1. Interchange of quantifiers

There are two kinds of quantifiers: universal: $\forall$, meaning "for all," and existential: $\exists$, meaning " there is" or "there exists." Within a quantifier, one may specify the kind of things the quantifier talks about, e.g.,

$$(\forall x : x \text{ is an integer}), \quad (\exists y : y \text{ is a real number}), \quad (\forall x \in \mathbb{Z}), \quad (\exists y \in \mathbb{R}),$$

etc.; here $\mathbb{Z}$ stands for the set of integers (positive, negative, or zero), and $\mathbb{R}$ stands for the set of reals. Two quantifiers of the same kinds mean either two universal quantifiers or two existential quantifiers, while two quantifiers of different kinds refer to one universal and one existential quantifier. Two quantifiers of the same kind are always interchangeable, but two quantifiers of different kinds are not. To see this, consider the following example:

$$(\forall x : x \text{ is licensed driver})(\exists y : y \text{ is a car})\ (x \text{ has driven } y).$$

This sentence is entirely reasonable, since it only says that "every licensed driver has driven a car," or, more precisely, "every licensed driver has driven at least one car." In real life, there are exceptions even to such a reasonable statements, but if you restrict your attention to life in a small town, the statement is most likely true. If one interchanges the quantifiers, the result is totally absurd:

$$(\exists y : y \text{ is a car})\ (\forall x : x \text{ is licensed driver})\ (x \text{ has driven } y).$$

This says that "there is a single car that every licensed driver has driven," and this is unlikely to be true even in a small town, unless everyone has gone to the same driving school within the last few years, and got a chance of practising on the same car.

Another, more mathematical, example is the following. Let x and y run over integers. Then the formula

$$(\forall x)(\exists y)[x > y]$$

is true, while the formula

$$(\exists y)(\forall x)[x > y]$$

is false. Indeed, the first formula is true. Given an arbitrary integer x, we can pick $y = x - 1$ to ensure that $x > y$. On the other hand, the second formula is not true. To see this, pick an arbitrary number y. Then $(\forall x)[x > y]$ is certainly not true; for example, $x > y$ is not true with $x = y - 1$.

2. More on logic

A sentence is a statement that is either true or false.[2] Sentences can be connected by the logic operations, also called sentential connectives *and*, *or*, *if ... then*, and *if and only if*, denoted in turn by $\&$, $\vee$, $\to$, and $\leftrightarrow$. Instead of $\&$, people often use the symbol $\wedge$. The logic operations only connect sentences (or, more generally, statements – see below). For

[2] That is, true or false in principle. If one talks about formal logic, then a sentence would be a declarative sentence. Some declarative sentences, however, cannot be considered sentences in the sense of logic, since they cannot be considered true or false even in principle. For example, the sentence "This sentence is false" cannot be taken as true (since it then says that it is false), and it cannot be taken as false (since then it says that it is true). In mathematical logic, what is a sentence or a statement is defined formally by describing the rules how a statement can be formed. The collection of these rules is called *syntax*.

example, the "or" in the sentence "You can have coffee or tea with your breakfast" is not a logic operation.[3]

The meaning of the sentential connectives can be illustrated by truth tables. Writing T for true and F for false (T and F are called *truth values*) we have for the operation $\&$, called *conjunction*:

A	B	$A \ \& \ B$
T	T	T
T	F	F
F	T	F
F	F	F

For the operation $\vee$, called *disjunction*, we have

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

As seen from the truth table, disjunction is always meant in the *inclusive* sense, that is $A \vee B$ is true unless both A and B are false. This differs from colloquial usage, where one often uses the word "or" in the exclusive sense, where "A or B in the exclusive sense" is true if exactly one of A and B is true.

For the operation $\to$, called *conditional*, we have

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

The meaning of the conditional is often differs from its colloquial use, where the meaning of "if A then B" is unclear in case A is false. In mathematics, one strictly follows the truth table above. For example, the sentence "if 2 by 2 is 5 then the snow is black" is a true sentence in mathematics.[4] In colloquial speech one would consider this sentence meaningless, or at best pointless. But it illustrates an important point: in the conditional, there does not need to be a causal connection between the constituents.[5] For the operation $A \to B$, instead of saying "if A then B", it is often more convenient to say "A only if B". In case of this latter sentence, the colloquial meaning approaches more closely the mathematical meaning of the conditional. Namely, "A only B" means that A is allowed to be true only in case B is also true; indeed, when A is true and B is false, the truth table entry for $A \to B$ shows false.

One occasionally reverses the arrow in the conditional, using the symbol $A \leftarrow B$ meaning

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[3]However, on can rewrite this as a logic operation by saying that "You can have coffee with your breakfast or you can have tea with your sentence" (although by doing this, one probably changes the meaning, since in logic, "or" is meant in the inclusive sense (allowing to have both coffee and tea), while on a restaurant menu the meaning is probably exclusive (not allowing to have both coffee and tea without paying extra).

[4]This example is due to the German mathematician David Hilbert. Note that the sentence "if 2 by 2 is 5 then the snow is white" is also true.

[5]That is, 2 by 2 being 5 does not cause the snow to be black. As for the constituents, or, with a more technical word, *operands*, in the conditional $A \to B$, A called the antecedent and B is called the consequence (the latter is a somewhat misleading name, since the name seems to imply a causal connection).

"A if B", or "if B then A":

A	B	$A \leftarrow B$
T	T	T
T	F	T
F	T	F
F	F	T

The conditional $A \leftarrow B$ (or $B \to A$) is usually called the *converse* of $A \to B$. Finally, the truth table of the operation $\leftrightarrow$, called *biconditional*, is

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

$A \leftrightarrow B$ is expressed as "A if and only if B", or, sometimes, as 'A iff B" (but it is not clear how one should pronounce the word "iff"). The word "iff" was introduced by Paul Halmos. "A if and only if B" is short for saying "A if B and A only if B"; formally, for $(A \leftarrow B)\ \&\ (A \to B)$. It is easy to check that the truth table for this formula is the same as the one given for the biconditional above.

These logic operations are called *binary* operations, since they involve two constituents, called *operands*, A and B in the above truth tables. The letters A and B used in the above formulas are often called *sentential variables*, i.e., variables that can either be true or false.

Negation. The operation "not" is called *negation*. "Not A means that "it is not the case that A", or, more simply, "A is not true". This is called a *unary* operation, since it has only one operand. The truth table for negation is

A	$\neg A$
T	F
F	T

Tautologies. A tautology is a logic expression (an expression involving the logical operations just defined) that is always true, whether or not the sentential variables in it are true or false. Examples for tautologies are

$$(\neg(A\ \&\ B)) \leftrightarrow ((\neg A) \vee (\neg B)).$$

For better readability, one can drop several pairs of parentheses here, to write

$$\neg(A\ \&\ B) \leftrightarrow \neg A \vee \neg B.$$

To make sense of this way of writing the formula, one can assigns priority to the logic operations in the order $\neg$, $\&$, $\vee$, $\leftarrow$, $\to$, $\leftrightarrow$, meaning that one first try to perform the operations with higher priority.[6] Most people consider the priority between $\&$ and $\vee$, and between $\leftarrow$ and $\to$ unclear, so it is best to use parentheses to avoid misunderstanding. To check that the above formula is indeed a tautology, one can use the truth table to evaluate

[6]This is similar to the rule in algebra that $\cdot$ (multiplication) has higher priority than $+$ (addition). That is, the formula $2+3\cdot 5$ means $2+(3\cdot 5)$, and not $(2+3)\cdot 5$. $+$ and $-$ have equal priority, so in the expression $2+3-5-6+8$ one performs the operations from left to write, i.e., this expression means $(((2+3)-5)-6)+8$ In computer science, one often uses the word *precedence* instead of priority; computer scientists unambiguously assign higher precedence to to logical and than to logical or; in mathematics, this precedence is not always taken for granted.

it for each choice of A and B to find that the formula is always true. One interpretation of the above tautology is that $\neg(A \;\&\; B)$ and $\neg A \vee \neg B$ mean the same thing. This is because the biconditional is true exactly when the two sides have the same truth value; so the above formula being always true means that the two sides on the biconditional in it always have the same truth value. The above formula is one of the two *De Morgan identities.* The other De Morgan identity is the tautology

$$\neg(A \vee B) \leftrightarrow \neg A \;\&\; \neg B.$$

Another simple tautologies are

$$(A \to B) \leftrightarrow \neg A \vee B;$$

we used parentheses on the left here, since the sometimes $\to$ and $\leftrightarrow$ is considered to have equal priority. When one wants to establish the *implication*[7] $A \to B$, one often takes advantage of the tautology

$$(A \to B) \leftrightarrow (\neg B \to \neg A),$$

and proves the implication $\neg B \to \neg A$ instead. The conditional $\neg B \to \neg A$ is called the *contrapositive* of $A \to B$. A further tautology is

$$\neg(A \leftrightarrow B) \leftrightarrow (\neg A \leftrightarrow B).$$

One often writes $A \not\leftrightarrow B$ instead of $\neg(A \leftrightarrow B)$. The truth table of $A \not\leftrightarrow B$ is

A	B	$A \not\leftrightarrow B$
T	T	F
T	F	T
F	T	T
F	F	F

One might call the operation $A \not\leftrightarrow B$ *exclusive or*, since it reflects the meaning the word "or" often used colloquially. However, it is best to avoid this term "exclusive or", since in mathematics the word "or" is always used in the inclusive sense, as defined by the logic operation $A \vee B$. The expression $A \not\leftrightarrow B$ in mathematics is often given as "A if and only if not B", reflecting the fact that this this expression is equivalent to (i.e., true exactly the same time as) the expression $A \leftrightarrow \neg B$; that is, the fact that

$$(A \not\leftrightarrow B) \leftrightarrow (A \leftrightarrow \neg B)$$

is a tautology.

Open statements. An *open statement* is a statement that has (zero or more) variables in it; when one gives values to these variables. For example, to say that "x is greater than 2" (where x denotes an unspecified real number) is on open statement. One can tell its truth value only after one specifies what real number x is. In mathematical logic, one describes the rules how open statements are formed; the collection of these rules are called *syntax.* However, want to discuss matters somewhat informally, so we will avoid a detailed discussion of syntax. Occasionally, open statements will be denoted by script capital letters such as $\mathcal{A}$ or $\mathcal{B}$.

[7] One often calls a conditional an implication, especially in informal usage, since instead of "if A then B" one might say "A implies B". However, logical implication has a meaning separate from, though related to, that of the conditional. Therefore, some consider calling a conditional an implication objectionable.

Negating quantified statements. The formula $\neg(\forall x)\mathcal{A}$ is the same as $(\exists x)\neg\mathcal{A}$; here $\mathcal{A}$ is some open statement.[8] That is, saying that "it is not true that for all x $\mathcal{A}$ holds" means that "there is an x for which A does not hold". One writes this by saying that

$$\neg(\forall x) \equiv (\exists x)\neg;$$

here $\equiv$ means that what are written on the two sides are equivalent, i.e., they mean the same thing, i.e. they can replace each other.[9] Similarly, the formula $\neg(\exists x)\mathcal{A}$ is the same as $(\forall x)\neg\mathcal{A}$. That is, to say that "it is not true that there exists an x for which $\mathcal{A}$ holds" means that "for all x it is not true that $\mathcal{A}$ holds". This can be expressed by the rule

$$\neg(\exists x) \equiv (\forall x)\neg.$$

One can use these rules and the tautologies mentioned above to move negation inside a formula. For example,

$$\neg(\forall\epsilon > 0)(\exists\delta > 0)(\forall p \in (0,1))(\forall q \in (0,1))\left(|p-q| < \delta \to \left|\frac{1}{p} - \frac{1}{q}\right| < \epsilon\right). \tag{1}$$

Here p, q, ϵ, and δ denote real numbers,[10] and $(0,1)$ denotes the interval $\{t : 0 < t < 1\}$. What this formula expresses is irrelevant for our present purpose.[11] The quantifiers $(\forall\epsilon > 0)$ is called a *restricted quantifier*, since it says that ϵ ranges over positive real numbers instead of all real numbers (where unrestricted variables range in the present case). Similarly, $(\forall\delta > 0)$, $(\exists p \in (0,1)$ are restricted quantifiers. The important point at present is that the above rules of interchanging negation and quantifiers are also true for restricted quantifiers.

The above formula can be transformed as follows:

$$\begin{aligned}
&\neg(\forall\epsilon > 0)(\exists\delta > 0)(\forall p \in (0,1))(\forall q \in (0,1))\left(|p-q| < \delta \to \left|\frac{1}{p} - \frac{1}{q}\right| < \epsilon\right)\\
&\quad\equiv \neg(\forall\epsilon > 0)(\exists\delta > 0)(\forall p \in (0,1))(\forall q \in (0,1))\left(|p-q| < \delta \to \left|\frac{1}{p} - \frac{1}{q}\right| < \epsilon\right)\\
&\quad\equiv (\exists\epsilon > 0)\neg(\exists\delta > 0)(\forall p \in (0,1))(\forall q \in (0,1))\left(|p-q| < \delta \to \left|\frac{1}{p} - \frac{1}{q}\right| < \epsilon\right)\\
&\quad\equiv (\exists\epsilon > 0)(\forall\delta > 0)\neg(\forall p \in (0,1))(\forall q \in (0,1))\left(|p-q| < \delta \to \left|\frac{1}{p} - \frac{1}{q}\right| < \epsilon\right)\\
&\quad\equiv (\exists\epsilon > 0)(\forall\delta > 0)(\exists p \in (0,1))\neg(\forall q \in (0,1))\left(|p-q| < \delta \to \left|\frac{1}{p} - \frac{1}{q}\right| < \epsilon\right)\\
&\quad\equiv (\exists\epsilon > 0)(\forall\delta > 0)(\exists p \in (0,1))(\exists q \in (0,1))\neg\left(|p-q| < \delta \to \left|\frac{1}{p} - \frac{1}{q}\right| < \epsilon\right)\\
&\quad\equiv (\exists\epsilon > 0)(\forall\delta > 0)(\exists p \in (0,1))(\exists q \in (0,1))\left(|p-q| < \delta \;\;\&\;\; \neg\left|\frac{1}{p} - \frac{1}{q}\right| < \epsilon\right)\\
&\quad\equiv (\exists\epsilon > 0)(\forall\delta > 0)(\exists p \in (0,1))(\exists q \in (0,1))\left(|p-q| < \delta \;\;\&\;\; \left|\frac{1}{p} - \frac{1}{q}\right| \ge \epsilon\right).
\end{aligned}$$

[8] Presumably, $\mathcal{A}$ depends on x, so one might be tempted to write $\mathcal{A}(x)$ instead. However, writing $\mathcal{A}(x)$ really does not make things clearer.

[9] We do not regard $\equiv$ as a logical connective here. It can be considered as a rule according to which we can change (or transform) formulas without changing their meanings. That is, it expresses a transformation rule; see below.

[10] One also says that these variables *range over* real numbers.

[11] It expresses the statement that the function $f(x) = 1/x$ is not uniformly continuous in the interval $(0,1)$, discussed below in these notes.

Restricted quantifiers. If E is a set, and $\mathcal{A}$ is an open statement, quantifiers of the form $(\forall x \in E)$ and $(\exists x \in E)$ are called *restricted quantifiers.* More generally, given an open statement $\mathcal{A}$, one can consider the restricted quantifiers ($\forall x : \mathcal{A}$ and $\exists x : \mathcal{A}$). If $\mathcal{B}$ is another open statement, then the formula $(\forall x : \mathcal{A})\mathcal{B}$ says that "for all x such that $\mathcal{A}$ holds we (also) have $\mathcal{B}$, and for $(\exists x : \mathcal{A})\mathcal{B}$ means that "for all x for which $\mathcal{A}$ holds we also have $\mathcal{B}$". It is easy to see that

$$(\forall x : \mathcal{A})\mathcal{B} \equiv (\forall x)(\mathcal{A} \to \mathcal{B}),$$

and

$$(\exists x : \mathcal{A})\mathcal{B} \equiv (\exists x)(\mathcal{A} \ \ \& \ \ \mathcal{B}).$$

Therefore, restricted quantifiers are not strictly necessary, but they are often convenient to use. They frequently make formulas simpler, and, a very important point, the rules discussed above involving the interchange of negation and quantifiers are also true for restricted quantifiers. Finally, for a set E,

$$(\forall x \in E) \equiv (\forall x : x \in E)$$

and

$$(\exists x \in E) \equiv (\exists x : x \in E).$$