A Nonstandard Theory of Topological Groups

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We shall take the theory of topological groups in [3] and show how certain structures receive natural nonstandard characterizations. We shall assume familiarity with [3].

Let $\mathcal{G}$ be an abstract group. Let X be a transitive set containing $\mathcal{G}$. Put $A = X \cup P(X) \cup P(P(X)) \cdots$ [where $P(X)$ = the power set of X]. If $\{a_i\}$ is an enumeration of A, let $\mathfrak{M}$ be the structure $\langle A, \epsilon, \{a_i\}\rangle$. Let $\mathfrak{M}^* = \langle A^*, e^*, \{a_i\}\rangle$ be an enlargement of $\mathfrak{M}$ in the sense of Robinson. Most of our results will be results about such an $\mathfrak{M}^*$. Notice that we do not need to include any group structure or topology on $\mathcal{G}$ explicitly, as it comes with $\mathfrak{M}$. If we consider two or more groups we shall assume they are simultaneously embedded in the transitive set X.

Occasionally we shall need to assume that $\mathfrak{M}^*$ is κ-saturated for a suitable cardinal κ. The situation is even more interesting if we assume that $\mathfrak{M}^*$ is saturated because any two structures definable in $\mathfrak{M}^*$, if elementarily equivalent, will be saturated and of the same power, and hence isomorphic. Generally, such an isomorphism will not be in the model $\mathfrak{M}^*$.

If we are assuming the axiom of choice, then assuming the existence of a saturated extension $\mathfrak{M}^*$ of $\mathfrak{M}$ does not involve any additional loss of generality. For consider the sets constructible in $\mathfrak{M}$. They form a model of $ZF + AC$ in which $\mathfrak{M}$ and quantifiers over $\mathfrak{M}$ retain their usual meaning but in which the G.C.H. holds for cardinals larger than $\overline{\overline{A}}$. Hence there are plenty of saturated structures, elementarily extending $\mathfrak{M}$.

Since the structure $\mathfrak{M}^*$ is not a standard model of set theory, its elements are not, intuitively, sets. Let us agree that the elementary embedding $\mathfrak{M} \to \mathfrak{M}^*$ is the identity map on A. Then for $a \in A$ the symbol a will denote both the element $a \in A^*$ and the subset $\{x \mid x \in A^*$

and $\mathfrak{M} \models x \in a\}$ of A^*. The symbol a^* will denote the set $\{x \mid \mathfrak{M}^* \models x \in a\}$. If a is infinite and $\mathfrak{M}^*$ is κ-saturated, $\kappa > \overline{\overline{A}}$, then $\overline{\overline{a^*}}$ will be at least κ and hence the subset $a \subseteq A^*$ will never be of the form b^* for any $b \in A$. In other words, all infinite subsets of A are external in $\mathfrak{M}^*$.

We shall make no distinction between ϵ and ϵ^*.

Suppose now that $\mathcal{G}$, x, e is a group. Then $\mathcal{G}^*$, x^*, e will also be a group elementarily extending $\mathcal{G}$; $\mathcal{G} \prec \mathcal{G}^*$. Now suppose $\mathfrak{F}$ a family of subsets of $\mathcal{G}$ such that $A \in \mathfrak{F} \to e \in A$. Define:

$$J_1 = \bigcap A^*: A \in \mathfrak{F},$$

$$J_0 = \text{subgroup of } \mathcal{G}^* \text{ generated by } J_1 \text{ and } \mathcal{G},$$

$$\mathfrak{I} = \{A: A \subseteq \mathcal{G} \text{ and } p \in A \to J_1 \cdot p \subseteq A^*\}.$$

***Theorem* 1.** *$\mathcal{G}$, $\mathfrak{I}$ is a topological group iff J_1 is a normal subgroup of J_0. The topology is Hausdorff iff*

$$J_1 \cap \mathcal{G} = \{e\}.$$

Moreover, $J_1 = \mu(e)$.

Proof. It is clear from the definition of $\mathfrak{I}$ that $\phi, \mathcal{G} \in \mathfrak{I}$ and $\mathfrak{I}$ is closed under finite intersections. Suppose now that, for each α, $A_\alpha \in \mathfrak{I}$. Suppose $p \in \bigcup A_\alpha$. Then there is a β such that $p \in A_\beta$. Hence $J_1 \cdot p \subseteq A_\beta^* \subseteq (\bigcup A_\alpha)^*$. Thus $\mathfrak{I}$ is a topology on $\mathcal{G}$.

We claim that if $A \in \mathfrak{F}$, then $e \in \text{int}(A)$.

For let $\mathcal{K}$ consist of finite intersections of members of $\mathfrak{F}$. Then for each $A \in \mathcal{K}$ there is a $B \in \mathcal{K}$ such that $B \cdot B \subseteq A$. Otherwise the formulas $p \in B$, $q \in B$, $p \cdot q \notin A$, with B ranging over $\mathcal{K}$, would be finitely satisfiable in $\mathfrak{M}$ and hence simultaneously satisfiable in $\mathfrak{M}^*$ by p, $q \in J_1$ such that $pq \notin A^*$. But this contradicts $pq \in J_1 \cdot J_1 \subseteq J_1 \subseteq A^*$.

Now, for $A_0 \in \mathcal{K}$, choose $A_1, A_2, \cdots \in \mathcal{K}$ such that $A_{i+1}{}^2 \subseteq A_i$. Then if $V = \bigcup_{n=1}^{\infty} A_1 \cdot A_2 \cdots A_n$, then $V \subseteq A_0$, and if $p \in V$, then $J_1 \cdot p = p \cdot J_1 \subseteq V^*$. Thus V is open in $\mathfrak{I}$ and $e \in V \subseteq A = A_0$, so $e \in \text{int}(A)$.

This gives $J_1 = \mu(e)$. For we have

$$\mu(e) = \bigcap A^*: e \in \text{int}(A) \subseteq \bigcap A^*: A \in \mathfrak{F} = J_1.$$

But if $e \in A$ and $A \in \mathfrak{I}$, then $J_1 \subseteq A^*$. Hence

$$J_1 \subseteq \bigcap A^*: A \in \mathfrak{I} \quad \text{and} \quad e \in A = \mu(e).$$

Thus $J_1 = \mu(e)$. Since J_1 is a normal subgroup of J_0, multiplication will be continuous at all standard points iff it is so at e. But let U be any standard neighborhood of e. Then $U \supseteq J_1$. Hence $U \supseteq J_1 \cdot J_1^{-1}$. But J_1 contains an open (nonstandard) neighborhood of e. Hence $\mathfrak{M}^* \models$

there are open neighborhoods V, W of e such that $V \cdot W^{-1} \subseteq U$. Hence $\mathfrak{M} \models$ there are $\cdots \subseteq U$. Thus $\mathcal{G}$, $\mathfrak{T}$ is a topological group.

The topology is Hausdorff iff monads are pairwise disjoint iff e is the only standard point in its own monad J_1.

The fact that these conditions are necessary was proved by Robinson.

***Theorem* 2.** *Given $\mathcal{G}$, $\mathfrak{T}$ a topological group. Then $\mathcal{G}$, $\mathfrak{T}$ is locally compact iff there is a neighborhood U of e such that $U^* \subseteq J_0$.*

Proof. Trivial. J_0 are precisely the near-standard elements of $\mathcal{G}^*$.

1. Local Isomorphism

Let $\mathcal{G}, e, \mathcal{G}', e'$, be two topological groups. They are locally isomorphic if there is a map f from a neighborhood U of e to a neighborhood V of e', which preserves products and inverses and is a homeomorphism between U and V.

***Theorem* 3.** *$\mathcal{G}, e, \mathcal{G}', e'$ are locally isomorphic iff there is a standard one-to-one function f which maps $\mu(e)$ isomorphically onto $\mu(e')$.*

Proof. Clearly, if a local isomorphism f exists between $\mathcal{G}$ and $\mathcal{G}'$, then it must create a one-to-one correspondence between small neighborhoods of e and e'. Thus

$$\begin{aligned} f[\mu(e)] &= \bigcap f(U)^*\colon U \text{ standard, open, } e \in U, \\ &= \bigcap V^*\colon V \text{ standard, open, } e' \in V, \\ &= \mu(e'). \end{aligned}$$

Conversely, suppose there is a standard, one-to-one function f mapping $\mu(e)$ isomorphically onto $\mu(e')$. Then the set A defined by

$$A = \{(p, q)\colon f(p), f(q), f(pq), f(qp), f(p^{-1}), f(q^{-1}) \text{ are defined and } f(pq) = f(p)f(q) \cdots\}$$

is a standard subset of $\mathcal{G}^* \times \mathcal{G}^*$ containing $\mu(e) \times \mu(e)$ and hence must contain a standard neighborhood $V \times V$ of (e, e) in the product topology. Otherwise the formula $(p, q) \notin A$ and the formulas $(p, q) \in V \times V$: V ranging over standard neighborhoods of e are finitely satisfiable and hence simultaneously satisfiable, giving $(p, q) \in \mu(e) \times \mu(e) - A$. Restrict f to V.

$f[V]$ is a standard set containing $\mu(e')$ and is therefore a neighborhood of e'. To show now that f is a local isomorphism suppose $U_1 \subseteq V$ is a standard open set and $p \in U_1$. Then U_1p^{-1} is a neighborhood of e and by the argument above

$$f[U_1p^{-1}] = (f[U_1] \cdot f(p)^{-1}) \cap f[V]$$

is a neighborhood of e' and $f[U_1]$ is a neighborhood of $f(p)$. Thus $f[U_1]$ is a neighborhood of each of its points, and is therefore open.

Thus f is open. By a symmetrical argument f is continuous. Thus f is the required local isomorphism.

Remark. The requirement in the theorem above that f be standard cannot be dropped. For suppose $\mu(e)$, $\mu(e')$ are elementarily equivalent and $\mathfrak{M}^*$ is saturated. Now consider a model $\mathfrak{M}_1$ of the same power as $\mathfrak{M}$ and such that $\mathfrak{M} \prec \mathfrak{M}_1 \prec \mathfrak{M}^*$ and $\langle \mathfrak{M}_1, \mu_{\mathfrak{M}_1}(e)\rangle \prec \langle \mathfrak{M}^*, \mu_{\mathfrak{M}^*}(e)\rangle$. Such a model exists by the downward Löwenheim–Skolem theorem. Now take a saturated extension of $\langle \mathfrak{M}_1, \mu_{\mathfrak{M}_1}(e)\rangle$ of the same power as $\mathfrak{M}^*$. Let it be $\langle \mathfrak{M}_2^*, \mu_{\mathfrak{M}_2}(e)\rangle$. Then $\mathfrak{M}_2^*$, $\mathfrak{M}^*$ being saturated of the same power, are isomorphic, with an isomorphism that preserves $\mathfrak{M}$ and hence $\mu(e)$. Thus $\mu(e)$ is saturated in $\mathfrak{M}^*$, even though it is not definable in it. But now the $\mu(e)$, $\mu(e')$ above must be isomorphic. This gives at most $2^{\aleph_0}$ isomorphism types. However, the number of local isomorphism types for topological groups is much larger.

2. Construction of the Dual Group

Let $\mathcal{G}$, 0 be an Abelian topological group, $\mathcal{H}$, $0'$ a connected Abelian group. Robinson has given a nonstandard proof that if U is a neighborhood of 0 in $\mathcal{H}$, then $\bigcup_{n \in N} U^n = \mathcal{H}$.

Consider $\mathrm{Hom}(\mathcal{G}, \mathcal{H})$ = all continuous homomorphisms from $\mathcal{G}$ to $\mathcal{H}$. These form a group in the obvious way, letting

$$(f + g)(x) = f(x) + g(x).$$

The constant function $0'$ is the identity in $\mathrm{Hom}(\mathcal{G}, \mathcal{H})$, denoted O.

For the topology we take the following to be the neighborhoods of O. Given a compact $C \subseteq \mathcal{G}$ and an open $U \subseteq \mathcal{H}$, $0' \in U$, the corresponding neighborhood of O is the set $\{f: f[C] \subseteq U\}$.

With this definition $\mu(O)$ will be $\{f: f[J_0] \subseteq \mu(0')\}$. By Theorem 1, $\mathrm{Hom}(\mathcal{G}, \mathcal{H})$ is a topological group.

Suppose now that $\mathcal{H}$ = the complex numbers of modulus 1, under multiplication and $\mathcal{G}$, is locally compact. To show that $\mathrm{Hom}(\mathcal{G}, \mathcal{H})$ is locally compact, notice first that every element of $\mathcal{H}^*$ is near-standard. Let U be a symmetric neighborhood of 1 in $\mathcal{H}$, $\bar{U} \subsetneq \mathcal{H}$, C a compact neighborhood of 0 in $\mathcal{G}$. Then let

$$A^* = \{f : f(C) \subseteq U\}^*$$
$$= \{f : f(C^*) \subseteq U^*\}.\ A^* \text{ is a neighborhood of } 0.$$

If $f \in A^*$, let $g = {}^0f$. We have to show g continuous. First $g(C) \subseteq \bar{U} \subsetneq \mathcal{K}$.

Suppose W, a symmetric neighborhood of $0'$, $= 1$. Then $\exists n$, $W^n \supseteq U$. Choose U_1 such that $U_1{}^n \subseteq C$. Then

$$g(U_1) \subseteq W, \text{ otherwise}$$
$$g(U_1{}^n) = (g(U_1))^n \text{ would have points not in } \bar{U}.$$

Hence g is continuous.

Now $f - g^*$ clearly takes J_0 in $\mu(0') = \mu(1)$. So f is near-standard. Hence $\mathrm{Hom}(\mathcal{G}, \mathcal{K})$ is locally compact.

From now on we write $\mathcal{G}' = \mathrm{Hom}(\mathcal{G}, \mathcal{K})$, where $\mathcal{K}$ = complex numbers of modulus 1.

***Theorem* 4.** *If $\mathcal{G}$ is compact, $\mathcal{G}'$ is discrete; if $\mathcal{G}$ is discrete, $\mathcal{G}'$ is compact.*

Proof. Suppose $\mathcal{G}$ is compact. We have to show that $\mu(O)$ in $\mathcal{G}'^*$ is just O. But

$$\mu(O) = \{f : f(J_0) \subseteq \mu(1)\}$$
$$= \{f : f(\mathcal{G}^*) \subseteq \mu(1)\}$$

Suppose now $f(x) \neq 1$ for some x. Then there is an $n \in N^*$, $f(x^n) \not\subseteq \mu(1)$. Hence $f \not\subseteq \mu(O)$.

Suppose $\mathcal{G}$ is discrete. To show that every $f \in \mathcal{G}'^*$ is near-standard, given f take $g = {}^0f$. g is automatically continuous. Then $f - g^* \in \mu(O)$. Hence f is near-standard.

3. Construction of the Haar Measure

Let $\mathcal{G}$, $\mathfrak{I}$ be a locally compact group. Let $V \subseteq \mu(e)$ be an infinitesimal neighborhood of e. Fix a compact standard set C and given any compact, standard, set C' define

$$m(C') = {}^0\left[\frac{l(C')}{l(C)}\right],$$

where $l(A)$ = least member of N^* such that there is an internal covering of A^* by l sets of the form $V^* \cdot p$.

If C has interior, then it follows from the Heine–Borel property that $l(C')/l(C)$ is finite and $m(C')$ is defined.

***Theorem* 5.** (a) $m(A \cdot p) = A$.

(b) *If A, B are disjoint,* $m(A \cup B) = m(A) + m(B)$.

Proof

(a) is obvious from definition.

(b) follows from the fact that a set of the form $V^* \cdot p$ cannot simultaneously intersect A^* and B^*, otherwise ${}^0p$ would exist and be in $A \cap B$.

Suppose now that $\mu(e)$ is a normal subgroup of $\mathcal{G}^*$. Then e has small neighborhoods invariant under inner automorphisms. In fact, given a standard neighborhood U,

$$\bigcap_{p \in \mathcal{G}^*} p^{-1} \cdot U^* \cdot p \supseteq \bigcap_{p \in \mathcal{G}^*} p^{-1} \cdot \mu(e) \cdot p \supseteq \mu(e).$$

Hence

$$\bigcap_{p \in \mathcal{G}^*} p^{-1} \cdot U^* \cdot p \text{ is an } \mathfrak{M}^*\text{-neighborhood of } e.$$

But

$$\bigcap_{p \in \mathcal{G}^*} p^{-1} \cdot U^* \cdot p = \left(\bigcap_{p \in \mathcal{G}} p^{-1} \cdot U \cdot p\right)^*$$

so

$$\bigcap_{p \in \mathcal{G}} p^{-1} \cdot U \cdot p \text{ is an } \mathfrak{M}\text{-neighborhood of } e.$$

Thus e has arbitrarily small neighborhoods invariant under inner automorphisms. Take an infinitesimal one, V. The corresponding measure is right-invariant. To see left-invariance note that

$$l(p \cdot A) = l(p \cdot A \cdot p^{-1} \cdot p) = l(p \cdot A \cdot p^{-1}) = l(A)$$

using the invariance of V in the last step. Thus we get

$$m(p \cdot A) = m(A)$$

and

***Theorem* 6.** *If $\mu(e)$ is a normal subgroup of $\mathcal{G}^*$, then $\mathcal{G}$ is unimodular.*

The converse is also true. We leave the proof to the reader.

References

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