Algebraic Theories in Quantum Field Theory and Quantum Algebra

A Proposal

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August 29, 1999

1 Introduction

Over the past few years, there have been several major innovations in quantum field theory (QFT) and quantum algebra (QA). Following Graeme Segal's seminal definition of conformal field theory, many researchers studying QFT, have now begun using categories and functors in a new and fundamental way. The categories (loosely) correspond to segments of space-time while the functors take segments of space-time to linear maps of Hilbert spaces. In QA, researchers are working with many new structures that are radically different from standard algebraic structures. Many of these structures are neither commutative nor associative and have relations that hold only "up to homotopy."

Both QFT and QA have created many new and diverse mathematical structures that need to be studied from a level of generality that encompasses them all. We propose to study these two fields of mathematical physics from the point of view of a category theoretic version of universal algebra called *functorial semantics*. Functorial semantics deals with algebraic theories that describe algebraic structures. Our goal is to generalize functorial semantics enough to place both of these fields into a single formalism. Placing them within one formalism will help us understand the deep connections within each area and between both areas. We will use the powerful tools available in functorial semantics to better understand and prove theorems about the structures in QFT and QA.

We have found that properly generalized algebraic theories are well suited for describing the plethora of diverse structures of QA. Furthermore, categories that are the common fare of QFT such as topological cobordisms, conformal cobordisms and symplectic cobordisms are, in fact, examples of generalized algebraic theories. The study of these two aspects of algebraic theories will identify connections between them and advance both fields.

2 Functorial Semantics

In order to understand how our work will relate to QFT and QA, we must begin by reviewing some classical categorical universal algebra. The formalism that we use to describe algebraic structures is Lawvere's functional semantics ([10],[11] and e.g. [2]). This formalism has proven to be quite useful in modern algebra. Functional semantics is based on the language and tools of category theory. To certain types of algebraic structures, there corresponds a category, **T**, called an algebraic theory, that describes such a structure. Product preserving functors from **T** to an ambient category are algebras/models of the structure in the ambient category. Homomorphisms between algebras are natural transformations between the product preserving functors.

People who study category theory have used functorial semantics because of its powerful tools; its flexibility of using different ambient categories; and because it gives a canonical way of finding an algebraic theory for every algebraic structure. Functorial semantics is sufficiently general to describe many different structures in many different ambient categories. Our goal is to further generalize functorial semantics so that we can use it and its powerful tools to describe and prove theorems about the structures that are found in modern mathematical physics.

2.1 Algebraic Theories

An algebraic theory **T** is a category whose objects are the natural numbers and in which n is isomorphic to the n-ary product of 1, i.e. $n \simeq 1^n$. The morphisms in the category correspond to operations of an algebraic structure. For example, the theory of groups, $\mathbf{T}_{Groups}$ is an algebraic theory generated by the following morphisms: $+ : 2 \longrightarrow 1$ corresponding to the 2- ary multiplication in a group, $-(\ \ ) : 1 \longrightarrow 1$ corresponding to the unary inverse, and $e : 0 \longrightarrow 1$ corresponding to the nullary unit of the group. These morphisms must satisfy the following relations:

$$
\begin{tikzcd}
3 \arrow[r, "id\times"] \arrow[d, "+\times id"'] & 2 \arrow[d, "+"] \\
2 \arrow[r, "+"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[d, "-()\times id"'] & 1 \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "!"] & 2 \arrow[d, "id\times -()"] \\
 & 0 \arrow[d, "e"] & \\
2 \arrow[r, "+"'] & 1 & 2 \arrow[l, "+"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
1\times 0 \arrow[r, "id\times e"] \arrow[dr, "\simeq"'] & 2 \arrow[d, "+"] & 0\times 1 \arrow[l, "e\times id"'] \arrow[dl, "\simeq"] \\
 & 1 &
\end{tikzcd}
$$

where Δ is the usual "diagonal" map and ! is the unique map from 1 to 0. These morphisms generate all the other morphisms in $\mathbf{T}_{Groups}$. For example there is a unique $+$ map from 3 to 1 that is the extension of the associative addition. In some sense, $\mathbf{T}_{Groups}$ is the "shape" or "template" of groups. One can visualize a small part of this "shape" as

$$\begin{tikzcd}
\cdots \arrow[r, "+"] & 4 \arrow[r, "+"] & 3 \arrow[r, "+"] \arrow[rr, bend left, "+"] \arrow[rr, bend right, "+"'] & \overset{=}{\underset{=}{2}} \arrow[r, "+"] & 1 \arrow[loop above, "-"] & 0 \arrow[l, "e"']
\end{tikzcd}$$

where the two morphisms from 3 to 1 are the two (equal) ways of associating three objects.

Another example is $\mathbf{T}_{Rings}$ which is similar to $\mathbf{T}_{Groups}$ but has the 2-ary operation $\bullet : 2 \longrightarrow 1$ and the nullary operation $u : 0 \longrightarrow 1$ satisfying the usual ring relations. In the theory of rings, we insist that the two operations $+ : 2 \longrightarrow 1$ and $+ \circ twist : 2 \longrightarrow 2 \longrightarrow 1$ are, in fact the same (abelian.) One can go on to give the definitions of other theories of algebraic structures such as monoids, commutative monoids, boolean rings, R-modules for an arbitrary ring R, Jordan algebras, lattices, etc.

Given an algebraic theory, $\mathbf{T}$, we can easily create the category of algebras/models of the theory. Algebras are simply product preserving functors from $\mathbf{T}$ to the category $\mathbf{Set}$. For example, $F : \mathbf{T}_{Groups} \longrightarrow \mathbf{Set}$ has $F(1) = X$ where X is some set. Since F is product preserving and $2 \simeq 1^2$, $F(2) = X^2$. In general $F(n) = X^n$ and $F(0) = X^0 = *$, the one object set. Morphisms go where they are supposed to i.e. $F(+ : 2 \longrightarrow 1) \quad = \quad + : X^2 \longrightarrow X$. These operations on X must satisfy the same relations that $\mathbf{T}_{Groups}$ satisfy. Thus we have a similar picture:

$$\begin{tikzcd}
\cdots X^4 \arrow[r, "+"] & X^3 \arrow[r, "+"] \arrow[rr, bend left, "+"] \arrow[rr, bend right, "+"'] & \overset{=}{\underset{=}{X^2}} \arrow[r, "+"] & X^1 \arrow[loop above, "-"] & \{*\}. \arrow[l, "e"']
\end{tikzcd}$$

In a sense, the product preserving functors from $\mathbf{T}_{Groups}$ are representations of the "shape" of groups.

Every product preserving functor determines an algebra and every algebra determines a product preserving functor. Given two such algebras, a homomorphism from one to the other corresponds to a natural transformation from one product preserving functor to the other. For a given theory $\mathbf{T}$, product preserving functors and natural transformations between them form the category of algebras, $\mathbf{Alg}(\mathbf{T}, \mathbf{Set})$.

Notice that our use of the category of $\mathbf{Set}$ was not really important. The only property that we used about $\mathbf{Set}$ is that it has finite products. We could have used any other category, $\mathbf{C}$, with finite products. For example, $\mathbf{Alg}(\mathbf{T_{Groups}}, \mathbf{Top})$ is the category of topological groups. If we use the category of differentiable manifolds $\mathbf{DiffMan}$, then $\mathbf{Alg}(\mathbf{T_{Groups}}, \mathbf{DiffMan})$ is Lie groups. Let $\mathbf{Coalg}$ denote the category of coassociative, cocommutative coalgebras where the tensor product is the Cartesian product. The category $\mathbf{Alg}(\mathbf{T_{Groups}}, \mathbf{Coalg})$ is nothing more than cocommutative Hopf algebras. There are many more examples using other ambient categories and other theories.

2.2 Algebraic Functors

Algebraic theories do not stand alone. An algebraic theory-morphism is a functor $G : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ that is the identity on objects i.e. $G(n) = n$. There is, for example, (ex. A) an obvious inclusion of the theory of abelian groups into the theory of rings. (ex. B) There is a surjection of the theory of groups onto the theory of abelian groups. Given $G : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ and a $\mathbf{T}_2$ algebra $F : \mathbf{T}_2 \longrightarrow \mathbf{C}$, we can compose and get a $\mathbf{T}_1$ algebra $F \circ G$. And so there is an induced functor $G^* : \mathbf{Alg(T_2, C)} \longrightarrow \mathbf{Alg(T_1, C)}$. Such functors are called algebraic functors. Using the above examples, (ex. A) we have the forgetful functor from the category of rings to the category of abelian groups; (ex. B) there is an inclusion of the category of abelian groups into the category of groups.

One of the main theorems of functorial semantics states that every algebraic functor has a left adjoint. Such adjunctions are pervasive throughout all of algebra. (ex. A) The left adjoint to the forgetful functor from the category of rings to the category of abelian groups is the functor that assigns the free ring to an abelian group; (ex. B) the left adjoint of the inclusion of abelian groups into groups is the abelianization functor ($G \mapsto G/[G,G]$). There are many other examples of this adjunction. All forgetful functors have free left adjoint functors: free symmetric algebras, free tensor algebras, etc.

Category theory provides us with a mechanism for constructing the left adjoint to every algebraic functor in a unified and clear way. Given $G : \mathbf{T}_1 \longrightarrow \mathbf{T}_2$ and a $\mathbf{T}_1$ algebra $F : \mathbf{T}_1 \longrightarrow \mathbf{C}$ one completes the following diagram

$$\begin{tikzcd} \mathbf{T}_2 \arrow[rr, dashed, "Lan_G(F)"] & & C \\ & \mathbf{T}_1 \arrow[ul, "G"] \arrow[ur, "F"'] & \end{tikzcd}$$

with a special type of limit called a Kan extension. (The final section of Saunders Mac Lane's classic book [15] is appropriately titled "All Concepts Are Kan Extensions.") So even though all of the above examples of left adjoints seem different, they are, in fact, constructed the same way.

2.3 Algebraic Theory Reconstruction

Functorial semantics shows us how to reconstruct the algebraic theory from a category of algebras. For example, the theory of groups can be reconstructed from the category of groups. Whereas there are many ways of presenting the theory of groups, this reconstruction gives a canonical description of the theory of groups. The reconstruction again uses Kan extensions. For every "nice" category of algebras there is a forgetful functor to **Set**. This forgetful functor has a free left adjoint. One can look at the images of the finite sets under this left adjoint. The theory is reconstructed as the full subcategory of the category of algebras whose objects are the free algebras on finite sets. Equivalently, The morphisms in the theory from n to m is the set of natural transformations $NAT(U^n, U^m)$ where U is the forgetful functor from the category of algebras to **Set**.

This reconstruction gives a beautiful duality between theories (syntax) and algebras (semantics.) However, this reconstruction does something more. In some sense, the reconstructed theory is the "right" theory to work with. It is the theory generated by the algebras. It is a type of "minimal model" of all theories of a certain structure.

3 Generalizations of Functorial Semantics

Functorial semantics has been around more then thirty years and has been generalized to many area of algebra and computer science. There have been generalizations to many sorted algebras; algebras with infinitary operations [12]; algebras in a monoidal closed category [3]; etc. Computer scientists have long since coopted algebraic theories for their own use (Wagner [23] is a survey article about ordered theories, iteration theories, rational theories, iterative theories etc.) Such computer science generalizations have been used in context-free grammars, flowchart semantics, recursion schemata, recursively defined domains, etc.

The fact that functorial semantics have been found in such diverse fields is a testament to the power of the ideas found within. We feel that the following families of generalizations will be of use in QFT and QA.

3.1 From One-dimensional Theories to Two-dimensional Theories

With the study of quantum algebra comes the study of categories with extra structure. Categories with extra structure are categories with operations that satisfies certain equations strictly and certain equations up to natural isomorphisms. There is a plethora of such categories with extra structure: symmetric, monoidal, closed, balanced, pivotal, spherical, tortile tensor, etc. The fact that some operations on categories are equal up to a natural isomorphism demands a new dimension in functorial semantics.

The new dimension is formulated in the language of 2-categories. Whereas a category has objects: $x, y, z, \ldots$ and morphisms $f : x \longrightarrow y, f' : x \longrightarrow y, g : z \longrightarrow z', \ldots$ between the objects, a 2-category also has 2-morphisms (or 2-cells) between morphisms $\alpha : f \Longrightarrow f', \beta : g \Longrightarrow g' \ldots$. Just as an algebraic theory is a category whose objects are the natural numbers, so too, a 2-theory is a 2-category whose objects are still the natural numbers. Algebras for such a 2-theory will be product preserving 2-functors from the 2-theory to the 2-category **Cat**. Morphisms in the 2-theory will correspond to functors between products of a category e.g. $+ : 2 \longrightarrow 1$ will correspond to a functor $\oplus : \mathbf{C}^2 \longrightarrow \mathbf{C}$. 2-cells correspond to natural transformations between functors e.g. α in

$$\begin{tikzcd}
3 \arrow[r, "+\times id"] \arrow[d, "id\times +"'] & 2 \arrow[d, "+"] \\
2 \arrow[r, "+"'] \arrow[ur, Rightarrow, "\alpha"] & 1
\end{tikzcd}$$

will correspond to the associativity (iso)morphism between the two ways of associating three objects in $\mathbf{C}$.

Again, one need not take algebras only in **Cat**. One can also take algebras in any 2-category that has finite products. For example, let $\mathbf{Top_h}$ be the 2-category (actually bi-category) of topological spaces, continuous maps and homotopies between continuous maps. 2-theories can then be used to describe many structures on topological spaces like H-spaces, homotopy-everything-spaces, A_∞-spaces, etc.

A large part of two dimensional functorial semantics has already been done by the author in [26]. In this paper, many examples of 2-theories and their algebras are given. The relationship between theories and 2-theories is exploited to give many examples of each. The paper also shows how one can "combine" two 2-theories to get a new 2-theory, called the Kronecker product, such that the algebras for the new 2-theory has both structures of the old 2-theories. We also show how to reconstruct the 2-theory from the 2-category of algebras.

The core of [26] deals with coherence questions. Ever since Mac Lane's classic paper [14], coherence questions have played a major role when studying categories with additional structure. Coherence deals with the relationship between two operations on a category. Whereas when dealing with sets, two operations can either be equal or not equal, when dealing with categories, many more options exist. Between any two operations on a category, there can be no relation, there can be a morphism, there can be an isomorphism, or there can be a unique isomorphism. Much effort has been exerted to characterize when one structure can be replaced by another. These theorems have been proved in an *ad hoc* fashion. The paper [26] looks at the coherence from the 2-theory-morphism perspective. Every 2-theory-morphism G between two 2-theories induces an algebraic 2-functor G^* and its left 2-adjoint Lan_G. This left 2-adjoint is constructed using a quasi-Kan extension (a 2-categorical version of Kan extensions.) However, the left 2-adjoint can be of differing strengths. The strength of the unit, $\eta : id \longrightarrow (G^* \circ Lan_G)$, and the counit, $\varepsilon : (Lan_G \circ G^*) \longrightarrow id$, tell us to what extent one structure can be replaced by another. Whereas in 1-dimensional functorial semantics, the left adjoint is an equivalence of categories if and only if the theory-morphism is an isomorphism, in 2-dimensional functorial semantics there are many different intermediate levels of the adjunction. A 2-theory morphism can induce a weak-, quasi-, strict-, equivalence- or bi-equivalence- adjunction. Different levels of adjunction tell us how much one structure can be replaced by another structure. For example, there is an obvious 2-theory morphism from the the 2-theory of monoidal categories to the 2-theory of strict monoidal categories. This 2-theory-morphism induces an inclusion from the 2-category of strict monoidal categories into the 2-category of monoidal categories. The quasi-Kan extension assigns to every monoidal category, a strict monoidal category. The unit of this adjunction is an equivalence at each monoidal category. And so we can treat every monoidal category as if it were strict. Another example is the 2-theory-morphism from the 2-theory of braided monoidal categories to the 2-theory of symmetric monoidal categories. This 2-theory morphism induces an inclusion of symmetric monoidal categories into braided monoidal categories. The quasi-Kan extension assigns to every braided monoidal category, a symmetric monoidal category. However, in this case the unit of the adjunction is a *quasi*-equivalence and hence we can not treat every braided monoidal category as if it were symmetric.

This process gives us a recipe for solving coherence questions: one must look at the two 2-theories and at the 2-theory-morphisms between them; determine the strength of the adjunction induced by this 2-theory morphism; and "replace" one structure by the other structure to the extent that the adjunction allows. This recipe gives us a universal and organic manner of handling coherence questions. We hope that this paper would be useful for coherence theory.

3.2 From Cartesian Theories to Monoidal Theories

One is tempted to say that if $\mathbf{T}_{Monoids}$ is the theory of monoids and **AbGroups** is the category of abelian groups, then $\mathbf{Alg}(\mathbf{T_{Monoids}}, \mathbf{AbGroups})$ is the category of rings. Alas, this is false. To get the category of rings, one needs to generalize functorial semantics in a different way. A ring is an abelian group with a multiplicative structure. However the multiplication on a ring R is not given by $\bullet : R \times R \longrightarrow R$, rather it is given by $\bullet : R \otimes R \longrightarrow R$ where $\otimes$ is the tensor product of abelian groups. Similar problems occur throughout algebra. For example, a k-algebra for a given field k is a monoid in the category of k-vector spaces with respect to the tensor product (not the Cartesian product) of vector spaces. With these problems in mind, we are led to generalize the notion of an algebraic (Cartesian) theory to the notion of a monoidal theory.

A monoidal theory is a strict monoidal category whose objects are the natural numbers and whose monoidal structure on objects is simply addition i.e. $m \otimes n = m + n$. (The definition of a monoidal 2-theory is obvious.) Algebras for monoidal theories can occur in any $\mathbf{C}$ with a monoidal structure. We shall assume for simplicity sake that $\mathbf{C}$ is, in fact, a strictly symmetric monoidal (closed) structure (although leaving out this assumption gets one into some very interesting territory [28].) Algebras for a monoidal theory are no longer Cartesian product preserving functors, rather they are tensor product preserving functors (i.e. $F(n) = X^{\otimes n}$). As will be shown below, the generalization from Cartesian theories to monoidal theories brings the entire field of quantum groups under the aegis of algebraic theories.

We would like to further generalize monoidal theories by enriching the underlying category of the theory over abelian groups. That means that the hom-sets will have the structure of an abelian group. We further require that the algebras/functors preserve the abelian group structure. This would allow one to describe even more algebraic structures such as Lie algebras where the definition requires (Jacobi identity) an abelian group structure.

Presently, we are studying the following important question: to what extent can a monoidal theory be reconstructed from the category of monoidal algebras. We hope that given enough restrictions on **C** and its monoidal structure, some type of reconstruction theory will work.

3.3 From One-sorted Theories to Bi-sorted Theories

Many of the structures that are put on a category are closed. That means there is a hom functor $HOM : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathbf{C}$ with some properties. One-sorted algebraic 2-theories can not handle this structure since we do not have a way of talking about the opposite category. We can however generalize this to a bi-sorted 2-theory. The objects of a one-sorted 2-theory are the natural numbers, or equivalently, the free monoid on one generator. In contrast, the objects of a bi-sorted 2-theory form the free monoid on two generators. Let the monoid be generated by λ and ρ. A typical object then looks like $\rho^3\lambda^2\rho\lambda^3$. Algebras for a bi-sorted 2-theory will be in a 2-category **D** that has finite products and an involution $I : \mathbf{D} \longrightarrow \mathbf{D}$. Algebras for a bi-sorted 2-theory will preserve the monoid structure (i.e. $F(xy) = F(x) \times F(y)$ for $x, y \in FreeMonoid(\lambda, \rho)$ as well as $F(\lambda) = I(F(\rho))$. The example in mind is **Cat** with the involution operator taking a category to it's opposite. Bi-sorted theories will be able to handle all sorts of structures on a category.

Two-sorted theories differ from bi-sorted theories in that the objects of a two-sorted theory are the free *commutative* monoid on two generators. It seems that bi-sorted theories are the right type of theories to look at.

One can combine any all of the above generalizations and get a bi-sorted monoidal 2-theory. Surprisingly enough, such theories are found in quantum field theory.

4 Algebraic Theories in Quantum Field Theory

Graeme Segal's ground-breaking paper [18] has caused a revolution in modern quantum field theory. He defines a 2-dimensional conformal field theory (cft) as a type of functor from a category $\mathbf{T}_{conf}$ to a category of Hilbert spaces or vector spaces. The objects of $\mathbf{T}_{conf}$ are parameterized families of oriented circles. Morphisms are Riemann surfaces whose boundaries are the families of circles. $\mathbf{T}_{conf}$ has a strict monoidal closed structure given by placing one family of circles next to the other. Similarly, one Riemann surface can be placed next to another. The closure structure is given by inverting the orientation of the circles and changing the orientation of the surfaces. A cft is a monoidal closed functor from $\mathbf{T}_{conf}$ to Hilbert spaces satisfying certain additional conditions.

Our main point is that $\mathbf{T}_{conf}$ *is a type of algebraic theory. It is a monoidal bi-sorted algebraic theory.* The objects of $\mathbf{T}_{conf}$ are really elements of the free monoid generated by λ and ρ (two orientations). The Riemann surfaces are "operations" in the monoidal theory. Whereas when working in classical algebras, the usual ambient category is **Set**, here in QFT, the ambient category is Hilbert spaces. Conformal field theories are simply algebras/models of $\mathbf{T}_{conf}$ in the ambient category of Hilbert spaces. In some sense, $\mathbf{T}_{conf}$ is the "shape" or "template" for 2-dimensional cft.

Witten & Co. (see e.g. [4]) went on to describe other QFTs like topological quantum field theory $\mathbf{T}_{tqft}$. (Warning: "$\mathbf{T}$" and the last "t" both stand for theory; however they come from different perspectives.) The objects of $\mathbf{T}_{tqft}$ are the same as $\mathbf{T}_{conf}$. The morphisms, however, do not have Riemann structures. They are simply smooth topological spaces. $\mathbf{T}_{tqft}$ is also a bisorted monoidal algebraic theory.

Others have gone on to study different types of QFTs. Eliashberg & Co. study symplectic quantum field theory. J. Morava [17] studies topological gravity (the morphisms are cobordisms with with diffeomorphism-induced equivalence classes of Riemannian metrics.) Many researchers (Morava, Tillman) have thought of cobordism categories as a 2-category. The 2-cells are diffeomorphisms or isotopies between the 1-cells that preserve the 0-cells. These can also be thought of as generalized algebraic theories. They are types of bi-sorted monoidal **2**-theories.

All this is for 2-dimensional QFTs. We can however talk of d-dimensional QFTs where the objects comprise the free monoid generated by the closed d-dimensional spaces (with structure, if needed) and the morphisms are d+1-dimensional cobordisms between them (and perhaps 2-cells between those). These are also generalized algebraic theories i.e. multi-sorted monoidal (2-)theories.

There are many different types of cobordism categories in low-dimensional topology. **Braids**, **Links**, **Ribbons**, **Tangles** etc are all examples of such categories of cobordisms. Such categories can all be seen as generalized algebraic theories for 0+1-dimensional QFTs.

Now that we set up the analogies of QFT and universal algebra, the questions simply flow forth. There are many different functors between these 2-theories. For example there are forgetful functors $U : \mathbf{T}_{conf} \longrightarrow \mathbf{T}_{tqft}$ and $U' : \mathbf{T}_{syplec} \longrightarrow \mathbf{T}_{tqft}$. What type of coherence results fall out of such 2-theory-morphisms? What does the "free" tqft for a given cft look like? Via precomposition, U induces a functor from the category of topological quantum field theories to the category of conformal field theories. Does this functor have a (quasi-)left adjoint? What can we say about the (quasi-)adjoint functors induced from the inclusion of the d-dimensional tqft into the $d+1$-dimensional tqft? What are the universal properties of these theories with respect to each other? What type of restrictions can be put on the cobordism categories that can make the adjoints stronger? Questions can also be asked for the low-dimensional topology cobordisms. For example **Links** inject into **Ribbons**. What can we say about the relationship between the invariants described by these two categories? These questions could and should be asked in the language of generalized functorial semantics. Our goal is to answer them in this language as well.

4.1 Tannaka Duality and Theory Reconstruction

Tannaka duality is about the relationship between a category of (co)modules of an algebra and the original algebra. This result is related to Yetter's [29] monoidal reconstruction theory which deals with many different types of cobordism categories. We conjecture that Tannaka duality's construction of a algebra from its category of modules is nothing more then a souped-up version of a theory reconstruction from a category of algebras.

One can see why we think this conjecture is true by looking at the category of R-modules for some ring R. In classical functorial semantics, the theory of R-modules is given as the theory of abelian groups with the addition of a morphism $r : 1 \longrightarrow 1$ for every $r \in R$. Following sections 2.3, if one wanted to reconstruct just the ring R (rather then the whole theory of R-modules) one simply looks at the natural transformation $NAT(U^1, U^1)$ where U is the forgetful functor from the category or R-modules to **Set**. In other words, $R \approx NAT(U^1, U^1) = End(U)$. Endomorphisms of the forgetful functor constitute the central point of Tannaka Duality.

5 Algebraic Theories in Quantum Algebra

Quantum algebra is a loosely connected family of fields that study certain algebraic structures that were inspired by questions in modern quantum physics. Quantum groups were created to study the quantum inverse scattering method. Quasi-Hopf algebras help us with the Knizhnik-Zamolodchikov equations. Homotopy Lie algebras and Homotopy Gerstenhaber Algebras have been found to be deeply connected to string theory.

Our goal is to formalize these structures in generalized functorial semantics. A quantum group is not, in fact, a group. That is, it is not a group in any category. However when one generalizes the notion of a group to be *monoidal* preserving functor from $\mathbf{T}_{Groups}$ then quantum groups are a group in the category of coassociative coalgebras. Notice that in this category, the Cartesian product is not the same as the tensor product (we need cocommutative coalgebras for that). Many of the exotic structures in QA can be described by monoidal theories.

Many homotopy structures (e.g. $A_\infty, G_\infty, L_\infty$ etc) can be described by chain theories, that is, theories whose morphisms are chain complexes. These might also be called dg theories. Algebras should be taken in the category of chain complexes. Chain theories are a special type of monoidal theory. Once we formalize them in the language of monoidal theories, we will better be able to work with them and their interactions with other structures.

There are many constructions in the field of quantum groups that we would like to formalize and characterize. Drinfeld's quantum double construction (see e.g. [8] and [16]) shows how to go from a finite dimensional Hopf algebra H to a braided Hopf algebra $D(H)$. $D(H)$ is used extensively to find invariants in low-dimensional topology. We conjecture that $D(\quad)$ can be seen as a quasi-Kan extension (see section 2.2) along the inclusion monoidal-theory-functor from the theory of Hopf algebras to the theory of braided Hopf algebras. Whereas we doubt that this monoidal-theory-functor induces a strict adjunction, there must be some type of quasi-adjunction that can be stated. Upon finding this quasi-adjunction, we would be able to find the universal properties of $D(H)$ and hence the universal properties of the invariants created by $D(H)$. Street [19] has similar constructions to create cobraided bialgebras, cobalenced bialgebras and cotortile Hopf algebras. We would also like to analyze these constructions from the monoidal-theory-functor point of view.

In addition to these constructions on Hopf algebras (or bialgebras), there are constructions that mimic the above constructions from the representation theoretic view. Let H be a Hopf algebra. Let $Comod_f(H)$ denote the category of finite-dimensional comodules on H. $Comod_f(H)$ has a monoidal structure. Given a monoidal category M, one can construct the center of M, denoted $Z(M)$ (see [8], [7]) which is a braided tensor category. $Z(Comod_f(H))$ is equivalent to $Comod_f(D(H))$ as a braided monoidal category. We would like to look at these constructions from the 2-theoretic point of view. While $Z(\quad)$ and the forgetful functor from braided monoidal categories to monoidal categories are not strict adjoint, it is conjectured that they are *quasi*-adjoint. Street [19] has similar constructions to find the balanced and tortile category "over" a monoidal category. Our aim would be to find the universal properties of these constructions.

5.1 Operads Vs. Algebraic Theories

Researchers have until now used operads to describe the algebraic structures in QA. An operad is a sequence $\mathcal{O}(0), \mathcal{O}(1), \ldots, \mathcal{O}(n), \ldots$ where $\mathcal{O}(n)$ is a vector space (set, topological space, category, etc) that describes the n-ary operations. These $\mathcal{O}(i)$ are related to each other in a complicated way. We would like to present algebraic theories as an alternative or complement to operads. The relationship between operads and algebraic theories was dealt with years ago by J. Frank Adams. In his "Infinite Loop Spaces" [1], Adams shows that topological operads (operads where each $\mathcal{O}(n)$ is a topological space) are equivalent to topological algebraic theories (denoted "PROPs" by Mac Lane.) In [25], we followed Adams in showing that a categorical operad is equivalent to an algebraic 2-theory. Basically, given an algebraic theory, $\mathbf{T}$, one gets an operad by setting $\mathcal{O}(n) = Hom_{\mathbf{T}}(n, 1)$. Conversely, Given an operad, $\mathcal{O}$, one builds an algebraic theory by writing $Hom_{\mathbf{T}}(m, n)$ as a "disjoint union" of n objects from $\mathcal{O}(m)$. We conjecture that with enough provisos, one can write every operad as a monoidal algebraic theory. The converse of this is, in fact, false. Hence, monoidal algebraic theories are more powerful than operads.

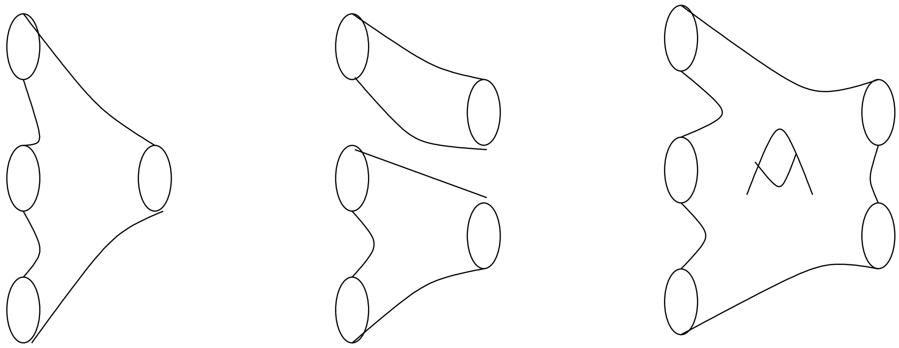
There are many reasons why we believe that algebraic theories are better suited than operads for discussing QA structures.

1. Algebraic theories deal well with co-operations as well as operations. A morphism $n \longrightarrow 1$ is an n-ary operations and a morphism $1 \longrightarrow n$ is an n-ary co-operation. Since algebraic theories employ both operations and co-operations at one time, they can easily deal with bialgebras, Hopf algebras, Drinfeld algebras etc. This would negate the necessity of introducing cooperads.

2. Algebraic theories can handle "poly-operations" (operations from n to m where $m \geq 2$) that can not be decomposed into regular operations. When dealing with Cartesian operations, there seems to be nothing gained by this since $Hom(n, m) \simeq Hom(n, 1)^m$. However, when dealing with the tensor product, one can easily find some examples of a "poly-operation" from $A \otimes A$ to $A \otimes A$ that can not be decomposed into two operations.

3. Algebraic theories are more transparent then operads. One can give the definition of an algebraic theory in a few words whereas the definition of an operad is a few pages of technical diagrams and discussions.

4. Using algebraic theories, one can make the connection with QFT discussed in section 4. This can not be done with operads as the following three cobordisms show:



(a) (b) (c)

(a) is described by an element in $\mathcal{O}(3)$. (b) can be described by an element in $\mathcal{O}(1)$ tensored with an element of $\mathcal{O}(2)$. However, even though (c) is a legitimate cobordism, it can not be described by an operad. It can, however, be described by an algebraic theory. It is simply an element of $Hom_{\mathbf{T}}(3, 2)$ for some cobordism theory.

5.2 Deformation Theory

The study of algebraic structures themselves rather then their algebras/models really takes off with deformation theory. In Maxim Kontesevich's celebrated work on deformation quantization of Poisson manifolds, a central formality theorem is proven. This theorem says that the small disk operad is homotopy equivalent to its cohomology. We would like to outline some directions that we would like to follow for studying deformation theory from the view of generalized functorial semantics. Due to space considerations, we address this subsection to the cognoscenti of deformation theory. We use the language and notation of Kontsevich's restatement [9] of Tamarkin's reproof [20] of Kontsevich's theorem.

1. One of the central notions of deformation theory is the Hochschild complex. For an associative algebra, the Hochschild complex has a cup product and a Gerstenhaber bracket that has different properties and relations. Gerstenhaber already looked at bialgebras and found that the Hochschild complex is a bicomplex with much structure. Markl, Shnider and Sternberg have looked at the Hochschild complex of Drinfeld algebras, Majid algebras, quantum groups etc. We would like to extend these results and functorially describe the structure of the Hochschild complex for every type of algebra given by an algebraic theory. Our aim is to understand the relationship between an algebraic structure and its associated Hochschild complex.

2. The object of interest for deformation theory is the algebra of functions on a smooth manifold. This is an associative algebra. If we formalize the constructions of deformation theory from the universal algebraic point of view, we can extend deformation theory to Hopf algebras, quantum groups, non-associative algebras and noncommutative algebras etc.

3. One of the more mysterious aspects of deformation theory is the concept of a d-algebra from the generalized Deligne conjecture. A 0-algebra is a complex. A 1-algebra is an A_∞-algebra. The Hochschild complex of an associative algebra is a 2-algebra. The definition of a d-algebra is given but seems hard to "get a handle on". It looks as if the notion of a d-algebra can be given inductively as the Kronecker product of generalized algebraic theories. The Kronecker product [5] of algebraic theories are a way of combining two algebraic structures. Given two theories $\mathbf{T}_1$ and $\mathbf{T}_2$, the Kronecker product of them is denoted $\mathbf{T}_1 \otimes \mathbf{T}_2$ and has the following universal property:

$$\mathbf{Alg}(\mathbf{T_1} \otimes \mathbf{T_2}, \mathbf{Set}) \quad \simeq \quad \mathbf{Alg}(\mathbf{T_1}, \mathbf{Alg}(\mathbf{T_2}, \mathbf{Set})).$$

Category theory has given us a way of constructing the Kronecker product of two algebraic theories. In [26] we generalized the the construction to 2-theories. We would like to generalize the constructions to chain-theories. We conjecture that the theory of d+1-algebras can be constructed as follows:

$$\mathbf{T}_{d+1-algebras} \quad \simeq \quad \mathbf{T}_{d-algebras} \otimes \mathbf{T}_{1-algebras}.$$

4. The swiss-cheese algebraic theory (operad see Voronov's [22]) seems to be the result of the Kronecker product of the small interval algebraic theory (operad) and the small disk algebraic theory (operad).

5. We would like to place a closed Quillen model structure on the category of **ChainTheories**. The weak equivalences are quasi-isomorphisms that is morphisms that induce an isomorphism in homology. We believe that a large part of deformation theory can be stated in this generality.

6. What exactly is the relationship between an X_∞-algebra and an X-algebra? Is it similar to "coherefication" of [26]?

6 Connections and Conclusions

Throughout this proposal we have really been discussing many different types of algebraic theories. In general an algebraic theory is a strict monoidal category whose objects are a free monoid and whose monoidal structure is given by the monoid multiplication. The morphisms can be many different things: sets, categories, chain complexes etc. Every type of morphism determines the type of theory. 1-theories, 2-theories, monoidal-theories, monoidal-2-theories chain-complex-theories, abelian-theories etc. All these different theories and their respective theory-morphisms and theory-natural transformations etc form (1,2,3-) categories called **1Theories**, **2Theories**, **MonoidalTheories**, **Monoidal2Theories**, **ChainTheories**, **AbelianTheories** etc. There are theories where the morphisms are topological spaces called topological theories or PROPs. They form **TopTheories**. All the cobordism theories that were discussed within the context of QFT form a category **CobordismTheories**. There are, of course, many more theory types not listed. For each theory type there is a way of getting a category of algebras: product (monoidal, abelian, continuous etc) preserving functors (2-functors, closed etc).

However these theory types do not stand alone. There are many functors (2-functors, 3-functors, monoidal functors etc) between them that are used throughout QFT and QA. We shall list a few examples.

1. There is an inclusion of **1Theories** into **MonoidalTheories** since a Cartesian product is a type of monoidal product. Similarly for **2Theories** and **Monoidal2Theories**.

2. In [26] the following relationship between the (2-)category of **1Theories** and the (3-)category of **2Theories** was discussed. Analogous to the relationship between sets and topological spaces, we have the following adjunctions:

$$\begin{tikzcd}[column sep=huge]
\mathbf{1Theories} \arrow[r, shift left=1.5, "d"] \arrow[r, bend right=40, "c"'] \arrow[r, phantom, shift left=5, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=5, "\perp"] & \mathbf{2Theories}. \arrow[l, bend right=40, "\pi_0"'] \arrow[l, shift left=1.5, "U"]
\end{tikzcd}$$

$c(T)$ is the 2-theory with the same 1-cells as T and a unique 2-cell between nontrivial 1-cells. $d(T)$ has the same 1-cells as T and only trivial 2-cells. $U(\mathbf{T})$ forgets the 2-cells of $\mathbf{T}$. $\pi_0(\mathbf{T})$ is a quotient theory of $\mathbf{T}$ where two 1-cells are set equal if there is a 2-cell between them. The units and counits of these adjunctions are of interest. $\mu : \mathbf{T} \longrightarrow d\pi_0\mathbf{T}$ is the 2-theory-morphism corresponding to "strictification": every 2-cell becomes the identity. Similarly, $\mu : \mathbf{T} \longrightarrow cU\mathbf{T}$ might be called "coherification": a 2-theory is forced to be coherent. $\varepsilon : dU\mathbf{T} \longrightarrow \mathbf{T}$ is the injection of the 1-theory into the 2-theory. There should be many other types of adjunctions similar to this for other types of theories.

3. The homology functor can be extended to $H_* :$ **TopTheories** $\longrightarrow$ **ChainTheories**. For example the small discs theory goes to the homotopy

Gerstenhaber algebra theory and the small interval theory goes to the A_∞ theory.

4. One of the main ideas in quantum groups is that the structure of an algebra (coalgebras, bialgebra, Hopf algebra, quasi-Hopf quasi triangular algebra etc) A is reflected in the structure of the category of modules (comodules, bimodules, bicrossed modules etc) of A. Hence there is a functor Rep : **MonoidalTheories** $\longrightarrow$ **2Theories** that takes a monoidal-theory $\mathbf{T}$ to the 2-theory of the structure of its category of (co)modules for an algebra of T. For example, if A is an old-fashioned algebra, then the category of modules is simply a category. If we add a coassociative comultiplication to A, then the category of modules inherits a strict monoidal structure. If the algebra has an involution (R-matrix, Drinfeld weak comultiplicaton structure, etc) then the category of modules will have duality (braiding, monoidal structure, etc). We would like to formalize this functor. Can we describe this functor in a universal way without talking about each structure individually? Is there some type of inverse of this functor? Do we really gain anything by going from the algebra with structure to the category with extra structure? Or can every theorem about categories with extra structure be understood on the algebra with structure level?

5. To every cobordism theory, David Yetter's monoidal reconstruction theory [29] relates a different type of algebra. Can this be formalized as a functor from from **CobordismTheories** to **MonoidalTheories**? Universal properties? What is the relation of this with Tannaka duality?

6. Many of the constructions that happen within one theory type are mimicked in other theory types. For example: a) There should be a Kronecker product in **ChainTheories** and in **TopTheories**. Homology (see 3.) should preserve Kronecker product. b) The quantum double construction of a Hopf algebra is induced by a map in **MonoidalTheories**. The center construction on a monoidal category is induced by a morphism in **2Theories**. Rep should somehow preserve this morphism. c) There are different versions of the bar constructions that should be preserved by these functors.

We close with a few broad statements about our goal. Whereas ring theorists must deal with rings and group theorists must deal with groups, people who study QFT and QA must deal with many different types of algebraic structures. Universal algebra is the study of all algebraic structures. There is a definite need for a study of universal algebra with an eye towards QFT and QA to understand and classify these structures.

The power of category theory is its ambiguity. Whereas number theory deals with numbers and topology deals with topological spaces, category theory deals with everything. The wide variety of different types of structures that are arise in QFT and QA can not be handled without the language and tools of category theory. Our goal of studying the structures of QFT and QA with a categorical version of universal algebra promises to be a significant and enlightening endeavor.

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