APPROXIMATION OF NUMBERS BY FRACTIONS[1]

The number π expressing the ratio of the circumference to the diameter of the circle is approximately 3.141,592,653,589,793... . π can be approximated by common fractions as

$$\frac{22}{7},\quad \frac{333}{106},\quad \frac{355}{113},\quad \frac{103993}{33102}.$$

As a measure of how good this approximations are, consider the following approximate equations:

$$\pi-\frac{22}{7}\approx -.001{,}264{,}489\approx -.0619\cdot\frac{1}{7^2},\quad \pi-\frac{333}{106}\approx 8.321{,}963\cdot 10^{-5}\approx 0.935\cdot\frac{1}{106^2},$$
$$\pi-\frac{355}{113}\approx -2.667{,}642\cdot 10^{-7}\approx -0.0034\cdot\frac{1}{113^2},\quad \pi-\frac{103993}{33102}\approx 5.778{,}906\cdot 10^{-10}\approx .633\cdot\frac{1}{33102^2};$$

the goodness of each of these approximations is measured in terms of 1 divided by the square of the denominator of the approximating fraction. The following theorem will explain why this way of measuring the goodness of the approximations make sense. The theorem is proved by a direct application of the pigeon hole principle:

Theorem. *Let x be a real number and n a positive integer. Then there are integers k and l such that $1\le l\le n$ and*

$$|lx-k|\le\frac{1}{n+1}. \tag{1}$$

Noting that $l\ne 0$, the inequality here can also be written as

$$\left|x-\frac{k}{l}\right|\le\frac{1}{l(n+1)}.$$

In other words, given any positive integer n, a real number x can always be approximated by a common fraction whose denominator l is $\le n$ such that the error of the approximation is less than or equal to

$$\frac{1}{l(n+1)}.$$

Since $1\le l\le n$, this error is less than $1/l^2$.

Proof. For a real number y, denote by $[y]$ the *integer part* of y. That is, $[y]$ is the largest integer $m\le y$. Further, denote by $\{y\}$ the *fractional part* of y; that is, $\{y\}\overset{def}{=}y-[y]$.[2] Clearly, $0\le\{y\}<1$.

Assume that no k and l satisfying the assertion of the theorem exist. Then for any integer l with $1\le l\le n$ we must have

$$\frac{1}{n+1}<\{lx\}<\frac{n}{n+1}. \tag{2}$$

[1]Notes for Course Core Studies 3.11 at Brooklyn College of CUNY. Attila Máté, April 25, 2010.

[2]The symbol $\overset{def}{=}$ describes an equation where the left-hand side is defined by means of the expression on the right-hand side.

Indeed, writing $s = [lx]$, we have $\{lx\} = lx - s$. Thus, if the first inequality fails then, noting that $lx \ge s$, we have

$$0 \le lx - s \le \frac{1}{n+1},$$

the inequality (1) claimed in the theorem is satisfied with $k = s$. If the second inequality fails then, noting that $lx - s = \{lx\} < 1$, we have

$$\frac{n}{n+1} \le lx - s < 1.$$

Subtracting 1 from all the members of the inequality,[3] we obtain

$$-\frac{1}{n+1} \le lx - (s+1) < 0.$$

In this case, inequality (1) claimed in the theorem is satisfied with $k = s + 1$.

Given that (2) is satisfied, each of the n numbers $\{1x\}$, $\{2x\}$, $\{3x\}$, ..., $\{nx\}$ belongs to at least[4] one of the $n - 1$ intervals

$$\left[\frac{1}{n+1}, \frac{2}{n+1}\right], \left[\frac{2}{n+1}, \frac{3}{n+1}\right], \left[\frac{3}{n+1}, \frac{4}{n+1}\right], \dots \left[\frac{n-2}{n+1}, \frac{n-1}{n+1}\right], \left[\frac{n-1}{n+1}, \frac{n}{n+1}\right].$$

Since there are n numbers and $n - 1$ intervals here, there must be (at least) one among these intervals to which (at least) two of these numbers belong. That is, there are integers p and q with $1 \le p < q \le n$ such that $\{px\}$ and $\{qx\}$ belong to the same one among these intervals.[5] Since the length of each of these intervals is $1/(n+1)$, we then must have

$$|\{qx\} - \{px\}| \le \frac{1}{n+1};$$

Writing $r = [px]$ and $s = [qx]$, we have $\{px\} = px - r$ and $\{qx\} = qx - s$; hence the above inequality becomes

$$|(q-p)x - (r-s)| \le \frac{1}{n+1};$$

here p, q, r, s are integers. Writing $l = q - p$ and $k = r - s$, this inequality becomes identical to (1), the inequality we wanted to show. As $0 \le p < q \le n$, the inequality $1 \le l \le n$ also follows. $\square$

[3]The members of the inequality are the expressions on the left-hand side, the middle, and on the right-hand side, separated by the inequality signs.

[4]The interval $[a, b]$ is the set of points $\{x : a \le x \le b\}$. In case $\{px\}$ is one of the numbers

$$\frac{1}{n+1}, \quad \frac{2}{n+1}, \quad \frac{3}{n+1}, \quad \dots, \quad \frac{n-2}{n+1},$$

then $\{px\}$ belongs to each of the two intervals whose one endpoint is $\{px\}$.

[5]Saying that $1 \le p < q \le n$ amount to saying that there are two distinct integers p and q with $1 \le p \le n$ and $1 \le q \le n$ such that $\{px\}$ and $\{qx\}$ belong to the same interval, and the notation is so chosen that the smaller integer is denoted by p and the larger one by q.