Analysis of Algorithms

BACKGROUND

Mathematical Background

- What is the complexity of an algorithm?

- Growth of functions:
  - The "O, Ω, Θ, o, ω" notation.

- Proofs:
  - By contradiction.
  - By induction.
  - The Pigeonhole Principle.

- Recursive formulas and the Master Theorem.

- Some useful formulas.

Algorithm Complexity

- How much of resources the algorithm requires?
  - Usually: time and space (memory).
- Complexity: as a function of the input length.
  - Usually an integer $n > 0$.
  - Usually a monotonic non-decreasing function.

Worst Case and Average Case Complexity

- $T(n)$ is a worst case complexity:
  - If for all inputs of length n the complexity is $T(n)$.

- $T(n)$ is an average case complexity:
  - If the average complexity over all length n inputs is $T(n)$.
  - Follow some distribution of the inputs (usually the uniform distribution).

Bounds

Upper bounds: A function $f(n)$ such that

- "$T(n) \leq f(n)$" for all n.

Lower bounds: A function $g(n)$ such that

- "$T(n) \geq g(n)$" for all n.

Tight bounds: A function $h(n)$ such that

- "$T(n) \approx h(n)$" for all n.

Performance Evaluation of Algorithms

Theoretical analysis:

- All possible inputs.
- Independent of hardware/software implementation.
- High level language.

Experimental Study:

- Some typical inputs.
- Depends on hardware/software implementation.
- A real program.

Growth of Functions

- Objective: A language to express that Algorithm A is better than or worse than or equivalent to Algorithm B.
- Need to define "$\leq$" between functions measuring the growth of functions.
- Independence of the hardware/software environment: Turing machine, RAM machine, classroom model, today computers, and future super-computers.
- Ignore constants that can be affected by changing the environment.

Examples of Function Growth

Running Time	1 second	1 minute	1 hour
$400n$	2,500	150,000	9,000,000
$20n \lceil \log n \rceil$	4,096	166,666	7,826,087
$2n^2$	707	5,477	42,426
n^4	31	88	244
2^n	19	25	31

Maximum size of a problem that can be solved in one second, one minute, and one hour, for various running times measured in microseconds.

Examples of Function Growth

Running Time	New Maximum Size
$400n$	$256m$
$20n \lceil \log n \rceil$	$\approx 256((\log m)/(7 + \log m))m$
$2n^2$	$16m$
n^4	$4m$
2^n	$m + 8$

Increase in the maximum size of a problem by using a computer that is 256 times faster than the previous one.

Each entry is given as a function of m, the previous maximum problem size.

The “O, Ω, Θ, o, ω” Notation

Big-Oh: $f(n) = O(g(n))$ if $f(n)$ asymptotically less than or equal to $g(n)$.

Big-Omega: $f(n) = \Omega(g(n))$ if $f(n)$ asymptotically greater than or equal to $g(n)$.

Big-Theta: $f(n) = \Theta(g(n))$ if $f(n)$ asymptotically equal to $g(n)$.

Little-oh: $f(n) = o(g(n))$ if $f(n)$ asymptotically strictly less than $g(n)$.

Little-omega: $f(n) = \omega(g(n))$ if $f(n)$ asymptotically strictly greater than $g(n)$.

Big-Oh

- $f(n) = O(g(n))$:
  - There exists a real constant $c > 0$ and an integer constant $n_0 > 0$ such that $f(n) \leq cg(n)$ for every integer $n \geq n_0$.

	$f(n) = O(g(n))$	$g(n) = O(f(n))$
$g(n)$ grows faster	YES	NO
$f(n)$ grows faster	NO	YES
same growth	YES	YES

Big-Omega and Big-Theta

- $f(n) = \Omega(g(n))$:
  - There exists a real constant $c > 0$ and an integer constant $n_0 > 0$ such that $f(n) \geq cg(n)$ for every integer $n \geq n_0$.

- $f(n) = \Theta(g(n))$:
  - There exist 2 real constants $c', c'' > 0$ and an integer constant $n_0 > 0$ such that $c''g(n) \leq f(n) \leq c'g(n)$ for every integer $n \geq n_0$.

Some Propositions

1. $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$.
2. $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$.
3. $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ iff $f(n) = \Theta(g(n))$.
4. $f(n) = O(g(n))$ and $g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$.
5. $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$.
6. $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$.

Examples

- ⋆ $3n = \Theta(n/2)$.
- ⋆ $1000000n = \Theta(n/100000)$.
- ⋆ $n\log_2 n/100000 = \Omega(10000000n)$.
- ⋆ $\log_2(n) = \Theta(\log_{10}(n))$.
- ⋆ $a_d n^d + a_{d-1}n^{d-1} + \cdots + a_1 n + a_0 = \Theta(n^d)$
  - for constants $a_0, a_1, \ldots, a_d$ and $a_d > 0$.

More Propositions

- ⋆ For any real constant c:
  - $O(cf(n)) = O(f(n))$.
  - $O(f(n)/c) = O(f(n))$.
  - $O(c) = O(1)$.
- ⋆ $O(f(n)) + O(g(n)) = O(f(n) + g(n)) = O(\max\{f(n), g(n)\})$.
- ⋆ $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$.

Little-oh and Little-omega

- ⋆ $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$:
  - For any constant $c > 0$ there exists an integer constant $n_0 > 0$ such that $f(n) \leq cg(n)$ for every integer $n \geq n_0$.
- ⋆ $f(n) = \omega(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$:
  - For any constant $c > 0$ there exists an integer constant $n_0 > 0$ such that $f(n) \geq cg(n)$ for every integer $n \geq n_0$.

Examples and Propositions

Examples:

- ⋆ $\log_2 n = o(\sqrt{n})$.
- ⋆ $n^3 = \omega(n^2)$.
- ⋆ $10^{100} n = o(n^2/10^{100})$.

Propositions:

- ⋆ $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$.
- ⋆ $f(n) = o(g(n))$ and $g(n) = o(h(n)) \Rightarrow f(n) = o(h(n))$.
- ⋆ $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n))$.

Hierarchy of Functions

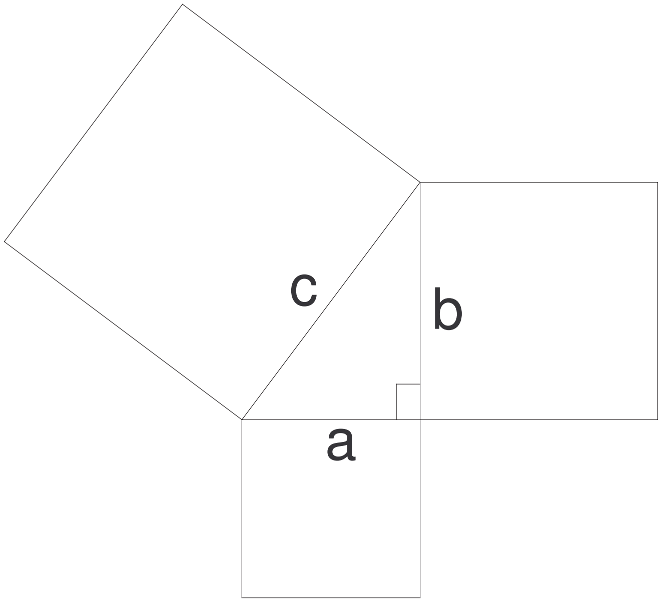
constant	1	
log star	log^*n	
loglog	$\log\log n$	
logarithmic	$\log n$	
poly-logarithmic	$\log^k n$	constant integer $k > 1$
sub-linear	n^ε	constant $0 < \varepsilon < 1$
linear	n	
above-linear	$n \log n$	
quadratic	n^2	
cubic	n^3	
polynomial	n^k	constant integer $k > 1$
super-polynomial	$n^{\log n}$	
exponential	2^n	
factorial	$n!$	
super-exponential	$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$	n powers

What is a Proof?

Definition I: The cogency of evidence that compels acceptance by the mind of a truth or a fact.

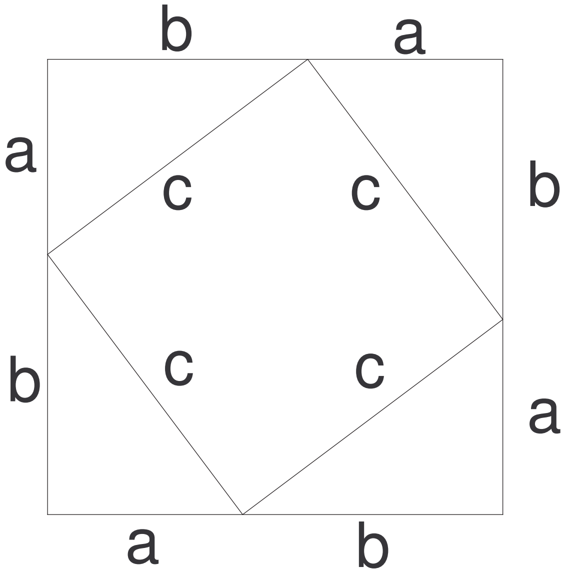
Definition II: The process or an instance of establishing the validity of a statement especially by derivation from other statements in accordance with principles of reasoning.

The Pythagorean Theorem



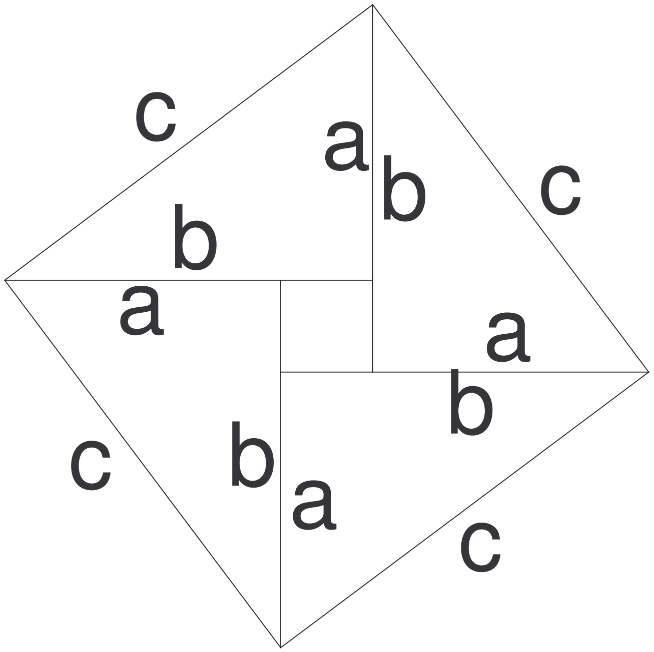
Theorem: $c^2 = a^2 + b^2$ for a right triangle with short sides of length a and b and long side of length c.

Proof I



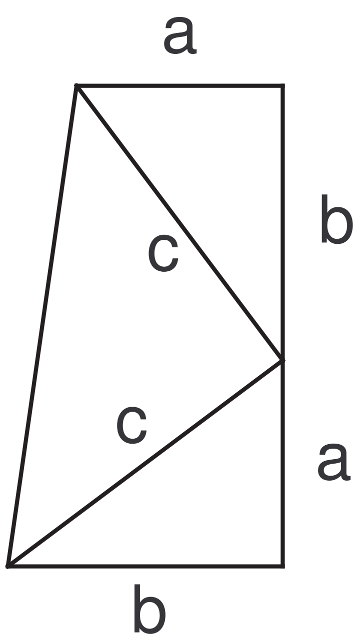
$$(a+b)^2 = c^2 + 4\frac{ab}{2} \Rightarrow a^2 + b^2 = c^2$$

Proof II



$$c^2 = 4\frac{ab}{2} + (b-a)^2 \;\Rightarrow\; c^2 = b^2 + a^2$$

Proof III



$$\frac{a+b}{2}(a+b) = \frac{c^2}{2} + 2\frac{ab}{2} \;\Rightarrow\; a^2 + b^2 = c^2$$

Prime Numbers

Theorem: There are infinitely many prime numbers.

Proof:

- ⋆ Let $p_1 < p_2 < \cdots < p_n$ be a set of n primes.
- ⋆ Let $Q = p_1 p_2 \cdots p_n + 1$.
- ⋆ If Q is a prime, then a new prime is found.
- ⋆ Otherwise, Q is a product of two or more primes.
    - The Fundamental Theorem of Arithmetic.
- ⋆ None of these primes can be $p_1, \ldots, p_n$.
- ⋆ Therefore, a new prime is found.
- ⋆ This process can continue infinitely often to find infinitely many primes.

The Numbers

An integer number: One of the (counting) numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

- Z (Z^+, Z^-) – The set of all (positive, negative) integers.

A rational number: A number that can be expressed as a fraction p/q where p (the numerator) and q (the denominator) are integers and $q \neq 0$.

- Q (Q^+, Q^-) – The set of all (positive, negative) rationals.

A real number: A number of the form $n.b_1b_2b_3\ldots$ where n is an integer and $b_i \in \{0, 1\}$ for all $i = 1, 2, \ldots$.

- $\Re$ $(\Re^+, \Re^-)$ – The set of all (positive, negative) reals.

Equality Between Sets

Definition: Two finite or infinite sets S and T have the same cardinality ($S \approx T$) if and only if there is a one-to-one mapping $f : S \to T$ from S to T.

Mapping: For all $s \in S$ there exists $t \in T$ such that $f(s) = t$.

From S: $s_1 \neq s_2 \;\Rightarrow\; f(s_1) \neq f(s_2)$.

To T: For all $t \in T$ there exists $s \in S$ such that $f(s) = t$.

Remark: For finite sets $S = T$ is equivalent to $S \approx T$.

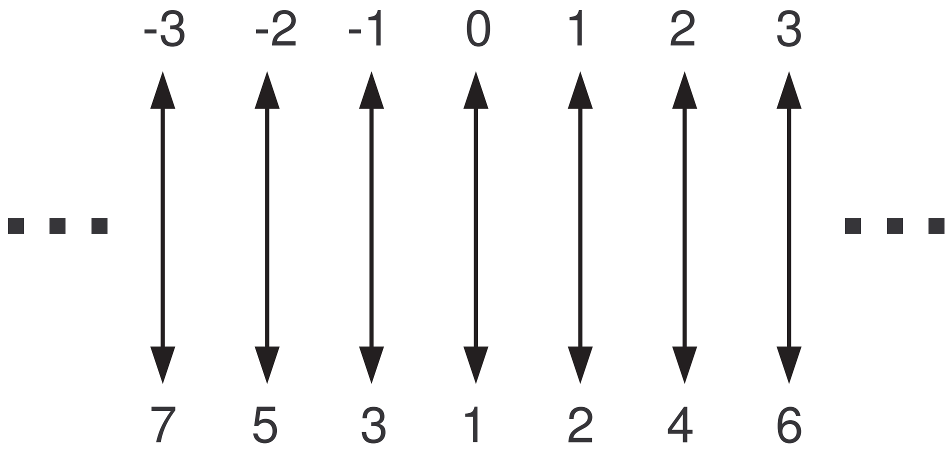
Examples

- $Z \approx Z^+$ although it "seems" like Z is twice as large as Z^+.
- $Q \approx Z$ although there are infinite number of rational numbers between any two consecutive integers.
- $\Re \not\approx Q$ although there are infinite number of rational numbers between any two real numbers.
  - Since $Q \subset \Re$ (every rational number is also a real number), it follows that $Q \leq \Re$ from a cardinality point of view.

$Z \approx Z^+$

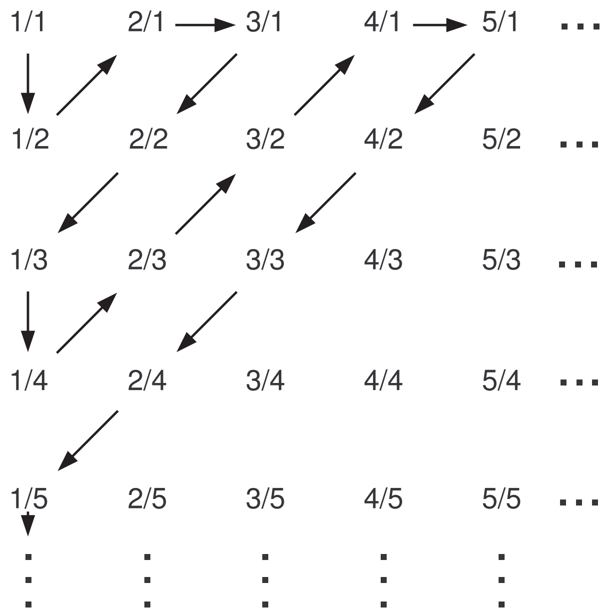
The mapping $f : Z \rightarrow Z^+$:

- $f(n) = 2n$ for a positive integer n.
- $f(-n) = 2n + 1$ for a positive integer n.
- $f(0) = 1$.



$Z^+ \approx Q^+$

The mapping: From Z^+ to Q^+ by following the arrows.



1/1 2/1 3/1 4/1 5/1 ...
1/2 2/2 3/2 4/2 5/2 ...
1/3 2/3 3/3 4/3 5/3 ...
1/4 2/4 3/4 4/4 5/4 ...
1/5 2/5 3/5 4/5 5/5 ...
⋮ ⋮ ⋮ ⋮ ⋮

$Z^+ \not\approx \Re^+$

- ⋆ Assume that a mapping $f : Z^+ \longrightarrow \Re^+$ exists.

$$\begin{aligned}
f(1) &= n_1.b_1^1 b_1^2 b_1^3 \ldots b_1^i \ldots \\
f(2) &= n_2.b_2^1 b_2^2 b_2^3 \ldots b_2^i \ldots \\
f(3) &= n_3.b_3^1 b_3^2 b_3^3 \ldots b_3^i \ldots \\
&\vdots \\
f(i) &= n_i.b_i^1 b_i^2 b_i^3 \ldots b_i^i \ldots
\end{aligned}$$

- ⋆ Let $x = n_1.\bar{b}_1^1 \bar{b}_2^2 \bar{b}_3^3 \ldots \bar{b}_i^i \ldots$ where $\bar{b}_j^j = 1 - b_j^j$.
- ⋆ $x \neq f(i)$ for any $i = 1, 2, \ldots$.
- ⋆ Therefore f is not a valid mapping: a contradiction.

A Proof by Contradiction

Theorem: The number $\sqrt{2}$ is an irrational number.

Proof:

- Assume towards contradiction that $\frac{p}{q} = \sqrt{2}$ for two positive integers p and q that have no common divisor.
- Squaring the equation implies $p^2 = 2q^2$.
- Therefore, p must be even.
- Let $p = 2r$ for a positive integer r.
- It follows that $2q^2 = (2r)^2 = 4r^2 \;\Rightarrow\; q^2 = 2r^2$.
- Therefore, q must be even.
- A contradiction since both p and q are even.

A False Proof

$$\begin{aligned}
a &= b + c \\
(a-b)a &= (a-b)(b+c) \\
a^2 - ab &= ab + ac - b^2 - bc \\
a^2 - ab - ac &= ab - b^2 - bc \\
a(a-b-c) &= b(a-b-c) \\
a &= b
\end{aligned}$$

Another False Proof

$$
\begin{aligned}
i &= i \\
\sqrt{-1} &= \sqrt{-1} \\
\sqrt{\frac{1}{-1}} &= \sqrt{\frac{-1}{1}} \\
\frac{\sqrt{1}}{\sqrt{-1}} &= \frac{\sqrt{-1}}{\sqrt{1}} \\
\sqrt{1} \times \sqrt{1} &= \sqrt{-1} \times \sqrt{-1} \\
1 &= -1
\end{aligned}
$$

Induction

- ⋆ Let P_n be a statement for all the positive integers $(n = 1, 2, 3, \ldots)$.

- ⋆ If the following 2 properties hold:

  1. P_1 is true.
  2. P_{k+1} is true if P_k is true for each positive integer k.

- ⋆ Then P_n is true for all n.

Why Induction Works?

- ⋆ Let S be the set of all numbers n for which P_n is false.
- ⋆ Let k be the minimum number in S.
- ⋆ $k > 1$ since by the first property of the induction definition, P_1 is true.
- ⋆ By the minimality of k, P_{k-1} is true and P_k is false.
- ⋆ A contradiction to the second property of the induction definition.

A Summation Problem

- ⋆ Prove that for any integer $n \geq 1$:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

- ⋆ Define:
  - $L(n) = 1 + 2 + 3 + \cdots + n.$
  - $R(n) = \frac{n(n+1)}{2}.$

- ⋆ Prove that $L(n) = R(n)$ for $n \geq 1$.

Verifying the Claim

⋆ First check small values of n:

n	$L(n)$	$R(n)$
1	$1 = 1$	$\frac{1 \cdot 2}{2} = 1$
2	$1 + 2 = 3$	$\frac{2 \cdot 3}{2} = 3$
3	$1 + 2 + 3 = 6$	$\frac{3 \cdot 4}{2} = 6$
$\vdots$	$\vdots$	$\vdots$
10	$1 + 2 + 3 + \cdots + 10 = 55$	$\frac{10 \cdot 11}{2} = 55$

A Direct Proof

Idea: Compute the value of $2L(n)$.

Example:
$$
\begin{aligned}
2(1+2+3) &= (1+2+3)+(3+2+1) \\
&= (1+3)+(2+2)+(3+1) \\
&= 3\cdot 4 \\
&= 12 \\
&= 2\frac{3\cdot 4}{2}.
\end{aligned}
$$

A Direct Proof

$$\begin{aligned} 2 \cdot L(n) &= (1 + \cdots + n) + (n + \cdots + 1) \\ &= (1 + n) + (2 + (n-1)) + \cdots + (n+1) \\ &= (n+1) + (n+1) + \cdots + (n+1) \\ &= n(n+1) \end{aligned}$$

This implies that: $L(n) = \frac{n(n+1)}{2} = R(n)$.

A Proof by Induction

Prove correctness for $n+1$ assuming correctness for n:

$$\begin{aligned}
L(n+1) &= 1+2+\cdots+n+(n+1) \\
&= L(n)+(n+1) \\
&= R(n)+(n+1) \\
&= \frac{n(n+1)}{2}+(n+1) \\
&= \frac{(n+1)(n+2)}{2} \\
&= R(n+1)
\end{aligned}$$

The Pigeonhole Principle

Simple form: If $k+1$ objects are placed into k boxes then there exists at least one box containing 2 or more objects.

Dual form: If k objects are placed into $k+1$ boxes then there exists at least one empty box.

Generalized form: If K objects are placed into k boxes then there exists at least one box containing $\lceil K/k \rceil$ or more objects.

Hand Shaking

Problem: In a group of n people handshaking takes place. No-one shakes their own hand and no pair of people shake hands more than once. Show that there must be two people who have shaken the same number of hands.

Proof:

- ⋆ For $1 \leq i \leq n$, person P_i shakes $0 \leq h_i \leq n-1$ hands.
- ⋆ There cannot be 2 people P_i and P_j such that $h_i = 0$ and $h_j = n-1$.
- ⋆ There are only $n-1$ possible values for $h_1, h_2, \ldots, h_n$.
- ⋆ By the pigeonhole principle at least 2 people have shaken the same number of hands.

Friends and Enemies

Problem: In a group of 6 people, each pair of individuals consists of 2 friends or 2 enemies. Show that there are either 3 mutual friends or 3 mutual enemies in the group.

Proof:

- ⋆ Let A be one of the 6 people.
- ⋆ By the generalized pigeonhole principle either A has 3 friends or A has 3 enemies.
- ⋆ Assume A has 3 friends B, C, D.
- ⋆ If any 2 of B, C, D are friends then these 2 and A are 3 mutual friends.
- ⋆ Otherwise B, C, D are 3 mutual enemies.
- ⋆ Similar arguments hold if B, C, D are the enemies of A.

Discussing Topics

Problem: 17 people discuss 3 topics where each pair discuss only 1 topic. Prove that there are at least 3 people who discuss among themselves the same topic.

Proof:

⋆ Let A be one of the 17 people.

⋆ By the generalized pigeonhole principle there exists a topic T discussed by A with at least 6 people.

⋆ If among these 6 people there are B and C who discuss T, then A, B, C discuss T among themselves.

⋆ Otherwise, these 6 people discuss only the other 2 topics.

⋆ By the previous problem, there are 3 people who discuss among themselves the same topic.

Induction vs. Pigeonhole Principle

Induction: . . . then for all . . .

Pigeonhole Principle: . . . then there exists . . .

A Simple Recursive Formula

- Consider the following recursive formula:
  - ⋆ $T(1) = 0.$
  - ⋆ $T(n) = 2T(n/2) + n.$
- Compute the solution for small powers of 2:
  - ⋆ $T(2) = 2T(1) + 2 = 2.$
  - ⋆ $T(4) = 2T(2) + 4 = 8.$
  - ⋆ $T(8) = 2T(4) + 8 = 24.$
  - ⋆ $T(16) = 2T(8) + 16 = 64.$

Guessing the Solution

- Guess $T(n) = n \log_2 n$ for $n = 2^k$ (power of 2).

- Verify the guess for small numbers:
  - ⋆ $1 \log_2 1 = 0$.
  - ⋆ $2 \log_2 2 = 2$.
  - ⋆ $4 \log_2 4 = 8$.
  - ⋆ $8 \log_2 8 = 24$.
  - ⋆ $16 \log_2 16 = 64$.

A Proof by Induction

$$\begin{aligned}
T(n) &= 2T(n/2) + n \\
&= 2(n/2)\log_2(n/2) + n \\
&= n(\log_2 n - 1) + n \\
&= n\log_2 n
\end{aligned}$$

A More Complicated Recursive Formula

- Consider the following recursive formula:
  - ⋆ $T(1) = a$.
  - ⋆ $T(n) = 2T(n/2) + bn$.

  For some constants a, b (independent of n).

- Compute the solution for small powers of 2:
  - ⋆ $T(2) = 2T(1) + 2b = 2b + 2a$.
  - ⋆ $T(4) = 2T(2) + 4b = 8b + 4a$.
  - ⋆ $T(8) = 2T(4) + 8b = 24b + 8a$.
  - ⋆ $T(16) = 2T(8) + 16b = 64b + 16a$.

Guessing the Solution

- Guess $T(n) = bn\log_2 n + an$ for $n = 2^k$ (power of 2).
- Verify the guess for small numbers:
  - ⋆ $b \cdot 1\log_2 1 + a \cdot 1 = a.$
  - ⋆ $b \cdot 2\log_2 2 + a \cdot 2 = 2b + 2a.$
  - ⋆ $b \cdot 4\log_2 4 + a \cdot 4 = 8b + 4a.$
  - ⋆ $b \cdot 8\log_2 8 + a \cdot 8 = 24b + 8a.$
  - ⋆ $b \cdot 16\log_2 16 + a \cdot 16 = 64b + 16a.$

A Proof by Induction

$$\begin{aligned}
T(n) &= 2T(n/2) + bn \\
&= 2(b(n/2)\log_2(n/2) + a(n/2)) + bn \\
&= bn(\log_2 n - 1) + an + bn \\
&= bn\log_2 n + an
\end{aligned}$$

Another Recursive Formula

- Consider the following recursive formula:
  - ⋆ $T(1) = a.$
  - ⋆ $T(n) = T(n/2) + b.$

  For some constants a, b (independent of n).

- Compute the solution for small powers of 2:
  - ⋆ $T(2) = T(1) + b = b + a.$
  - ⋆ $T(4) = T(2) + b = 2b + a.$
  - ⋆ $T(8) = T(4) + b = 3b + a.$
  - ⋆ $T(16) = T(8) + b = 4b + a.$

Guessing the Solution

- Guess $T(n) = b \log_2 n + a$ for $n = 2^k$ (power of 2).

- Verify the guess for small numbers:

  - ⋆ $b \cdot \log_2 1 + a = a.$
  - ⋆ $b \cdot \log_2 2 + a = b + a.$
  - ⋆ $b \cdot \log_2 4 + a = 2b + a.$
  - ⋆ $b \cdot \log_2 8 + a = 3b + a.$
  - ⋆ $b \cdot \log_2 16 + a = 4b + a.$

A Proof by Induction

$$\begin{aligned} T(n) &= T(n/2) + b \\ &= (b\log_2(n/2) + a) + b \\ &= b(\log_2 n - 1) + a + b \\ &= b\log_2 n + a \end{aligned}$$

Another Recursive Formula

- Consider the following recursive formula:
  - ⋆ $T(1) = 0.$
  - ⋆ $T(n) = 4T(n/2) + n.$
- Compute the solution for small powers of 2:
  - ⋆ $T(2) = 4T(1) + 2 = 2.$
  - ⋆ $T(4) = 4T(2) + 4 = 12.$
  - ⋆ $T(8) = 4T(4) + 8 = 56.$
  - ⋆ $T(16) = 4T(8) + 16 = 240.$

Guessing the Solution

- Guess $T(n) = n^2 - n$ for $n = 2^k$ (power of 2).

- Verify the guess for small numbers:

  - ⋆ $1^2 - 1 = 0.$
  - ⋆ $2^2 - 2 = 2.$
  - ⋆ $4^2 - 4 = 12.$
  - ⋆ $8^2 - 8 = 56.$
  - ⋆ $16^2 - 16 = 240.$

A Proof by Induction

$$\begin{aligned}
T(n) &= 4T(n/2) + n \\
&= 4((n/2)^2 - (n/2)) + n \\
&= 4(n^2/4) - 2n + n \\
&= n^2 - n
\end{aligned}$$

The Master Theorem: Assumptions

- ⋆ Let $a > 0$ and $b > 1$ and $d \geq 0$ be constants.
  - Independent of n.

- ⋆ Let $T(1) = \Theta(1)$.

- ⋆ Let $T(n) = aT(n/b) + \Theta(n^d)$ for $n > 1$.
  - n/b can be either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$.

The Master Theorem

1. If $d < \log_b a$
   then $T(n) = \Theta(n^{\log_b a})$.

2. If $d = \log_b a$
   then $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^d \log n)$.

3. If $d > \log_b a$
   then $T(n) = \Theta(n^d)$.

Example I

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 9T(n/3) + n \end{aligned}$$

- ⋆ $a = 9$.
- ⋆ $b = 3$.
- ⋆ $d = 1$.
- ⋆ $\log_b a = \log_3 9 = 2 > 1 = d$.

$\Longrightarrow$ Case 1: $T(n) = \Theta(n^2)$.

Example II

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(2n/3) + 1 \end{aligned}$$

- ⋆ $a = 1$.
- ⋆ $b = 3/2$.
- ⋆ $d = 0$.
- ⋆ $\log_b a = \log_{3/2} 1 = 0 = d$.

$\Longrightarrow$ Case 2: $T(n) = \Theta(\log n)$.

Example III

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 3T(n/4) + n \end{aligned}$$

- ⋆ $a = 3$.
- ⋆ $b = 4$.
- ⋆ $d = 1$.
- ⋆ $\log_b a = \log_4 3 \approx 0.793 < 1 = d$.

$\Longrightarrow$ Case 3: $T(n) = \Theta(n)$.

Proof Outline for the Master Theorem

- ⋆ Assume that n is a power of b.
- ⋆ There are $\log_b(n)$ levels to the recursion.
- ⋆ The kth level is made up of a^k subproblems.
- ⋆ Each subproblem at level k is of size n/b^k.
- ⋆ The total work done at level k is:

$$w(k) = a^k \cdot \Theta\left(\frac{n}{b^k}\right)^d = \Theta(n^d) \cdot \left(\frac{a}{b^d}\right)^k$$

Proof Outline for the Master Theorem

- ⋆ The numbers $w(0), w(1), \ldots, w(\log_b(n))$ form a geometric series with ratio a/b^d.
- ⋆ $w(0) = \Theta(n^d)$.
- ⋆ $w(\log_b(n)) = a^{\log_b(n)} = n^{\log_b(a)}$.
- ⋆ $T(n) = \sum_{k=0}^{\log_b(n)} w(k) = \Theta(n^d) \sum_{k=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^k$.
- ⋆ The sum depends on the ratio a/b^d.

Proof Outline for the Master Theorem

- ⋆ If $a/b^d < 1$ then the sum is dominated by the first term.
  - $T(n) = \Theta(w(0)) = \Theta(n^d)$.
- ⋆ If $a/b^d = 1$ then all $\Theta(\log(n))$ terms are equal to $\Theta(n^d)$.
  - $T(n) = \Theta(n^d \log(n))$.
- ⋆ If $a/b^d > 1$ then the sum is dominated by the last term.
  - $T(n) = \Theta(w(\log_b(n))) = \Theta\left(n^{\log_b(a)}\right)$.
- ⋆ Comparing a/b^d to 1 is equivalent to comparing a to b^d which is equivalent to comparing $\log_b(a)$ to d.

Analysis of Algorithms – Worst Case Complexity

Algorithm $\mathcal{A}$ has a worst case complexity $T(n)$:

- ⋆ To prove that $T(n) = O(f(n))$, show this for all inputs of size n for all n.
- ⋆ To prove that $T(n) = \Omega(f(n))$, show this for one input of size n for infinitely many n.
- ⋆ To prove that $T(n) = \Theta(f(n))$: show that $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$.

Analysis of Algorithms – Average Case Complexity

Algorithm $\mathcal{A}$ has an average case complexity $T(n)$ for a given distribution:

- ⋆ To prove that $T(n) = O(f(n))$, show this
  by averaging over all inputs of size n for all n.
- ⋆ To prove that $T(n) = \Omega(f(n))$, show this
  by averaging over all inputs of size n for infinitely many n.
- ⋆ To prove that $T(n) = \Theta(f(n))$:
  show that $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$.

Properties of Logarithms

For real numbers $a > 1$, $b > 1$, $x > 0$, and $y > 0$:

- ◇ $\log_a x = z \Leftrightarrow a^z = x$
- ◇ $\log_a 1 = 0$
- ◇ $\log_a a = 1$
- ◇ $\log_a x^y = y \log_a x$
- ◇ $\log_a xy = \log_a x + \log_a y$
- ◇ $\log_a \frac{x}{y} = \log_a x - \log_a y$
- ◇ $a^{\log_b x} = x^{\log_b a}$
- ◇ $\log_a b = \frac{1}{\log_b a}$
- ◇ $\log_a x = \frac{\log_b x}{\log_b a} = \log_a b \cdot \log_b x$

Summation Formulas

- ⋆ $\sum_{i=\ell}^{u} 1 = 1 + 1 + \cdots + 1 = u - \ell + 1$
  - ◇ $\sum_{i=1}^{n} 1 = n = \frac{1}{1}n^1$
- ⋆ $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2$
- ⋆ $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3}n^3$
- ⋆ $\sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + n^k \approx \frac{1}{k+1}n^{k+1}$

Summation Formulas

- ⋆ $\sum_{i=0}^{n} a^i = 1 + a + \cdots + a^n = \frac{a^{n+1}-1}{a-1} \quad (a > 1)$
  - ◇ $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$

- ⋆ $\sum_{i=0}^{n} a^i = 1 + a + \cdots + a^n = \frac{1-a^{n+1}}{1-a} \quad (a < 1)$
  - ◇ $\sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

- ⋆ $\sum_{i=0}^{\infty} a^i = 1 + a + a^2 + \cdots = \frac{1}{1-a} \quad (a < 1)$
  - ◇ $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$

Summation Formulas

$$\sum_{i=1}^{n}(i \cdot 2^i) = 1 \cdot 2 + 2 \cdot 2^2 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2$$

$$\begin{array}{cccccc}
2 & & & & & \\
4 & 4 & & & & \\
8 & 8 & 8 & & & \\
16 & 16 & 16 & 16 & & \\
\vdots & \vdots & \vdots & \vdots & & \\
2^n & 2^n & 2^n & 2^n & \cdots & 2^n \\
-- & -- & -- & -- & -- & -- \\
2(2^n - 1) & 4(2^{n-1} - 1) & 8(2^{n-2} - 1) & 16(2^{n-3} - 1) & \cdots & 2^n(2-1)
\end{array}$$

$$\begin{aligned}
\sum_{i=1}^{n}(i \cdot 2^i) &= \sum_{i=1}^{n} 2^i(2^{n-i+1} - 1) = \sum_{i=1}^{n}(2^{n+1} - 2^i) \\
&= n2^{n+1} - 2(2^n - 1) = (n-1)2^{n+1} + 2
\end{aligned}$$

Summation Formulas

$$\sum_{i=1}^{\infty} \frac{i}{2^i} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{i}{2^i} + \cdots = 2$$

$$\begin{array}{cccccc}
1/2 & & & & & \\
1/4 & 1/4 & & & & \\
1/8 & 1/8 & 1/8 & & & \\
\vdots & \vdots & \vdots & & & \\
1/2^i & 1/2^i & 1/2^i & \cdots & 1/2^i & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-- & -- & -- & -- & -- & -- \\
1 & 1/2 & 1/4 & \cdots & 1/2^{i-1} & \cdots
\end{array}$$

$$\sum_{i=1}^{\infty} \frac{i}{2^i} = \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

Summation Formulas

- ⋆ $\sum_{i=1}^{n} \log_2 i = \log_2 1 + \log_2 2 + \cdots + \log_2 n \approx n \log_2 n$

- ⋆ For $n = 2^k$:

$$\begin{aligned}\sum_{i=0}^{\log_2 n} \log_2 \left(\frac{n}{2^i}\right) &= \log_2 \left(\frac{n}{1}\right) + \log_2 \left(\frac{n}{2}\right) + \cdots + \log_2 \left(\frac{n}{n}\right) \\ &= 1 + 2 + \cdots + \log_2 n \\ &= \frac{\log_2 n(\log_2 n + 1)}{2} \approx \frac{\log_2^2 n}{2}\end{aligned}$$

Sum Manipulation Rules

- ⋆ $\sum_{i=\ell}^{u} c a_i = c \sum_{i=\ell}^{u} a_i$
- ⋆ $\sum_{i=\ell}^{u} (a_i \pm b_i) = \sum_{i=\ell}^{u} a_i \pm \sum_{i=\ell}^{u} b_i$
- ⋆ $\sum_{i=\ell}^{u} a_i = \sum_{i=\ell}^{m} a_i + \sum_{i=m}^{u} a_i$ for $\ell \leq m \leq u$
- ⋆ $\sum_{i=\ell+1}^{u} (a_i - a_{i-1}) = a_u - a_\ell$

Approximation of a Sum by a Definite Integral

- ⋆ For a monotonic non-decreasing function $f(x)$:

$$\int_{\ell-1}^{u} f(x)dx \leq \sum_{i=\ell}^{u} f(i) \leq \int_{\ell}^{u+1} f(x)dx$$

- ⋆ For a monotonic non-increasing function $f(x)$:

$$\int_{\ell}^{u+1} f(x)dx \leq \sum_{i=\ell}^{u} f(i) \leq \int_{\ell-1}^{u} f(x)dx$$

Harmonic Numbers

- ⋆ Harmonic numbers: $H(n) = \sum_{i=1}^{n} \frac{1}{i}$.
- ⋆ $f(x) = 1/x$ is a monotonic non-increasing function.

Upper Bound:

$$H(n) \leq 1 + \int_{1}^{n} \frac{dx}{x} = 1 + \ln(n) - \ln(1) = 1 + \ln(n).$$

Lower bound:

$$H(n) \geq \int_{1}^{n+1} \frac{dx}{x} = \ln(n+1) - \ln(1) = \ln(n+1).$$

Exact bound:

$$H_n = \ln(n) + \gamma + 1/(2n) - 1/(12n^2) + \epsilon_n/(120n^4)$$

for $0 < \epsilon_n < 1$ and $\gamma \approx 0.5772$ (Euler's constant).

Floor and Ceiling Formulas

For real x and integer n:

- ⋆ $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
- ⋆ $\lfloor x + n \rfloor = \lfloor x \rfloor + n$
- ⋆ $\lceil x + n \rceil = \lceil x \rceil + n$
- ⋆ $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$
- ⋆ $\lceil \log_2(n+1) \rceil = \lfloor \log_2 n + 1 \rfloor$

Modular Arithmetic

For integers n and m and a positive integer p:

$$(n + m) \bmod p = (n \bmod p + m \bmod p) \bmod p$$

$$(n \cdot m) \bmod p = (n \bmod p \cdot m \bmod p) \bmod p$$

Stirling's Formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

$$n \longrightarrow \infty \quad \Rightarrow \quad n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\log_2(n!) = \Theta(n \log_2 n)$$

Fibonacci Numbers

The sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

Definition:

$$F_k = \begin{cases} 0 & \text{for } k = 0, \\ 1 & \text{for } k = 1, \\ F_{k-1} + F_{k-2} & \text{for } k \geq 2. \end{cases}$$

Fibonacci Numbers - The Original Problem

Story: A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month.

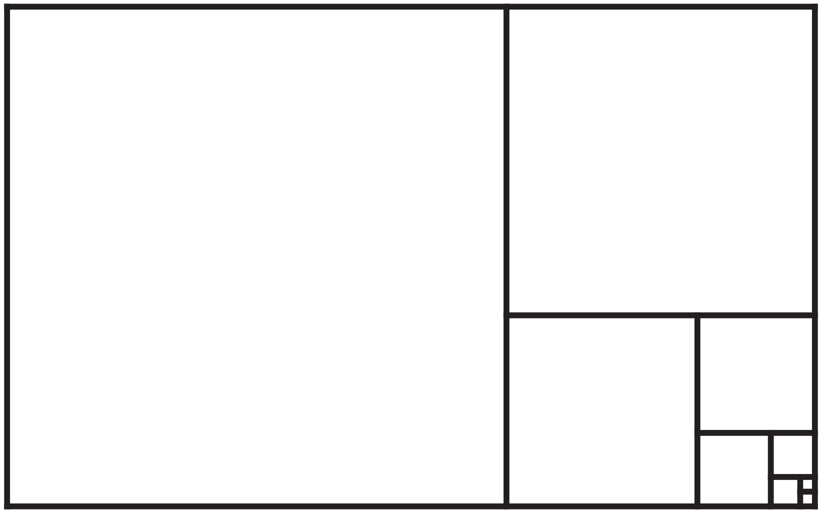
Problem: Find a recurrence relation for the number of pairs of rabbits on the island after k months, assuming that no rabbits ever die.

Solution: F_k rabbits on the island after k months.

The Golden Ratio

⋆ Find a rectangle of size $x \times 1$ such that after removing a 1×1 square the remaining rectangle has size $1 \times \frac{1}{x}$.

⇒ $x = 1 + \frac{1}{x}$.



Solving the Equation $x = 1 + \frac{1}{x}$

- ⋆ Start with $x_1 = 1$.
- ⋆ Define $x_i = 1 + \frac{1}{x_{i-1}}$ for $i > 1$.
  - – $x_2 = 1 + \frac{1}{1} = 2$.
  - – $x_3 = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$.
  - – $x_4 = 1 + \frac{1}{3/2} = \frac{5}{3} \approx 1.667$.
  - – $x_5 = 1 + \frac{1}{5/3} = \frac{8}{5} = 1.6$.
  - – $x_6 = 1 + \frac{1}{8/5} = \frac{13}{8} = 1.625$.
  - – $x_7 = 1 + \frac{1}{13/8} = \frac{21}{13} \approx 1.615$.
  - – $x_8 = 1 + \frac{1}{21/13} = \frac{34}{21} \approx 1.619$.
  - – $x_9 = 1 + \frac{1}{34/21} = \frac{55}{34} \approx 1.618$.

Solving the Equation $x = 1 + \frac{1}{x}$

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}}}}}}$$

Solving the Equation $x = 1 + \frac{1}{x}$

Proposition: $x_i = \frac{F_{i+1}}{F_i}$.

Proof: Initially $x_1 = 1 = \frac{1}{1} = \frac{F_2}{F_1}$. Assume the statement is correct for x_i and prove it by induction for x_{i+1}.

$$\begin{aligned} x_{i+1} &= 1 + \frac{1}{x_i} = 1 + \frac{1}{F_{i+1}/F_i} \\ &= 1 + \frac{F_i}{F_{i+1}} = \frac{F_{i+1} + F_i}{F_{i+1}} = \frac{F_{i+2}}{F_{i+1}}. \end{aligned}$$

The Golden Ratio

Definition: The positive root of the equation $x^2 - x - 1 = 0$.

Remark: This equation is equivalent to $x = 1 + \frac{1}{x}$.

Solution to the quadratic equation: $x = \frac{1 \pm \sqrt{5}}{2}$.

The positive root: $\phi = (1 + \sqrt{5})/2 \approx 1.618$.

The negative root: $\hat{\phi} = (1 - \sqrt{5})/2 = 1 - \phi \approx -0.618$.

k as a Function of the Fibonacci Number F_k

- ⋆ $F_k = (\phi^k - \hat{\phi}^k)/\sqrt{5}$.
- ⋆ $F_{k+1} = \phi F_k + \hat{\phi}^k$.
- ⋆ $|\hat{\phi}| < 1 \Rightarrow F_k = \frac{\phi^k}{\sqrt{5}}$ rounded to the nearest integer.

Corollary: $\log_\phi(F_k) + 1 \leq k \leq \log_\phi(F_k) + 2$

Corollary: $k = \Theta(\log(F_k))$.

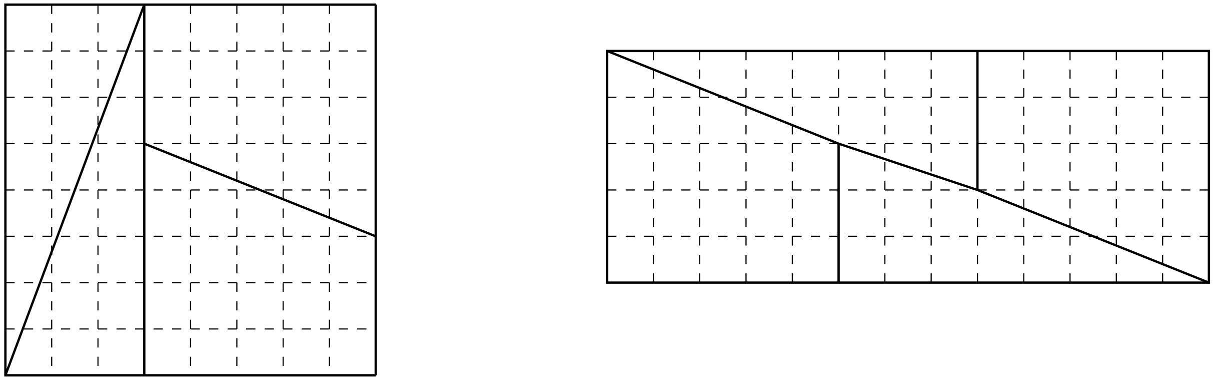
Interesting Facts

- ⋆ $F_{k-1} \cdot F_{k+1} - F_k^2 = (-1)^k$ for $k \geq 1$.
  - $F_5 \cdot F_7 - F_6^2 = 5 \cdot 13 - 8^2 = 1$.
  - $F_6 \cdot F_8 - F_7^2 = 8 \cdot 21 - 13^2 = -1$.

- ⋆ F_{mk} is a multiple of F_k for $k, m \geq 1$.
  - $F_{15} = 610$ is a multiple of $F_5 = 5$ and $F_3 = 2$.

- ⋆ $gcd(F_m, F_k) = F_{gcd(m,k)}$.
  - $gcd(F_{12}, F_{18}) = gcd(144, 2584) = 8 = F_6 = F_{gcd(18,12)}$.

A Paradox?



- The area of the left rectangle is $8 \times 8 = 64$.
- The area of the right rectangle is $5 \times 13 = 65$.
- But both rectangles contain the same 4 shapes!

Hint: $F_k / F_{k+2} \approx 1/\phi^2$ for all $k \geq 0$ ($\approx$ and not equal!).

Kilometers and Miles

- ⋆ One mile is approximately $1.609344 \approx \phi$ kilometers.
- ⋆ $F_{k_1} + F_{k_2} + \cdots + F_{k_h}$ kilometers are approximately $F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_h-1}$ miles.
- ⋆ $32 = 21+8+3 = F_8+F_6+F_4$ kilometers are approximately $F_7 + F_5 + F_3 = 13 + 5 + 2 = 20$ miles.
- ⋆ $32 = 21 + 8 + 3 = F_8 + F_6 + F_4$ miles are approximately $F_9 + F_7 + F_5 = 34 + 13 + 5 = 52$ kilometers.

Combinatorics

\# of permutations of an n-element set: $n!$

\# of k-combinations of an n-element set: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

\# of subsets of an n-element set: 2^n

\# of binary strings of length n: 2^n

\# of k-ary strings of length n: k^n

Binomial Coefficients

- ⋆ $C(n,k) = \binom{n}{k}$:
  - ◇ The number of size k subsets from the set $\{1, 2, \ldots, n\}$.
  - ◇ The coefficient of x^k in the expansion of $(1+x)^n$:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Recursive Definition:

$$C(n,k) = \binom{n}{k} = \begin{cases} 1 & \text{for } k = 0 \\ 1 & \text{for } k = n \\ \binom{n-1}{k-1} + \binom{n-1}{k} & \text{for } 0 < k < n \\ 0 & \text{otherwise} \end{cases}$$

The Pascal Triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Binomial Coefficients – Some Facts

- ⋆ $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$

- ⋆ $\binom{n}{k} = \binom{n}{n-k}$

- ⋆ $\left(\frac{n}{k}\right)^k \leq \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-k+1}{1} \leq (n-k+1)^k$

- ⋆ $\binom{n}{k} = \Theta(n^k)$ for a fix k