Paradoxes, Contradictions, and the Limits of Science

Many research results define boundaries of what cannot be known, predicted, or described. Classifying these limitations shows us the structure of science and reason.

Noson S. Yanofsky

Science and technology have always amazed us with their powers and ability to transform our world and our lives. However, many results, particularly over the past century or so, have demonstrated that there are limits to the abilities of science. Some of the most celebrated ideas in all of science, such as aspects of quantum mechanics and chaos theory, have implications for informing scientists about what cannot be done. Researchers have discovered boundaries beyond which science cannot go and, in a sense, science has found its limitations. Although these results are found in many different fields and areas of science, mathematics, and logic, they can be grouped and classified into four types of limitations. By closely examining these classifications and the way that these limitations are found, we can learn much about the very structure of science.

Discovering Limitations

The various ways that some of these limitations are discovered is in itself informative. One of the more interesting means of discovering a scientific limitation is through paradoxes. The word *paradox* is used in various ways and has several meanings. For our purposes, a paradox is present when an assumption is made and then, with valid reasoning, a contradiction or falsity is derived. We can write this as:

Assumption→Contradiction.

Because contradictions and falsehoods need to be avoided, and because only valid reasoning was employed, it must be that the assumption was incorrect. In a sense, a paradox is a proof that the assumption is not a valid part of reason. If it were, in fact, a valid part of reason, then no contradiction or falsehood could have been derived.

A classic example of a paradox is a cute little puzzle called the *barber paradox*. It concerns a small, isolated village with a single barber. The village has the following strict rule: If you cut your own hair, you cannot go to the barber, and if you go to the barber, you cannot cut your own hair. It is one or the other, but not both. Now, pose the simple question: Who cuts the barber's hair? If the barber cuts his own hair, then he is not permitted to go to the barber. But he is the barber! If, on the other hand, he goes to the barber, then he is cutting his own hair. This outcome is a contradiction. We might express this paradox as:

Village with rule→Contradiction.

The resolution to the barber paradox is rather simple: The village with this strict rule does not exist. It cannot exist because it would cause a contradiction. There are a lot of ways of getting around the rule: The barber could be bald, or an itinerant barber could come to the village every few months, or the wife of the barber could cut the barber's hair. But all these are violations of the rule. The main point is that the physical universe cannot have such a village with such a rule. Such playful paradox games may seem superficial, but they are transparent ways of exploring logical contradictions that can exist in the physical world, where disobeying the rules is not an option.

A special type of paradox is called a *self-referential paradox*, which results from something referring to itself. The classic example of a self-referential paradox is the *liar paradox*. Consider the sentence, "This sentence is false." If it is true, then it is false, and if it is false, then because it says it is false, it is true—a clear contradiction. This paradox arises because the sentence refers to itself. Whenever there is a system in which some of its parts can refer to themselves, there will be self-reference. These parts might be able to negate some aspect of themselves, resulting in a contradiction. Mathematics, sets, computers, quantum mechanics, and several other systems possess such self-reference, and hence have associated limitations.

Some of the stranger aspects of quantum mechanics can be seen as coming from self-reference. For example, take the dual nature of light. One can perform experiments in which light acts like a wave, and others in which it acts like a particle. So which is it? The answer is that the nature of light depends upon which experiment is performed. Was a wave experiment performed, or was a particle experiment performed? This duality ushers a whole new dimension into science. In classical science, the subject of an experiment is a closed system that researchers poke and prod in order to determine its properties. Now, with quantum mechanics, the experiment—and more important, the experimenter—become part of the system being measured. By the act of measuring the system, we affect it. If we measure for waves, we affect the system so that we

Noson S. Yanofsky is a professor of computer and information sciences at Brooklyn College of the City University of New York. He is a coauthor of Quantum Computing for Computer Scientists *(Cambridge University Press, 2008) and the author of* The Outer Limits of Reason: What Science, Mathematics, and Logic Cannot Tell Us *(MIT Press, 2013). Email: noson@sci.brooklyn.cuny.edu*

In the 1986 movie *Labyrinth*, the main character is faced with a riddle. One door in the scene above leads to her destination, the other to "certain death." She can only ask one of the door guards one question, and she's told that one of them always tells the truth, whereas the other always lies. Although she thinks she finds the correct answer, she still falls into a trap, likely because the labyrinth itself isn't fair. Riddles are often linked to paradoxes, and our mental constructs, including movies, can be full of contradictions. Finding out where such paradoxes exist in the real world can help us understand the limits of what science can and cannot know. (Image courtesy of the Jim Henson Company.)

cannot measure for particles and vice versa. This outcome is one of the most astonishing aspects of modern science.

The central idea of a paradox is the contradiction that is derived. Where the contradiction occurs tells us a lot about the type of limitation we found. The paradox could concern something concrete and physical. There are no contradictions or falsehoods in the physical universe. If something is true, it cannot be false, and vice versa. The physical universe does not permit contradictions, and hence, if a certain assumption leads to a contradiction in the physical universe, we can conclude that the assumption is incorrect.

Although contradictions and falsehoods cannot occur in the physical universe, they can occur in our mental universe and in our language. Our minds are not perfect machines and are full of contradictions and falsities. We desire contradictory things. We want to eat that second piece of cake and also to be thin. People in relationships simultaneously love and hate their partners. People even willfully believe false notions. Our language, a product of our mind, is also full of contradictions. When we meet a contradiction in mental and linguistic paradoxes, we essentially are able to ignore it, because it is not so strange to our already confused minds.

We cannot always be so cavalier about ignoring contradictions and

Because science and mathematics are constructed to mimic the contradiction-free physical universe, they also must not contain contradictions.

Rational Assumptions

One of the oldest stories about a limitation of science in classical times concerns the square root of two, $\sqrt{2}$. In ancient Greece, Pythagoras and his school of thought believed that all numbers are whole numbers or ratios of two whole numbers, called *rational numbers.* A student of Pythagoras, Hippasus, showed that this view of numbers is somewhat limiting and that there are other types of numbers. He showed that $\sqrt{2}$ is not a rational number and is, in fact, an *irrational number* (generally defined as any number that cannot be written as a ratio or fraction).

We do not know how Hippasus showed that the square root of two is irrational, but there is a pretty and simple geometric proof (attributed to American mathematician Stanley Tennenbaum in the 1950s) that is worth pondering. The method of proof is called a *proof by contradiction,* which is like a paradox. We are going to assume that the $\sqrt{2}$ is a rational number and then derive a contradiction:

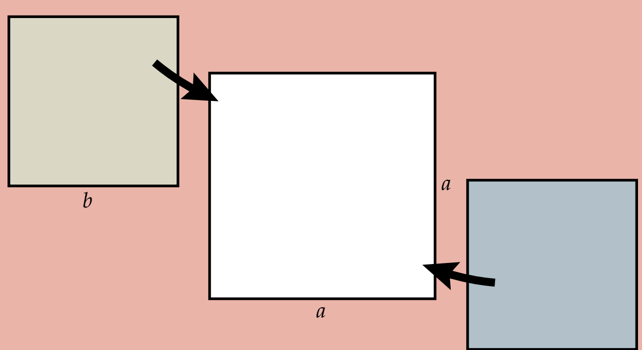
$\sqrt{2}$ is a rational number→Contradiction.

From this contradiction we can conclude that $\sqrt{2}$ is not a rational number.

First, assume that there are two positive whole numbers such that their ratio is the square root of two. Let us assume that the two smallest such whole numbers are a and b. That is, $\sqrt{2} = a/b$.

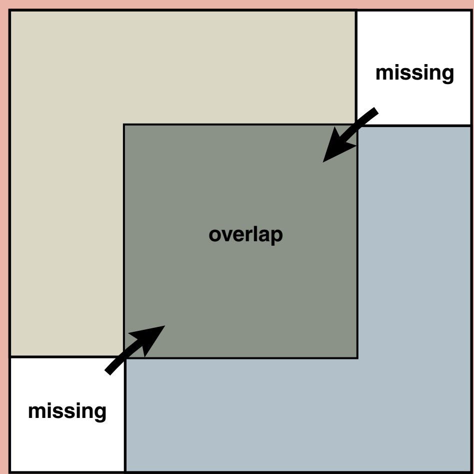
Squaring both sides of this equation gives us $2 = a^2/b^2$. Multiplying both sides by b^2 gives us $2b^2 = a^2$.

From a geometric point of view, this equation means that there are two smaller squares whose sides are each of size b,



and they are exactly the same size as a large square whose side is of size a. That is, if we put the two smaller squares into the larger square, they will cover the same area.

But when we actually place the two smaller squares into the larger, we find two problems. Firstly, we are missing



two corners. Secondly, there is overlap in the middle. So for the area of the larger square to equal the areas of the two smaller squares, the missing areas must equal the overlap. That is, 2(missing) = overlap.

But wait. We assumed that a and b were the smallest such numbers with which this result can happen; now we find smaller ones. So this result is a contradiction. There must be something wrong with our assumption that a and b are whole numbers. And thus the square root of two is not a rational number, but is irrational. Hippasus had shown that there was a number that did not follow the dictates of Pythagoras's science.

The followers of Pythagoras were fearful that the conclusion of Hippasus would be revealed and people would see the failings of the Pythagoras philosophy and religion. Legend has it that the other students of Pythagoras took Hippasus out to sea and threw him and his irrational ideas overboard.

falsities in human thought and speech. There are times when we must be more careful. Science is a human language that measures, describes, and predicts the physical world. Because science is constructed to mimic the contradiction-free physical universe, it also must not contain contradictions. Similarly, in mathematics, which is formulated by looking at the physical world, we cannot derive any contradictions. If we did, it would not be mathematics. When a paradox is derived in science or mathematics, it cannot be ignored, and science and mathematics must reject the assumption of the paradox. As an example of such a paradox, if we assume that the square root of two is a rational number, we get a contradiction (*see box above*). In this case, we must not ignore the paradox, but rather proclaim that the square root of two is not a rational number.

In addition to paradoxes, there are other ways of discovering limitations. Simply stated, one can piggyback off of a given limitation that shows that a certain phenomenon cannot occur, to show that another, even harder phenomenon also cannot occur. A simple example: When you are out of shape and climb four flights of steps, you will huff and puff. We can write this activity and its result as:

Climb four flights→Huff and puff.

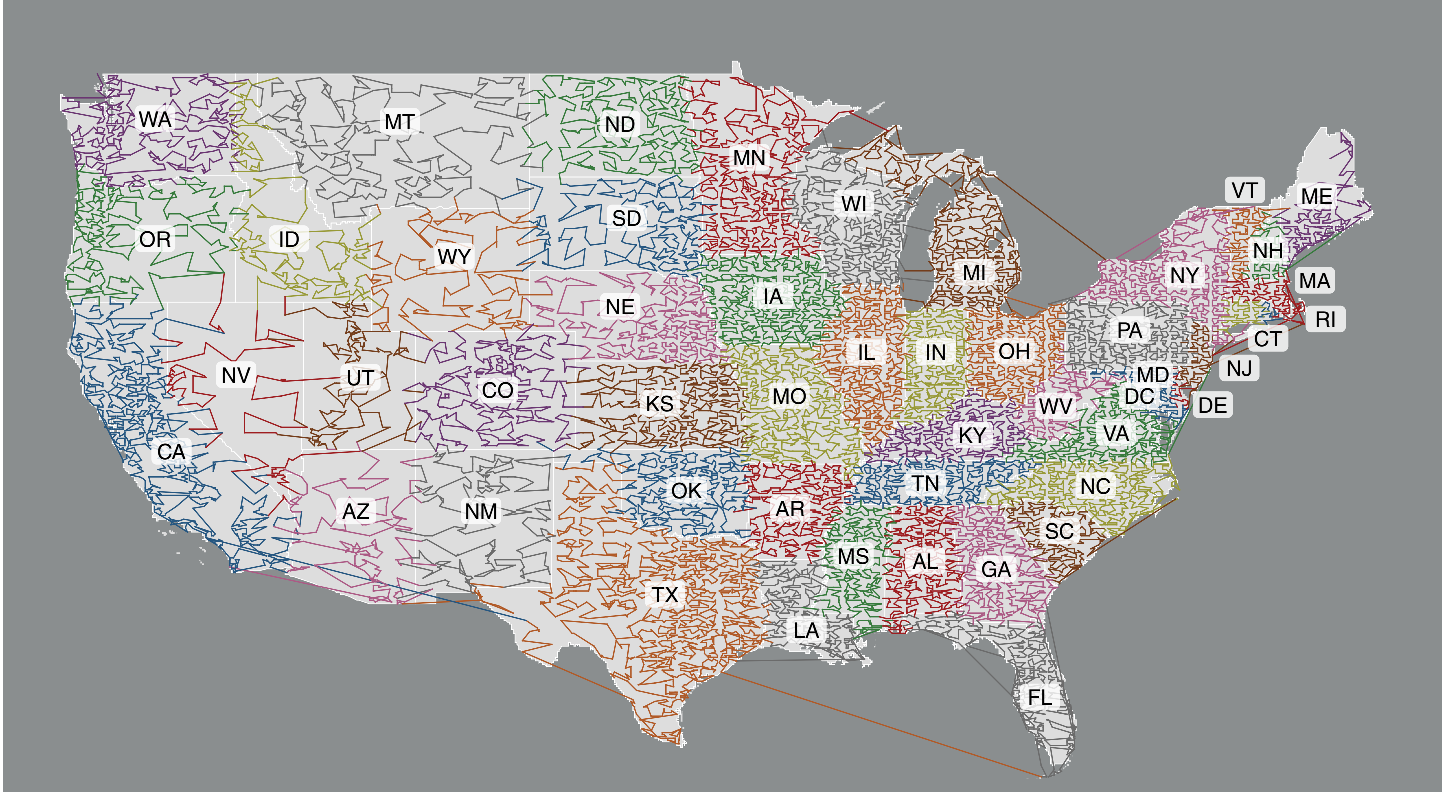
It is also obvious that if someone climbs five flights of steps, they also have climbed four flights of steps, that is:

Climb five flights→Climb four flights.

Combining these two implications gives us:

Climb five flights→Climb four flights →Huff and puff.

We conclude with the obvious observation that if you huff and puff after climbing four flights of steps, you will definitely huff and puff after climbing five flights of steps.



In a version of the traveling salesperson problem, computer scientist Robert Kosara created a map of an estimated best route for U.S. presidental candidates to visit all of the nation's ZIP codes. The map uses an approximation technique called *Hilbert curves*, and is reported to be about 75 percent optimal. (*For more on Hilbert curves, see "Crinkly Curves," May–June 2013.*) Finding the actual shortest route would take a computer essentially an infinite amount of time to compute. (Image courtesy of Robert Kosara, eagereyes.org.)

To generalize this simple example, assume that a limitation is found through a paradox:
Assumption-A→*Contradiction.*
Thus, Assumption-A is impossible. If we further show that:
Assumption-B→*Assumption-A,*
we can combine these two implications to get:
Assumption-B→ *Assumption-A*→ *Contradiction.*
This result shows us that because Assumption-A is impossible, then the second factor, Assumption-B, is also impossible.

With these methods of finding various limitations, we can define the four actual classes of limitations.

Physical Limitations
The first and most obvious type of limitation is one that says certain physical objects or processes cannot exist, like the village in the barber paradox.

Another example of a physical process that is impossible is time travel into the past. This limitation is usually

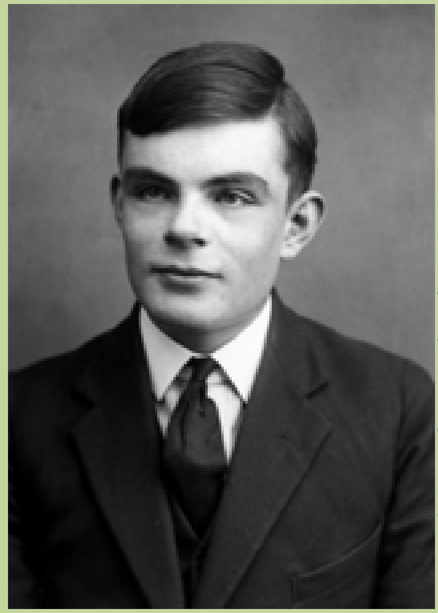
The Shortest Route

The Traveling Salesperson Problem is an easily stated computer problem that is an example of a practical limitation. Consider a traveling salesperson who wants to find the shortest route, from all possible routes, that will visit 10 different specified cities. There are many different possible routes the salesperson can take. There are 10 choices for the first city, nine choices for the second city, eight choices for the third city, and so on, down to two choices for the ninth city, and one choice for the tenth city. In other words, there are $10 \times 9 \times 8 \times ... \times 2 \times 1 = 10! = 3{,}628{,}800$ possible routes. A computer would have to check all these possible routes to find the shortest one. Using a modern computer, the calculation can be done in a couple of seconds. But what about going to 100 different cities? A computer would have to check $100 \times 99 \times 98 \times ... \times 2 \times 1 = 100!$ possible routes, which results in a 157-digit-long number: 93,326,215,443,944,152,681,699,238,856,266,700,490,715,968,264,381,621,468,592,963,895,217,599,993,229,915,608,941,463,976,156,518,286,253,697,920,827,223,758,251,185,210,916,864,000,000,000,000,000,000,000,000 potential routes.

For each of these potential routes, the computer would have see how long the route takes, and then compare all of them to find the shortest route. A modern computer can check about a million routes in a second. That computation works out to take 2.9×10^{142} centuries, a long time to find the solution.

Such a problem will not go away as computers get faster and faster. A computer 10,000 times faster, able to check 10 billion possible routes in a second, will still take 2.9×10^{138} centuries. Similarly, having many computers working on the problem will not help too much. Physicists tell us that there are 10^{80} particles in the visible universe. If every one of those particles were a computer working on our problem, it would still take 10^{62} centuries to solve it. The only thing that possibly can help this problem is finding a new algorithm to figure out the shortest route without looking through all the possibilities. Alas, researchers have been looking for decades for such a magic algorithm. They have not found one, and most computer scientists believe that no such algorithm exists.

The traveling salesperson problem can be solved for small inputs. Even for large inputs, a program can be written that will solve it, but the program will demand an unreasonable amount of time to determine the solution. Although there does exist a shortest possible route, the knowledge of that route is inherently beyond our ability to ever know, making it a practical limitation. (But see image above for a sample of one estimate of the answer.)



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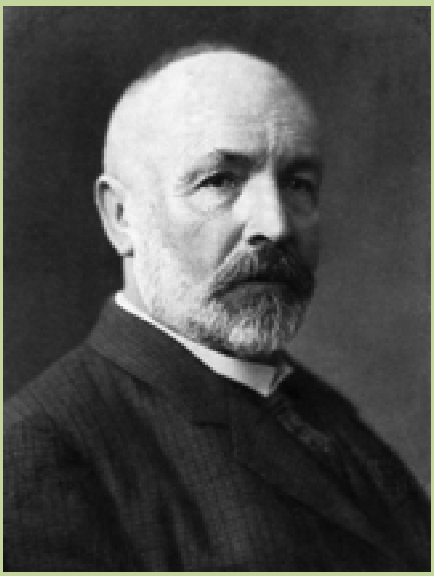

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Across various fields of science, mathematics, and logic, over the past century, some of the greatest minds have discovered instances in which research cannot find the answer, demonstrating a limit on science itself. Albert Einstein (*top left*) showed limitations related to relativity theory. Kurt Gödel (*with Einstein*) found that there are mathematical statements that are true but not provable. Alan Turing (*above*) showed limits to what computers can compute. Georg Cantor (*far left*) showed that not all infinite sets are the same size, altering the concept of infinity. And Bertrand Russell (*left*) showed a limit in what types of mathematical sets can exist, without causing a contradiction that cannot be present in mathematics. (Albert Einstein and Kurt Gödel photograph by Oskar Morgenstern, courtesy of The Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ.)

shown through a self-referential paradox that is often called the *grandfather paradox*. In it, a person goes back in time and kills his bachelor grandfather. Thus his father will not be born, the time traveler himself will not be born—and hence the time traveler will not be able to kill his grandfather. One need not be homicidal to obtain such a paradox: In the 1985 movie *Back to the Future*, the main character starts to fade out of existence because he traveled back in time and accidentally stopped his mother and father from getting married. A time traveler need only go back several minutes and restrain the earlier version of himself from getting into the time machine.

What is different about events in time travel that cause these paradoxes? Usually, an event affects another, later event: If I eat a lot of cake, I will gain weight. With the time travel paradox, an event affects itself. By killing his bachelor grandfather, the time traveler ensures that he cannot kill his bachelor grandfather. The event negates itself. The simple resolution to the grandfather paradox is that, in order to avoid contradictions, time travel is impossible. Alternatively, if perchance time travel is possible, it is impossible to cause such a contradiction.

Another example of a limitation that shows the impossibility of a physical process is the *halting problem*. Before engineers actually built modern computers, Alan Turing showed that there are limitations to what computers can perform. In the 1930s, prior to helping the Allies win World War II by breaking the Germans' Enigma cryptographic code, he showed what computers cannot do by way of a self-referential paradox. As anyone who deals with computers knows, sometimes a computer "gets stuck" or goes into an "infinite loop." It would be nice if there were a computer that could determine whether a computer will get stuck in an infinite loop. Essentially, we are asking computers to be self-referential. Turing showed that no such computer could possibly exist. He showed that if such a computer could exist, he would make a computer that would negate its own "haltingness." Such a program would perform the following task: "When asked if I will halt or go into an infinite loop, I will give the wrong answer." However, computers cannot give wrong answers because they do exactly what their instructions tell them to do, hence we have a contradiction, which occurs because of the assumption that we made about a computer that can determine whether any computer will go into an infinite loop. That assumption is incorrect. Many other problems in computer science, mathematics, and physics are shown to be unsolvable by piggybacking off the fact that the halting problem is unsolvable.

There are many other examples of physical limitations. For instance, Einstein's special theory of general relativity tells us that a physical object cannot travel faster than the speed of light. And quantum theory tells us that the action of individual subatomic particles is probabilistic, so no physical process can predict how a given subatomic particle will act.

Mental Construct Limitations

Recall that although our minds are full of contradictions, we must, when dealing with science and mathematics, steer clear of them, and that means restricting certain mental and linguistic activities.

In the first years of elementary school, we learn an easy mental construct limitation: We are not permitted to divide by zero. Despite the reasons for this rule being so obvious to us now, let us justify it. Consider the equation $3 \times 0 = 4 \times 0$. Both sides of the equation are equal to zero and hence the statement is true. If you were permitted to divide by zero, you could cancel out the zeros on both sides of the equation and get $3 = 4$. This outcome is a clear falsehood that must be avoided.

A more advanced result in which one sees the mental construct limitation more clearly is in what's called *Russell's paradox*. In the first few years of the 20th century, British mathematician Bertrand Russell described a paradox that shook mathematics to its core. At the time, it was believed that all of mathematics could be stated in the language of *sets,* which are collections of abstract ideas or objects. Sets can also contain sets, or even have themselves as an element. This idea is not so far-fetched:

David Parker/Science Photo Library

A fractal illustration, which is self-similar across length scales and infintely complex, is used to illustrate the concept of *quantum chromodynamics*, the interactions between quarks and gluons, which make up protons, neutrons, and other subatomic particles. The forces between quarks and gluons are classified as colors—hence the name of the concept. Quantum mechanical properties at these subatomic scales are often at the heart of paradoxes and thus limitations on science.

Consider the set of ideas that are contained in this article. That set contains itself. The set of all sets that have more than three elements contains itself. The set of all things that are not red contains itself. The fact that sets can contain themselves makes the whole subject ripe for a self-referential paradox.

Russell said that we should consider all sets that do not contain themselves and call that collection R (for Russell). Now simply pose the question: Does R contain R? If R does contain R, then as a member of R that is defined as containing only those sets that do not contain themselves, R does not contain R. On the other hand, if R does not contain itself, then, by definition, it belongs in R. Again we arrive at a contradiction. The best method of resolving

The mathematical formulation of "This statement is not provable" negates its own provability.

Russell's paradox is to simply declare that the set *R* does not exist.

What is wrong with the collection of elements we called *R*? We gave a seemingly exact statement of which types of objects it contains: "those sets that do not contain themselves." And yet, we have declared that this collection is not a legitimate set and cannot be used in a mathematical discussion. Mathematicians are permitted to discuss the green apples in my refrigerator but are not permitted to discuss the collection *R*. Why? Because the collection *R* will cause us to arrive at a contradiction. Mathematicians must restrain themselves because we do not want contradictions in our mathematics.

In 1931, Austrian mathematician Kurt Gödel, then 25 years old, proved one of the most celebrated theorems of 20th-century mathematics. Gödel's Incompleteness Theorem shows that there are statements in mathematics that are true but are not provable. Gödel showed this result by demonstrating that mathematics can also talk about itself. Mathematical statements about numbers can be converted into numbers. Using this ability to self-reference, he formulated a mathematical statement that essentially says: "This mathematical statement is not provable." It's a mathematical statement that negates its own provability. If you analyze this statement carefully, you realize that it cannot be false (in which case it would be provable), and hence it would be true and contradictory. But since it is true, it must also be unprovable. Gödel showed that not everything that is true has a mathematical proof.

Throughout mathematics and science, there are many other examples of mental construct limitations. For instance, one cannot consider the square root of two to be a rational number (*see box on page 168*). Zeno's famous paradoxes, created by Greek mathematician Zeno of Elea around 450 BCE and involving such conundrums as motion being an illusion, can also be seen as examples of mental construct limitations.

There are reasons to believe that there is a lot more "out there" that we cannot know than what we can know.

Practical Limitations

So far we have seen limitations that show it is impossible for something or some process (physical or mental) to exist. In a practical limitation, we are dealing with things that are possible, albeit extremely improbable. That is, it is impossible to make some prediction or find some solution in a normal amount of time or with a normal amount of resources.

The classical example is the butterfly effect from chaos theory. The phrase comes from the title of a 1972 presentation by mathematician Edward Lorenz of the Massachusetts Institute of Technology: "Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" Lorenz was a meteorologist and a mathematician. He was discussing the fact that weather patterns are extremely sensitive to slight changes in the environment. A small flap of a butterfly's wing in Brazil might cause a change that causes a change that eventually causes a tornado in Texas. Of course, one should not go out and kill all the butterflies in Brazil; the butterfly flap might instead send a coming tornado off course and save a Texas city. The point of the study is that because there is no way we can keep track of the many millions of butterflies in Brazil, we can never predict the paths of tornados or of the weather in general. This thought experiment shows a limitation of our predictive ability.

Many other problems from chaos theory show limitations. Predicting tomorrow's lottery numbers is also beyond our ability. If you wanted to know the numbers, you would have to keep track of all the atoms in the bouncing ball machine—far too many for us to ever be able to do.

Perpetual motion machines are another example of a practical limitation.

Estimating the Unsolvable

How much is beyond our ability to solve? In general, such things are hard to measure. However, in computer science there is an interesting result along these lines. We all know of many different tasks that computers perform with ease. However, there are many problems that are beyond the ability of computers. We can examine whether there are more solvable problems than unsolvable problems.

First, a bit about infinite sets. Mathematicians have shown that there are different levels of infinity. The smallest infinity corresponds to the natural numbers: {0, 1, 2, 3 …}. We say that this set of numbers is "countably infinite." Although we can never finish counting the natural numbers, we can at least begin listing them. In contrast, the set of all real numbers—that is numbers such as –473.4562372... and pi—are "uncountably infinite." We cannot even begin to count them. After all, what is the first real number after 0? 0.000001? What about 0.0000000001? It can be shown easily that uncountably infinite sets are vastly larger than countably infinite sets.

Now let us turn to computers that solve problems. There are a countably infinite number of potential computer programs for solvable computer problems. In contrast, there are uncountably infinite computer problems. If one takes all the uncountably infinite computer problems and subtracts the countably infinite solvable problems, one is left with uncountably infinite unsolvable problems. Thus the overwhelming vast majority of computer problems cannot be solved by any computer. Computers can only solve a small fraction of all the problems there are.

There is essentially no way that one can make a machine that will continue to move without losing all its energy. One might be tempted to say that this limitation is really a physical one because it says that a perpetual motion machine cannot exist in the physical universe. But by the second law of thermodynamics, it is extremely improbable for there to be a machine that does not dissipate its energy. Improbable, but not impossible.

The theory of thermodynamics and statistical mechanics is about large groups of atoms and the heat and energy they can create. Because in such systems there are too many elements to keep track of, the laws in such theories are given as probabilities, and are ripe for finding other examples of practical limitations. In computer science, an example of a problem that is theoretically solvable, but for large inputs will never practically be solved, is called the *traveling salesperson problem* (*see box on page 169*). There are many more.

Limitations of Intuition

The fourth type of limitation is more of a problem with the way we look at the world. Science has shown that our naive intuition about the universe that we live in needs to be adjusted. There are many aspects of reality that seem obvious, but are, in fact, simply false.

One of the most shocking examples of this false perception comes from Einstein's special theory of relativity. The notion of *space contraction* says that if you are not moving and you observe an object moving near the speed of light, then you will see the object shrink. This observation is not an optical illusion: The object actually shrinks. Similarly, the phenomenon of *time dilation* says that when an object moves close to the speed of light, all the processes of the object will slow down. Of course, an observer traveling with the object will see neither space contraction nor time dilation. Thus our naive view that objects have fixed sizes and processes have fixed duration is faulty.

Some of the most counterintuitive aspects of modern science occur within quantum mechanics. Since the beginning of last century, physicists have been showing that the subatomic world is an extremely strange place. In addition to finding that the properties of things (such as a photon acting like a wave or a particle) depend on how they are measured, researchers have found that rather than a particle having a single position, it can be in many places at one time, a property called *superposition*. Indeed, not only position, but many other properties of a subatomic particle, might have many different values at the same time. Heisenberg's uncertainty principle tells us that objects do not have definitive properties until they are measured. A famous concept called Bell's theorem shows us that an action here can affect objects across the universe, which is called *entanglement*. (For more on Bell's theorem and entanglement, see "Quantum Randomness," July–August 2014.)

One might think that mathematics is always intuitive and that our intuitions in that field at least might never need to be adjusted. But this assumption is also not true. In the late 19th century, German mathematician Georg Cantor, a pioneer in set theory, showed us that our intuition about infinity is somewhat troublesome. The naive view is that all the infinite sets are the same size. Cantor showed that in fact there are many different sizes of infinite sets. (*See box on the opposite page.*)

In the sciences, whenever there is a paradigm shift, all of our ideas about a certain subject have to be readjusted. We have to look at phenomena from a new viewpoint.

The Unknowable

The classification of the limitations of science is only beginning, and many questions still arise. Is this classification complete, or are there other limitations that are of a different type? Is there a subclassification of each of the classes? How do the methods of finding the limitations correspond to the types of limitation? Are there results that are in more than one classification? Because some of the results in the other classes might also be counterintuitive, there might be some overlap between categories.

How widespread is this inability to know? Most scientists work in the areas in which progress in knowing happens every day. What about what cannot be known? In general, the concept is hard to measure. There are reasons to believe that there is a lot more "out there" that we cannot know than what we can know. (*See box on the opposite page for such a calculation in computer science.*) Nevertheless, it is hard to speculate. Isaac Newton said, "What we know is a drop, what we don't know is an ocean." Similarly, Princeton University theoretical physicist John Archibald Wheeler is quoted as saying, "As the island of knowledge grows, so does the shore of our ignorance." Newton and Wheeler were talking about what we do not know. What about what we cannot know?

Most of the limitations discussed here are less than a century old, a very short time in the history of science. As science progresses, it will become more aware of its own boundaries and limitations. By looking at these limitations from a unified point of view, we will be able to compare, contrast, and learn about these many different phenomena. We can understand more about the very nature of science, mathematics, computers, and reason.

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