indeed, we have

$$(UV)^*(UV) = (V^*U^*)(UV) = V^*(U^*U)V = V^*IV = V^*V = I.$$

2.3 Householder matrices

If $\mathbf{v}$ is a complex column vector with $\mathbf{v}^*\mathbf{v} = 1$, then the $n \times n$ matrix $H_{\mathbf{v}} = I - 2\mathbf{v}\mathbf{v}^*$ is called a Householder matrix. A Householder matrix is Hermitian and unitary. Indeed, given a Household matrix $H = H_{\mathbf{v}}$, we have

$$H^* = I^* - 2(\mathbf{v}\mathbf{v}^*)^* = I - 2(\mathbf{v}^*)^*\mathbf{v}^* = I - 2\mathbf{v}\mathbf{v}^* = H,$$

showing that H is Hermitian. Furthermore,

$$\begin{aligned} H^*H = HH = (I - 2\mathbf{v}\mathbf{v}^*)(I - 2\mathbf{v}\mathbf{v}^*) &= I - 2 \cdot 2\mathbf{v}\mathbf{v}^* + 4\mathbf{v}\mathbf{v}^*\mathbf{v}\mathbf{v}^* \\ &= I - 4\mathbf{v}\mathbf{v}^* + 4\mathbf{v}(\mathbf{v}^*\mathbf{v})\mathbf{v}^* = I - 4\mathbf{v}\mathbf{v}^* + 4\mathbf{v}\mathbf{v}^* = I; \end{aligned}$$

the parenthesis in the third term of the fifth member can be placed anywhere since multiplication of matrices (and vectors) is associative, and the fifth equation holds since $\mathbf{v}^*\mathbf{v} = 1$.

To simplify the discussion below, we will often write $\|\mathbf{x}\|$ for $\sqrt{\mathbf{x}^*\mathbf{x}}$; this is the induced norm associated with the canonical inner product described in (1.2); see (1.16) Given two vectors $\mathbf{x}$ and $\mathbf{y}$ with $\|\mathbf{x}\| = \|\mathbf{y}\|$, $\mathbf{x} \ne \mathbf{y}$, and such that $\mathbf{x}^*\mathbf{y}$ (which is a scalar) is real, there is a Householder transformation that maps $\mathbf{x}$ to $\mathbf{y}$ and $\mathbf{y}$ to $\mathbf{x}$. Indeed, if one takes

$$\mathbf{v} = \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|}, \tag{2.1}$$

then $H_{\mathbf{v}}$ maps $\mathbf{x}$ to $\mathbf{y}$. Indeed,

$$\begin{aligned} H_{\mathbf{v}}\mathbf{x} &= \left(I - 2\frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|} \cdot \frac{(\mathbf{x} - \mathbf{y})^*}{\|\mathbf{x} - \mathbf{y}\|}\right)\mathbf{x} = \left(I - 2\frac{(\mathbf{x} - \mathbf{y})(\mathbf{x}^* - \mathbf{y}^*)}{(\mathbf{x} - \mathbf{y})^*(\mathbf{x} - \mathbf{y})}\right)\mathbf{x} \\ &= \left(I - 2\frac{(\mathbf{x} - \mathbf{y})(\mathbf{x}^* - \mathbf{y}^*)}{\mathbf{x}^*\mathbf{x} - \mathbf{x}^*\mathbf{y} - \mathbf{y}^*\mathbf{x} + \mathbf{y}^*\mathbf{y}}\right)\mathbf{x} = \mathbf{x} - 2\frac{(\mathbf{x} - \mathbf{y})(\mathbf{x}^*\mathbf{x} - \mathbf{y}^*\mathbf{x})}{\mathbf{x}^*\mathbf{x} - \mathbf{x}^*\mathbf{y} - \mathbf{y}^*\mathbf{x} + \mathbf{y}^*\mathbf{y}}. \end{aligned}$$

In the denominator on the right-hand side we have $\mathbf{x}^*\mathbf{x} = \|\mathbf{x}\|^2 = \|\mathbf{y}\|^2 = \mathbf{y}^*\mathbf{y}$ in view of our assumption; further, our assumption that $\mathbf{x}^*\mathbf{y}$ is a real number implies that it equals its own Hermitian transpose, we have $\mathbf{x}^*\mathbf{y} = (\mathbf{x}^*\mathbf{y})^* = \mathbf{y}^*\mathbf{x}$; hence the denominator on the right equals $2\mathbf{x}^*\mathbf{x} - 2\mathbf{y}^*\mathbf{x}$. Thus, the right-hand side equals

$$\mathbf{x} - (\mathbf{x} - \mathbf{y}) = \mathbf{y},$$

showing that $H_{\mathbf{v}}\mathbf{x} = \mathbf{y}$, as we wanted to show.

Note that both requirements on $\mathbf{x}$ and $\mathbf{y}$ are essential. Indeed, given that H is Hermitian, if $\mathbf{y} = H\mathbf{x}$, we have

$$\mathbf{x}^*\mathbf{y} = \mathbf{x}^*H\mathbf{x} = (\mathbf{x}^*H^*\mathbf{x})^* = (\mathbf{x}^*H\mathbf{x})^*;$$

the last equation holds since H is Hermitian. Further,

$$\|\mathbf{y}\|^2 = y^*y = \mathbf{x}^*H^*H\mathbf{x} = \mathbf{x}^*\mathbf{x} = \|\mathbf{x}\|^2;$$

the third equation holds since $H^*H = I$ (i.e., that H is unitary).

Two square matrices A and B of the same size are called *unitarily equivalent* if there is a unitary matrix U such that $B = U^*AU$. A square matrix is called *upper triangular* if all its entries under the main diagonal are zero. It is called *lower triangular* if all its entries above the main diagonal are zero. It is called *diagonal* if all its entries outside the main diagonal are zero. We will prove the following

Theorem 2.1. *Every $n \times n$ matrix with complex entries is unitarily equivalent to an upper triangular matrix.*

Proof. Let $n \geq 1$ be an integer, and for i with $1 \leq i \leq n$ let $\mathbf{e}_i$ the ith n-dimensional unit column vector; that is all entries of $\mathbf{e}_i$ are 0 except for the ith entry, which is 1 – see (1.5). We will use induction on n; so assume that the assertion is true for square matrices of size smaller than $n \times n$. Let A be an $n \times n$ matrix, λ an eigenvalue of A, and $\mathbf{u}$ the corresponding eigenvector. Given any nonzero $\alpha \in \mathbb{C}$, the vector $\alpha\mathbf{u}$ is also an eigenvector for λ; so we may assume that assume that $\mathbf{u}^*\mathbf{u} = 1$ and that $\mathbf{u}^*\mathbf{e}_1$ is real. If $\mathbf{u} \neq \mathbf{e}_1$, then there is a Householder matrix H such that $H\mathbf{e}_1 = \mathbf{u}$; if $\mathbf{v} = \mathbf{e}_1$ then take $H = I$, in which case we still have $H\mathbf{e}_1 = \mathbf{u}$. We have $H^* = H^{-1}$, since H is unitary.[2.1] Hence

$$H^*AH\mathbf{e}_1 = H^*A\mathbf{u} = H^*\lambda\mathbf{u} = \lambda H^{-1}\mathbf{u} = \lambda\mathbf{e}_1,$$

so λ is an eigenvalue with eigenvector $\mathbf{e}_1$ of the matrix $B = H^*AH$.

For the the matrix $B = (b_{ij})$ we have

$$b_{i1} = \mathbf{e}_i^*B\mathbf{e}_1 = \mathbf{e}_i^*\lambda\mathbf{e}_1 = \lambda\mathbf{e}_i^*\mathbf{e}_1 = \lambda\delta_{i1}.$$

Therefore, B can be written as a block matrix

$$B = H^*AH = \begin{pmatrix} \lambda & \mathbf{b}^* \\ \mathbf{0} & B_1 \end{pmatrix},$$

where $\mathbf{0}$ is an $(n-1)$-dimensional zero column vector, $\mathbf{b}^*$ is the $(n-1)$-dimensional row vector consisting of the entries of the first row of B with the exception the first entry, and B_1 is an $(n-1) \times (n-1)$ matrix. By induction $C_1 = U_1^*B_1U_1$, where U_1 is a unitary matrix, and C_1 is an upper triangular matrix Then, defining the matrix C by the next equation, we have[2.2]

$$\begin{aligned}
C &= \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1 \end{pmatrix}^* H^*AH \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1^* \end{pmatrix} H^*AH \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1^* \end{pmatrix} B \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1^* \end{pmatrix} \begin{pmatrix} \lambda & \mathbf{b}^* \\ \mathbf{0} & B_1 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1^* \end{pmatrix} \begin{pmatrix} \lambda & \mathbf{b}^*U_1 \\ \mathbf{0} & B_1U_1 \end{pmatrix} = \begin{pmatrix} \lambda & \mathbf{b}^* \\ \mathbf{0} & U_1^*B_1U_1 \end{pmatrix} = \begin{pmatrix} \lambda & \mathbf{b}^*U_1 \\ \mathbf{0} & C_1 \end{pmatrix}.
\end{aligned}$$

[2.1] This is true whether H is a Householder matrix or $H = I$. In case $\mathbf{u} = \mathbf{e}_1$, nothing needs to be changed, and the choice $H = I$ made in this case will not change the matrix A. If one uses the method described here as a numerical algorithm, and $\mathbf{u}^*\mathbf{u} = 1$, and $\mathbf{u}^*\mathbf{e}_1$ is real, then instead of picking H as described, one would choose between the matrices H_1 and H_2 such that $H_1\mathbf{e}_1 = \mathbf{u}$; or $H_1\mathbf{e}_1 = -\mathbf{u}$, making the first choice if $\|\mathbf{e}_1 - \mathbf{u}\| \geq \|\mathbf{e}_1 + \mathbf{u}\|$, and the second one otherwise. With $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{y} = \mathbf{u}$ or $-\mathbf{u}$, this choice will maximize the denominator in equation (2.1) so as to minimize the numerical error.

[2.2] In forming the products of these block matrices it is important to ascertain that their partitionings are multiplicatively conformable; that is, if two submatrices need to be multiplied, the number of columns of the matrix on the left agrees with the number of rows of the matrix on the right. For this, the scalar on the top left (1 or λ) needs to be counted as a 1×1 matrix, and not as a scalar that can be multiplied by any matrix.

Since C_1 is an upper triangular matrix, we can see that C is also an upper triangular matrix. Observe that the matrix $\begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1 \end{pmatrix}$ is a unitary. Indeed, we have

$$\begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1 \end{pmatrix}^* \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1^* \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & U_1^* U_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & I_{n-1} \end{pmatrix} = I_n,$$

where I_k denotes the unit $k \times k$ unit matrix. Noting that the product of two unitary matrices is also unitary, as we remarked in Subsection 2.2. it follows that the diagonal matrix C is unitarily equivalent to A, the assertion of the theorem follows. □

A consequence of this is the following

Corollary 2.1. *Every $n \times n$ matrix with complex entries is unitarily equivalent to a lower triangular matrix.*

Proof. Let A be an $n \times n$ matrix with complex entries. According to Theorem 2.1, A^* is unitarily equivalent to an upper triangular matrix; i.e., for some unitary matrix U, the matrix $U^* A^* U$ is upper triangular. Hence its Hermitian transpose

$$(U^* A^* U)^* = U^* A^{**} U^{**} = U^* A U$$

is lower triangular. □

Corollary 2.2. *Every Hermitian matrix matrix is unitarily equivalent to a real diagonal matrix.*

Proof. Given a Hermitian matrix A, by the theorem we just proved there is a unitary matrix U^* such that $U^* A U$ is upper triangular. Since we have

$$(U^* A U)^* = U^* A^* U^{**} = U^* A U,$$

we can see that $U^* A U$ is Hermitian. Hence it must be a diagonal matrix with real entries. □

Theorem 2.1 is given as [1, (7.4.3) Theorem, p. 327]. Corollary 2.2 is given as [1, (7.4.4) Corollary, p. 331].

2.4 Gram–Schmidt orthogonalization

Let $n \geq 1$ be an integer, and let V be a real or complex n-dimensional inner product space. A system if vectors let $\mathcal{U} = (u_1, u_2, \ldots, u_n)$ of vectors in V is called *orthonormal* if $\langle u_i, u_j \rangle = \delta_{ij}$ for i, j with $1 \leq i, j \leq n$. Let $\mathcal{X} = (x_1, x_2, \ldots, x_n)$ be a basis. Then the following algorithm allows us to create an orthonormal basis $\mathcal{Y} = (y_1, y_2, \ldots, y_n)$ of V. Writing $\langle \cdot, \cdot \rangle$ for the inner product and $\| \cdot \|$ for the induced norm, for k with $1 \leq k \leq n$, assume that y_i for i with $1 \leq i < k$ has already been defined, let

$$z_k = x_k - \sum_{i=1}^{k-1} \langle y_i, x_k \rangle y_i; \tag{2.2}$$

for $k = 1$, the sum is empty, hence the equation gives $z_1 = x_1$. Assuming $z_k \neq 0$, put

$$y_k = \frac{1}{\|z_k\|} z_k; \tag{2.3}$$

if $z_k = 0$, the process cannot be continued.

Observe that, in fact $z_k = 0$ cannot happen. Indeed, if we were able to construct y_i for all i with $1 \le i < k$, then these equations show that y_i for such an i is in the span of the vectors $(x_j : 1 \le j < k)$. As $\mathcal{X}$ is linearly independent, the first of these equations show that we cannot have $z_k = 0$. Hence, there is no impediment in the construction of $\mathcal{Y}$.

Equation (2.3) shows that $\|y_k\| = 1$ for all k with $1 \le k \le n$. The show that $\mathcal{Y}$ is orthogonal, let k be an integer with $1 \le k \le n$, and assume that $\langle y_i, y_j \rangle = \delta_{ij}$ note that for any j and k with $1 \le j < k \le n$ we have

$$\begin{aligned}\langle y_j, y_k \rangle &= \frac{1}{\|z_k\|}\langle y_j, z_k \rangle = \frac{1}{\|z_k\|}\Big\langle y_j, x_k - \sum_{i=1}^{k-1} \langle y_i, x_k \rangle y_i \Big\rangle \\ &= \frac{1}{\|z_k\|}\Big(\langle y_j, x_k \rangle - \sum_{i=1}^{k-1} \langle y_i, x_k \rangle \langle y_j, y_i \rangle \Big) = \frac{1}{\|z_k\|}\Big(\langle y_j, x_k \rangle - \sum_{i=1}^{k-1} \langle y_i, x_k \rangle \delta_{ji} \Big) \\ &= \frac{1}{\|z_k\|}\Big(\langle y_j, x_k \rangle - \langle y_j, x_k \rangle \Big) = 0,\end{aligned}$$

as we wanted to show.

Observe the following about the method described above. According to equations (2.2) and (2.3), y_k is a linear combination of the vectors x_1, x_2, ..., x_k. Hence, $\mathcal{Y} = \mathcal{X}P$ for an $n \times n$ matrix $P = (p_{ij})$, then all entries p_{ij} are zero with $i > j$. That is P is an upper triangular matrix. None of the diagonal elements of P can be zero, since P is nonsingular. The simplest way to see this is that $\det P$ is the product of its diagonal elements, and $\det P \ne 0$ since P is nonsingular.[2.3]

Gram–Schmidt orthogonalization is discussed in [1, (7.3.2) Theorem, p. 318].

3 Hermitian forms

Let V be a finite-dimensional vector space over F, where F is either the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$.

Definition 3.1. A Hermitian form is a mapping $\langle \cdot, \cdot \rangle : V \times V \to F$ such that that satisfies Clauses (b) (c) and (d); that is, it satisfies

(b) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$,

(c) $\alpha \langle x, y \rangle = \langle x, \alpha y \rangle$ for all $\alpha \in F$ and $x, y \in V$,

(d) $\langle x, y \rangle + \langle x, z \rangle = \langle x, y + z \rangle$ for all $x, y, z \in V$.

3.1 Representation of Hermitian forms

Let $\mathcal{X} = (x_1, x_2, \ldots, x_n)$ be the basis of V (that is $n = \dim V > 0$), and let $\langle \cdot, \cdot \rangle$ be a Hermitian form on V. Let H be an $n \times n$ matrix whose entry at the (i, j) place is $\langle x_i, x_j \rangle$; that is,

$$\mathbf{e}_i^* H \mathbf{e}_i = \langle x_i, x_j \rangle; \tag{3.1}$$

[2.3] It is also easy to see from equations (2.2) and (2.3) that $p_{ii} \ne 0$ for all i with $1 \le i \le n$. It is not hard to show that the inverse of an upper triangular matrix is also upper triangular; hence $\mathcal{X} = \mathcal{Y}P^{-1}$ with an upper triangular matrix P^{-1}.

cf. (1.6); H is Hermitian, since $\langle x_j, x_i\rangle = \overline{\langle x_i, x_j\rangle}$ according to Clause (b) of Definition 3.1. Then, for arbitrary $u, v \in V$ we have

$$\langle u, v\rangle = (\mathcal{R}_{\mathcal{X}}u)^* H \mathcal{R}_{\mathcal{X}}v. \tag{3.2}$$

This can be established in a way similar to the establishing of equation (1.10); here, we derive this equation directly:

Writing

$$u = \sum_{i=1}^{n} \alpha_i x_i \qquad \text{and} \qquad v = \sum_{j=1}^{n} \beta_j y_j,$$

we have

$$\mathcal{R}_{\mathcal{X}}u = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \qquad \text{and} \qquad \mathcal{R}_{\mathcal{X}}v = (\beta_1, \beta_2, \ldots, \beta_n)^T.$$

Further

$$\begin{aligned}\langle u, v\rangle &= \Big\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{n} \beta_j y_j \Big\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{\alpha}_i \beta_j \langle x_i, y_j\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{\alpha}_i \beta_j \mathbf{e}_i^* H \mathbf{e}_j \\ &= \Big(\sum_{i=1}^{n} \alpha_i \mathbf{e}_i\Big)^* H \sum_{j=1}^{n} \beta_j \mathbf{e}_j = (\mathcal{R}_{\mathcal{X}}u)^* H \mathcal{R}_{\mathcal{X}}v,\end{aligned}$$

as we wanted to show.

Similarly as in equation (1.11), the matrix H defined in equation (3.1) is called the representation of the inner product $\langle\cdot,\cdot\rangle$ in the basis $\mathcal{X}$, and we write

$$H = \mathcal{P}_{\mathcal{X}}\langle\cdot,\cdot\rangle. \tag{3.3}$$

3.2 Change of bases

Write n for the dimension of V. Let $\mathcal{X}$ and $\mathcal{Y}$ be bases of the space V, and assume $\mathcal{Y} = \mathcal{X}P$ for a nonsingular $n \times n$ matrix P. Then, for $\mathbf{a} = \mathcal{R}_{\mathcal{Y}}u \in F_{n,1}$ we have $u = \mathcal{Y}\mathbf{a} = \mathcal{X}P\mathbf{a}$, and so $\mathcal{R}_{\mathcal{X}}u = P\mathbf{a} = P\mathcal{R}_{\mathcal{Y}}u$. Hence, for the H in (3.3) we have

$$\langle u, v\rangle = (\mathcal{R}_{\mathcal{X}}u)^* H \mathcal{R}_{\mathcal{X}}v = (P\mathcal{R}_{\mathcal{Y}}u)^* HP\mathcal{R}_{\mathcal{Y}}v = (\mathcal{R}_{\mathcal{Y}}u)^*(P^*HP)\mathcal{R}_{\mathcal{Y}}v.$$

Hence

$$\mathcal{P}_{\mathcal{Y}}\langle\cdot,\cdot\rangle = P^*\big(\mathcal{P}_{\mathcal{X}}\langle\cdot,\cdot\rangle\big)P. \tag{3.4}$$

Definition 3.2. Let K and H be $n \times n$ matrices over $F = \mathbb{C}$ or $F = \mathbb{R}$. We say that K is conjunctive to H if there is a nonsingular $n \times n$ matrix P over F such that $K = P^*HP$.

This definition is given in [1, (7.2.13) Definition, p. 315]. Formula (3.4) says that two Hermitian matrices representing the same Hermitian form in different bases are conjunctive to each other.

3.3 Orthogonalization

A system $\mathcal{X} = (x_1, x_2, \ldots, x_n)$ of vectors in V is called *orthogonal* with respect to the Hermitian form $\langle\cdot,\cdot\rangle$ if $\langle x_i, x_j\rangle = 0$ whenever $1 \le i < j \le n$ and $\mathcal{X}$ is linearly independent.[3.1] If in addition we have $\langle x_i, x_i\rangle = 1$, -1, or 0, we call $\mathcal{X}$ orthonormal.

[3.1] Since we allow that $\langle x_i, x_i\rangle = 0$, the linear independence of $\mathcal{X}$ does not follow from the other requirements.

Given a linearly independent basis $\mathcal{X} = (x_1, x_2, \dots, x_n)$ of V, one can define an orthonormal basis $\mathcal{Y} = (y_1, y_2, \dots, y_n)$ of V. The method Gram–Schmidt orthogonalization described in Subsection 2.4 needs to be modified to this end, because if one encounters a vector with zero "norm,"[3.2] that algorithm cannot be continues. So we describe a modified way of accomplishing our purpose.

Starting with $k = 0$, and continuing with $k = 1, 2, \dots$, we define the systems of vectors $\mathcal{X}_k$, $\mathcal{U}_k$, and $\mathcal{V}_k$ as follows: for $k = 0$ we put $\mathcal{X}_0 = \mathcal{X}$, and $\mathcal{U}_0$, and $\mathcal{V}_0$ will both be the empty system. For $k > 0$, we remove a vector x with $\langle x, x\rangle$ Then we remove a vector x with $\langle x, x\rangle \neq 0$ from $\mathcal{X}_{k-1}$ and normalize it; that is, form the vector

$$z = \frac{1}{\sqrt{|\langle x, x\rangle|}} x. \tag{3.5}$$

It is easy to see that we then have $\langle z, z\rangle = 1$ or $\langle z, z\rangle = -1$. Next, orthogonalize all the remaining vectors in $\mathcal{X}_{k-1}$ against z; that is, for every vector x' remaining in $\mathcal{X}_{k-1}$, replace x' with

$$\begin{aligned} x'' &= x' - \langle z, x'\rangle z \qquad \text{if} \quad \langle z, z\rangle = 1, \\ x'' &= x' + \langle z, x'\rangle z \qquad \text{if} \quad \langle z, z\rangle = -1. \end{aligned}$$

It then easily follows that $\langle x'', z\rangle = 0$. Let $\mathcal{X}_k$ be equal to $\mathcal{X}_{k-1}$ so modified, and then put z into $\mathcal{U}_{k-1}$ if $\langle z, z\rangle = 1$ and put it into $\mathcal{V}_{k-1}$ if $\langle z, z\rangle = -1$ to form $\mathcal{U}_k$ and $\mathcal{V}_k$. Assuming that the system obtained by merging the systems $\mathcal{X}_{k-1}$, $\mathcal{U}_{k-1}$, and $\mathcal{V}_{k-1}$ consists of n vectors and is linearly independent, it is easy to see that the system obtained by merging the systems $\mathcal{X}_k$, $\mathcal{U}_k$, and $\mathcal{V}_k$ will also consists of n vectors and is linearly independent. Therefore, it will be a basis of V.

This process can be continued as long as there is a vector x in $\mathcal{X}_{k-1}$ for which $\langle x, x\rangle \neq 0$.

If we have $\langle x, x\rangle = 0$ for all vectors x in $\mathcal{X}_{k-1}$, then we look for a pair of vectors z and w in $\mathcal{X}_{k-1}$ for which $\langle z, w\rangle \neq 0$. Then we have

$$\begin{aligned} \langle z + w, z + w\rangle &= \langle z, z\rangle + \langle z, w\rangle + \langle w, z\rangle + \langle w, w\rangle \\ &= \langle z, w\rangle + \langle w, z\rangle = \langle z, w\rangle + \overline{\langle z, w\rangle} = 2\Re\langle z, w\rangle; \end{aligned}$$

here the second equation holds since we assumed that $\langle z, z\rangle = \langle w, w\rangle = 0$. If V is a vector space over $\mathbb{R}$ then the right-hand side equals $\langle z, w\rangle$, so it is not zero by assumption. If V is a vector space over $\mathbb{C}$, then $\Re\langle z, w\rangle = 0$ may happen; but if so, then $\Im\langle z, w\rangle \neq 0$. Since we have

$$\begin{aligned} \langle z + iw, z + iw\rangle &= \langle z, z\rangle + \langle z, iw\rangle + \langle iw, z\rangle + \langle iw, iw\rangle \\ &= \langle z, z\rangle + i\langle z, w\rangle - i\langle w, z\rangle - i^2\langle w, w\rangle = \langle z, w\rangle + \langle w, z\rangle \\ &= i\langle z, w\rangle - i\overline{\langle z, w\rangle} = -2\Im\langle z, w\rangle; \end{aligned}$$

for the second equation, observe that $\bar{i} = -i$. For the fourth equation, note that if $\zeta = \alpha_\beta i$ where α and β are real, then

$$i\zeta - i\bar{\zeta} = i(\alpha + \beta i) - i(\alpha - \beta i) = 2i^2\beta = -2\beta = -2\Im\zeta.$$

So, to continue the process, we put $x = z + w$ or $x = z + iw$, making the choice such that $\langle x, x\rangle \neq 0$, and remove one of z and w from $\mathcal{X}_{k-1}$ to form $\mathcal{X}_k$.[3.3] Then we continue the process by

[3.2] we mean $\langle x, x\rangle$ for the vector x, but since this need not be a norm, we will avoid the work "norm" in what follows

[3.3] We can remove either one; the purpose of removing one vector is to maintain the linear independence of the system we are constructing. Observe that the system we obtained by merging the systems $\mathcal{X}_k$, $\mathcal{U}_k$, and $\mathcal{V}_k$ will be linearly independent if the same is true for the system obtained by merging the systems $\mathcal{X}_{k-1}$, $\mathcal{U}_{k-1}$, and $\mathcal{V}_{k-1}$.

normalizing x as in equation (3.5) to obtain z, we repeat the same step as above, beginning with orthogonalizing the remaining vectors in $\mathcal{X}_{k-1}$ against z. The procedure stops when we are unable to make the step to form the systems $\mathcal{X}_k$, $\mathcal{U}_k$, and $\mathcal{V}_k$. At that point, for all vectors x in $\mathcal{X}_{k-1}$ we will have $\langle x, x\rangle = 0$, and for any to vectors z and w in $\mathcal{X}_{k-1}$ we will have $\langle z, w\rangle = 0$. Then we can form $\mathcal{Y}$ by merging the systems $\mathcal{X}_{k-1}$, $\mathcal{U}_{k-1}$, and $\mathcal{V}_{k-1}$. It is clear that the system so obtained will be an orthonormal basis of V; indeed, it was true at all steps of the construction that the system obtained by merging the systems $\mathcal{X}_k$, $\mathcal{U}_k$, and $\mathcal{V}_k$ was linear independent. The orthonormality of the system follows from the fact that each of the remaining vectors in $\mathcal{X}_{k-1}$ was orthogonalized against the vector taken from it at each steps, and the vectors remaining in $\mathcal{X}_{k-1}$ at the end are orthogonal since the process cannot be continued.

Given an orthonormal basis $\mathcal{Y} = (y_1, y_2, \dots, y_n)$ of V, let n_+ be the number of vectors y in V with $\langle y, y\rangle = 1$, let n_- be the number of vectors y in V with $\langle y, y\rangle = -1$, and let n_0 be the number of vectors y in V with $\langle y, y\rangle = 0$, The triple (n_+, n_-, n_0) is called the *signature* of $\mathcal{Y}$. Clearly, we have $n_+ + n_- + n_0 = n$.

3.4 Uniqueness of the signature of orthonormal bases

Let $\mathcal{X} = (x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_l, z_1, z_2, \dots, z_m)$ be an orthonormal basis of V, where $\langle x_i, x_i\rangle = 1$ for all i with $1 \le i \le k$, $\langle y_i, y_i\rangle = -1$ for all i with $1 \le i \le l$, and $\langle z_i, z_i\rangle = 0$ for all i with $1 \le i \le m$. Similarly, let $\mathcal{Y} = (u_1, u_2, \dots, u_{k'}, v_1, v_2, \dots, v_{l'}, w_1, w_2, \dots, w_{m'})$ be an orthonormal basis of V, where $\langle u_i, u_i\rangle = 1$ for all i with $1 \le i \le k'$, $\langle v_i, v_i\rangle = -1$ for all i with $1 \le i \le l'$, and $\langle w_i, w_i\rangle = 0$ for all i with $1 \le i \le m'$. We will show that then $k = k'$, $l = l'$, and $m = m'$.

Writing n for the dimension of V, first note that $k+l+m = k'+l'+m' = n$, and so we need only to show the first two of these three equalities. To start with, we will show that $k \ge k'$. Indeed, assume on the contrary that $k < k'$. Then the system of vectors $\mathcal{U} = (u_1, u_2, \dots u_{k'}, y_1, y_2, \dots y_l, z_1, z_2, \dots z_m)$ contains $k' + l + m > n$ vectors, and so it is linearly dependent. Therefore, there are scalars β_1, β_2, $\dots$, β'_k, and γ_1, γ_2, $\dots$, γ'_k not all of which are zero such that

$$\sum_{i=1}^{k'} \alpha_i u_i + \sum_{j=1}^{l} \beta_j y_j + \sum_{r=1}^{m} \gamma_r z_r = 0.$$

Then we have

$$x = \sum_{i=1}^{k'} (-\alpha_i) u_i = \sum_{j=1}^{l} \beta_j y_j + \sum_{r=1}^{m} \gamma_r z_r \tag{3.6}$$

for some vector x. We cannot have $x = 0$; otherwise, the coefficients on both sides of the equation would have to be zero, since the systems appearing on the sides are linearly independent. As for the product $\langle x, x\rangle$, according to the first equation we have

$$\begin{aligned}\langle x, x\rangle &= \Big\langle \sum_{i=1}^{k'} (-\alpha_i) u_i, \sum_{i'=1}^{k'} (-\alpha_{i'}) u_{i'} \Big\rangle = \sum_{i=1}^{k'} \sum_{i'=1}^{k'} \overline{-\alpha_i}\,(-\alpha_{i'}) \langle u_i, u_{i'}\rangle \\ &= \sum_{i=1}^{k'} |\alpha_i|^2 \langle u_i, u_i\rangle = \sum_{i=1}^{k'} |\alpha_i|^2 > 0;\end{aligned}$$

here the third equation holds since $\langle u_i, u_{i'}\rangle = 0$ is $i \ne i'$; the fourth one holds since $\langle u_i, u_i\rangle = 1$. Finally, the inequality on the right holds since not all of the α_i are zero.

Similarly, making use of the equations $\langle y_j, y_j\rangle = -1$, $\langle y_j, y_{j'}\rangle = -0$ if $j \ne j'$, and $\langle y_j, z_r\rangle = \langle z_r, z_{r'}\rangle = 0$ for all j, r, and r' with $1 \le j \le l$ and $1 \le r, r' \le m$, the second equation for x gives

$$\langle x, x\rangle = -\sum_{j=1}^{l} |\beta_j|^2 \le 0.$$

These two inequalities for $\langle x, x\rangle$ contradict each other, showing the assumption $k < k'$ cannot be true, so we must have $k \ge k'$. The same argument interchanging the roles of the bases $\mathcal{X}$ and $\mathcal{Y}$ show that $k' \ge k'$; hence, indeed, $k = k'$.

The equation $l = l'$ can be shown in a similar way, instead of equation (3.6) using the equation

$$y = \sum_{j=1}^{l'} (-\beta) v_j = \sum_{i=1}^{k} \alpha_i x_i + \sum_{r=1}^{m} \gamma_r z_r$$

to show that $l' > l$ is not possible.

As we remarked above, having shown that $k = k'$ and $l = l'$, the equation $m = m'$ also follows since $k + l + m = k' + l' + m' = n$.

3.5 Sylvester's theorem

The result of the considerations above can be formulated for matrices as a theorem of James Joseph Sylvester for Hermitian matrices (cf. [1, (7.5.3) Theorem, p. 338]):

Theorem 3.1. *Let H be a Hermitian matrix. Then, there is a nonsingular matrix P such that P^*HP is a matrix all entries of which off the main diagonal are 0, while the entries of the diagonal are 1, -1, or 0. Furthermore, if Q^*HQ is another matrix matrix with the same properties, then the number of entries equaling 1, -1, or 0 are the same as those in P^*HP. Finally, it can also be arranged that all entries of 1 are on top, all entries of -1 are in the middle, and all entries of 0 are on the bottom.*

Given an $n\times n$ Hermitian matrix H, for the proof one introduces an Hermitian form $\langle\cdot,\cdot\rangle$ on an n-dimensional vector space V with basis $\mathcal{X}$ such that $H = \mathcal{P}_{\mathcal{X}}\langle\cdot,\cdot\rangle$. Then one construct an orthonormal basis $\mathcal{Y}$. With the nonsingular matrix such that $\mathcal{X} = \mathcal{Y}P$ we obtain the desired diagonal matrix $P^*HP = \mathcal{P}_{\mathcal{Y}}\langle\cdot,\cdot\rangle$. The order of the diagonal entries in this matrix can be changed arbitrarily by permuting the vectors in $\mathcal{Y}$. Finally, the number of entries of 1, -1, and 0 are uniquely determined by the uniqueness of the signature of the inner product, as described in Subsection 3.4.

3.6 Unitary equivalence to diagonal matrices and Sylvester's theorem

Given an $n \times n$ matrix Hermitian H matrix over complex numbers, Corollary 2.2 says that there is a unitary matrix U for which $D = (d_{ij}) = U^*HU$ is a diagonal matrix; it is easy to see that d_{ii} for $1 \le i \le n$ are the eigenvalues of H.[3.4] Writing $Q = (q_{ii})$ for the diagonal matrix with

$$q_{ii} = \begin{cases} |d_{ii}|^{-1/2} & \text{if } d_{ii} \ne 0, \\ 1 & \text{if } d_{ii} = 0. \end{cases}$$

[3.4] One way of seeing this is by noting that $U^* = U^{-1}$, and then observing that the characteristic polynomial of H is

$$\begin{aligned}\det(H - I\lambda) &= \det(U^{-1}\det(H - I\lambda)\det(U) = \det\bigl(U^{-1}(H - I\lambda)U\bigr) = \det(U^{-1}HU - I\lambda) \\ &= \det(D - I\lambda) = \prod_{i=1}^{n} (d_{ii} - \lambda).\end{aligned}$$

Then Q is nonsingular, and all diagonal elements of the matrix $D_1 = Q^*DQ$ are 1, -1, or 0.[3.5] As Q is a diagonal matrix all whose diagonal elements are real; hence $Q^* = Q$. That is $D_1 = (UQ)^*H(UQ)$, and so D_1 is conjunctive to H – cf. Definition 3.2. This is the existence part of Sylvester's theorem, except for the minor issue of the order of the entries in the diagonal of D_1. The diagonal entries of D_1 can be rearranged by using a permutation matrix P.[3.6] The matrix $D_2 = (UQP)^*H(UQP)$ is also conjunctive to H.

References

[1] Hans Schneider and George Philip Barker. *Matrices and Linear Algebra, 2nd ed.* Dover Publications, New York, 1973.

[3.5] Of course, $Q^* = Q$.

[3.6] A permutation matrix is a matrix every row and every column of which has exactly one entry equaling 1, all other entries being 0. If A is an arbitrary $n \times n$ matrix and P is an $n \times n$ matrix and P^* is another permutation matrix that is also its inverse. The matrix P^*A is a matrix obtained by permuting the rows of A, and AP is permuting the columns of A. If A is a diagonal matrix, then P^*AP is another diagonal matrix, obtained by rearranging the diagonal entries of A. By using an appropriate diagonal matrix, the diagonal entries of A can be rearranged in an appropriate way.