Resolving Paradoxes

Noson Yanofsky tells us how to deal with contradictions and the limitations of reason that arise from them.

We all have conflicting desires. We want to get promoted, but don't want to work too hard. We would like to date both Betty and Veronica (or both Bob and Vernon). We desire to stay thin, but also desire to eat that delicious piece of cake. All these conflicting wishes cause dilemmas. We are limited by the fact that we can only choose one of the options or we get a contradiction. As we mature we learn how to navigate away from contradictions and resolve our conflicts. Most of us learn to make choices, and some of us at times even make the right decisions. In general, our success depends on which of our conflicting choices we make.

Philosophers, scientists, and mathematicians also deal with contradictions in their work. They resolve these contradictions in various ways depending on how the contradictions occur. Although these contradictions occur in very diverse and unrelated fields, we can classify the types of contradictions that arise in these fields quite simply.

Existence Paradoxes

There is a cute story attributed to Bertrand Russell (1872-1970) that has puzzled people for decades. There exists an isolated village that has only one barber. Within this village some people go to the barber, and some cut their own hair. The village has the following rule that is strictly enforced: everyone must get their hair cut; furthermore, if you cut your own hair, then you do not go to the barber; and if you do not cut your own hair, you go to the barber. It's one or the other. While this seemingly innocuous rule works for most people in the village, there is one villager who causes a major problem for the authorities. Ask yourself the question: Who cuts the barber's hair? If the barber cuts his own hair, then, because of the rule, he cannot go to the barber: but he is the barber! If, on the other hand, the barber goes to the barber, then he is cutting his own hair, in violation of the rule. These two results present a contradiction. The contradiction is that the barber must cut his own hair and must not cut his own hair.

This mind-bending little story is a prototypical example of a paradox. That is, it is a thought experiment where an idea is assumed and a contradiction is logically derived from it. What's wrong with contradictions? A contradiction is a proposition (that is, an assertion) that something both is, and that it is not. We can also say that a contradiction is a proposition that is both true and false. There are no contradictions in the physical world. In the physical universe, a thing cannot both be and not be; or, a proposition is either true or it is false. What would it possibly mean for something to be both true and false? We cannot imagine the universe with a contradiction. Since contradictions are untenable, there must be something wrong with the assumption from which a contradiction is derived. In a sense, a paradox is a way of showing that a given assumption is not part of reason. Philosophers, scien-

tists, mathematicians, and logicians have all used paradoxes to demonstrate the validity or otherwise of assumptions, and so to demonstrate limitations to reason. They assume an idea, and if a contradiction or falsehood is derived from it, then the assumption is wrong. Otherwise, the assumption is logically legitimate.

Let us return to our little village and see how we can resolve the paradox. What is the assumption here?

Many have given simple solutions to the paradox: the barber is bald, or the barber's wife cuts his hair, or the barber quits his job as a barber and then cuts his hair before taking his job back again, etc. All these 'solutions' are just sneaking around the problem. In our village everyone needs to get his hair cut; no one else – not even the barber's wife – is permitted to cut someone else's hair, and the single barber cannot temporarily quit.

In fact, these ideas are shadows of the true resolution to the barber paradox, which is to realize that *the village described with its strict rule cannot exist*. The physical universe would not permit the existence of such a village, because it implies a contradiction. There are many isolated villages with one barber, but in them the rule can be violated in a myriad of ways: a barber from another city might visit, or the barber can be bald, or the barber asks his wife to cut his hair, or the barber cuts his own hair (so the rule does not hold), etc. All these different scenarios could happen to ensure that there are no contradictions in the universe. In short, the assumption of the barber paradox was that the village with this rule exists. This assumption is wrong.

There is another paradox that every fan of science fiction knows well. *The time travel paradox* or *the grandfather paradox* is about a time traveler who goes back in time and kills his grandfather before he'd met his grandmother. This action ensures that the time traveler's father will never be born, and hence the time traveler himself will never be born. If he is never born, he will never be able to go back in time and kill his bachelor grandfather; so, if he kills his bachelor grandfather, then he will not kill his bachelor grandfather; and if he does not kill his bachelor grandfather, then he can go back and kill his bachelor grandfather. This again is a contradiction.

Indeed, one does not need to be so homicidal to get such a result. All the time traveler has to do is go back in time two minutes before he gets into his time machine and make sure that his earlier self does not enter the machine. Stopping his earlier self from entering the time machine to stop himself from entering the machine will ensure that he cannot stop himself from entering the machine. Again, a contradiction. Most actions affect other actions. A time traveler has the unique ability to perform an action which affects itself. A paradox comes about if a time traveler performs an action that negates itself.

The time travel paradox is resolved with ease: the assumption that traveling backwards in time is possible is wrong; or even if backwards time travel were possible, still the time traveler would not be able to go back and perform an action that

contradicts itself. There is no way that the universe is going to let a time traveler kill his own bachelor grandfather.

Those two paradoxes are resolved in the same way. We came to a contradiction by assuming that a certain physical object or process exists. Once we abandon this assumption, we are free of the contradiction. These paradoxes show limitations of the physical world: a certain village cannot exist, or the actions of a time traveler are restricted. There are many similar examples of such physical paradoxes, and they are resolved in the same way.

Linguistic Paradoxes

Contradictions do not exist in the physical universe. However there are places where contradictions do exist: in our minds and language. This brings us to our second type of paradox. Unlike the universe, the human mind is not a perfect machine. It is full of contradictions, with conflicting desires and predictions. We want contradictory things. We love things that we know will be bad for us, and we have conflicting ideals in our lives. Similarly, our language, which expresses the mind's ideas, also expresses contradictions. We can say "I love her!" and "She drives me crazy!" within seconds of each other. Any teenager can easily list off all the contradictory statements that ever came out of their parents' mouths. These imperfections in what we think and say constitute our next class of paradoxes.

The most famous paradox in language is *the liar paradox*. The simple statement 'This statement is false' is apparently both true and false. If it is true, then it asserts that it is false; on the other hand, if it is false, then since it says it is false, it is true. Most declarative statements – that is, statements that assert something – are either true *or* false. Here, the statement is true *and* false. It is a contradiction.

What can be done with the liar paradox? Some philosophers say that 'This statement is false' is not a legitimate declarative statement. It is not actually declaring anything about anything. Statements that produce these sorts of problems are literally nonsensical. Some thinkers want to restrict the type of statements we can use so that paradoxes don't arise.

Many people find there's no reason to lose sleep over the liar paradox. After all, much that comes out of our mouths is nonsense. So, say that the liar statement is neither true nor false. Rather than trying to resolve the paradox by avoiding the contradiction, we ignore the paradox as 'just' nonsensical language. It is simply human speech without meaning.

Another paradox of language is called *the bald man paradox*. Consider a man who has absolutely no hair on his head. We can all agree that this man should be called 'bald'. What if a man has a single solitary hair on his head? Most of us would still agree that adding the single solitary hair won't change the description, and he will still be considered bald. From this we can make a more general rule: if a man is considered bald, then adding a single solitary hair does not change his status, and he is still considered bald. Using this rule over and over again, we infer that a man with two hairs is bald, a man with three hairs is bald, a man with four hairs is bald, ... a man with 100,000 hairs is bald. But we know that that last statement is false. So we have a contradiction. On the one hand, we can prove that a man with any amount of hair remains bald; and on the other hand, we know that there are some men who are simply not bald. What went wrong here?

Again, the resolution to this paradox (for historical reasons known as *the sorites paradox*) is simple. It is to recognize that there is no exact definition of the word 'bald', so there is no exact demarcation between bald and not bald. The word 'bald' is a human invention that does not *have* an exact definition. Similarly, other vague words – 'tall,' 'short,' 'smart', etc – do not have exact meanings. (In contrast, the phrase 'over six feet tall' does have an exact meaning.) We can therefore use the same solution for the bald man paradox as some do for the liar paradox: ignore it. Human beings have successfully used vague words for a long time. We can ignore the bald man paradox because it is about human language, which is inherently flawed.

Mathematical Paradoxes

So far we have seen that some paradoxes show us that there are physical objects or processes which cannot exist, while other paradoxes are about language and can be ignored. There is, however, a third class of paradoxes that come from language that cannot be ignored.

Mathematics is a language that human beings use to make sense of the world. Since math is a language, it can potentially have contradictions. However, because we want to use mathematics in science to talk about the physical universe, which does not have contradictions, we must make sure that the language of mathematics does not have contradictions, since we must ensure that our scientific predictions don't come to contradictory conclusions. If we conclude that a chemical process will only produce carbon monoxide, then we better make sure our calculations don't say that the process will produce carbon dioxide. Or if there are two ways to calculate how long it will take a projectile to return to Earth, we must ensure that the two methods produce the same results. Luckily, it is easy to see how to avoid contradictions in mathematics.

The first and simplest time we encounter contradictions in mathematics is in elementary school, when we are informed that we can divide any number by any other number *except zero*. It is taboo to divide a number by zero because if we divide by zero, we can derive a contradiction. Consider the true statement $0 \times 2 = 0 \times 3$. If we divide both sides by zero to 'cancel out' the zeros, we are left with $2 = 3$. So by dividing by zero we got a contradiction: $2 \neq 2$. So whereas physical paradoxes showed us that certain objects or processes cannot exist, paradoxes in mathematics show us that although we can perform certain operations, we should not, lest we come to a contradiction.

Mathematicians (and philosophers too) can use the fact that contradictions are not permitted to prove theorems. This is called *reduction to the absurd* or *proof by contradiction*: if you want to prove that a certain statement is true, assume it is false and derive a contradiction. Since there cannot be contradictions, the assumption of falsehood must be incorrect and therefore the original statement is true. Such proofs play a major role in modern mathematics.

Let's look at an example from the world of sets. Sets are collections of objects, and they come in many different forms: there is the set of green apples in my fridge; the set of cells in my body; the set of all real numbers, etc. There are also sets that contain sets. For example, a school can be thought of as a

set of classes. Each class, in turn, can be thought of as a set of students. So a school can be a set of sets of students.

Life gets interesting when we think of sets that contain themselves. There are real examples of such sets. Consider the set of ideas contained in this article. That set contains itself: the set of ideas contained in this article is an idea contained in this article. The set of all sets that have more than one element also contains itself. The set of all things that are not red contains itself.

Russell developed another paradox, this time concerning sets. Consider *all sets that do not contain themselves.* Let us call that collection R. Now pose the question: does R contain R? If R does contain R, then as a member of R, which is defined as containing only those sets that do not contain themselves, R does not contain R. On the other hand, if R does not contain itself, then, by definition, it does belong in R. Again we come to contradiction. This is called *Russell's paradox*.

By now we have enough familiarity with paradoxes to know that the obvious method of resolving Russell's paradox is to simply declare that the set R does not exist. However, things are not so simple here. Why can't the collection of elements we called R exist? We gave an exact statement of which types of objects it contains: 'those sets that do not contain themselves'. Yet we have declared that this collection is not a legitimate set and cannot be used in a mathematical discussion. Mathematicians are permitted to discuss the green apples in my refrigerator, but are not permitted to use the set R, because the set R will bring us to a contradiction.

Our final example of using a paradox in mathematics is one of the most important theorems in twentieth century mathematics: *Gödel's Incompleteness Theorem*. In mathematics, any provable statement must by definition be true, because here proof means 'mathematically demonstrated to be true'. Before Kurt Gödel (1906-78) came along, mathematicians assumed that any mathematical statement that is true is also provable. Gödel showed this assumption is false by formulating a mathematical statement similar to the liar statement, which essentially says of itself 'This mathematical statement is unprovable'.

Let's think about that for a moment. If the mathematical statement 'This mathematical statement is unprovable' is unprovable, then it is true. Alternatively, if 'This mathematical statement is unprovable' is provable, then it is false. But how can proving it lead us to a false statement? That is a contradiction, since all mathematical proofs are by definition proofs of true statements. We must conclude that Gödel's statement is both true and unprovable. A paradox has led us to a restriction of the power of mathematics: there are statements in mathematics that are true, but can never be proven.

We have seen three different types of paradoxes and their resolutions. Some paradoxes are about the physical universe and since contradictions cannot exist in the physical universe, such paradoxes show limitations to what can physically exist. In contrast, some paradoxes are about human languages which are full of contradictions, and we can ignore the contradictions. Finally, there are paradoxes in mathematics, where we must carefully watch our step so that we don't come to any contradictions. A surprisingly large part of the limitations of reason come from these three categories of paradoxes.

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