Algorithms

Assignment: Mathematical Background and Growth of Functions

Name: ....

Id: ....

Grade

Good Luck!

1. Rearrange the following 15 functions in a decreasing order of their growth:

$$\log_2(n) \qquad 3^n \qquad n^2 \qquad \sqrt{n} \qquad \frac{n}{\log_2(n)}$$

$$n^5 \qquad n! \qquad n \qquad 2^n \qquad n(\log_2(n))^2$$

$$n^{1/3} \qquad n\log_2(n) \qquad 1 \qquad \log_2^4(n) \qquad \log_2\log_2(n)$$

2. (a) Prove by induction that for any integer $n \geq 1$:

$$1 + 4 + 9 + 16 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6} \ .$$

(b) Prove by induction that for any real number $q > 1$ and integer $n \geq 0$:

$$1 + q + q^2 + q^3 + \cdots + q^{n-1} + q^n = \frac{q^{n+1} - 1}{q - 1} \ .$$

3. (a) A bag contains n white socks and n black socks. You take socks out of the bag one at a time until you have a pair of matching socks. How many socks do you need to take out of the bag to guarantee that you will a matching pair?

   Justify your answer using the Pigeonhole Principle.

(b) A bag contains n left shoes and n right shoes. You take shoes out of the bag one at a time until you have at least one left shoe and one right shoe. How many shoes do you need to take out of the bag to guarantee that you will have a matching pair?

   Justify your answer using the Pigeonhole Principle.

4. (a) Find the exact solution to the following recursive formula.
You may guess the solution and then prove it by induction.

$$
\begin{aligned}
T(1) &= 1 \\
T(n) &= T(n-1) + 3
\end{aligned}
$$

(b) Find the exact solution to the following recursive formula.
You may guess the solution and then prove it by induction.

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + (2n-1) \end{aligned}$$

(c) Find the exact solution to the following recursive formula.
You may guess the solution and then prove it by induction.

$$\begin{aligned} T(1) &= 3 \\ T(n) &= 3T(n-1) \end{aligned}$$

5. (a) Solve the following recursive formula using the master theorem.
You may assume that n is a power of 2.

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 8T(n/2) + n^2 \end{aligned}$$

(b) Solve the following recursive formula using the master theorem.
You may assume that $n = (4/3)^k$ for some integer k.

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(3n/4) + 10 \end{aligned}$$

(c) Solve the following recursive formula using the master theorem.
You may assume that n is a power of 2.

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n/2) + \sqrt{n} \end{aligned}$$

6. What is the *running time* of the following functions. Use the "O, Ω, Θ" notation.

(a) $f(n)$ (* $n > 0$ is an integer number *)

$c = 0$
for $i = 1$ **to** n **do**
  for $j = 1$ **to** n **do**
    for $k = 1$ **to** n **do**
      $c := c + 1$

(b) $f(n)$ (* $n > 0$ is an integer number *)

$c = 0$
for $i = 1$ **to** n **do**
  for $j = 1$ **to** i **do**
    for $k = 1$ **to** j **do**
      $c := c + 1$

(c) $g(x)$ (* $x > 1$ is a real number *)

while $x > 1$ **do**
  $x := x/3$

7. An input file contains all the integers from 1 to n except one. The numbers may appear in any order. The goal is to find the missing number. A solution may **scan** the input only **once**.

   (a) Describe a simple solution that requires $O(n)$-bits memory.

   (b) Describe a "tricky" solution that requires $O(\log n)$-bits memory.