Algorithms

Assignment Graphs: Solutions
1. Consider the Petersen graph.

   (a) Three symmetric ("nice") drawings of the Petersen graph:

   Additional drawings: [http://mathworld.wolfram.com/PetersenGraph.html](http://mathworld.wolfram.com/PetersenGraph.html)
(b) The adjacency matrix of the Petersen graph:

\[
\begin{array}{cccccccccc}
& A & B & C & D & E & F & G & H & I & J \\
A & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
B & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
C & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
D & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
E & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
F & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
G & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
H & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
I & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
J & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

(c) The adjacency lists of the Petersen graph:

\begin{align*}
A & \rightarrow (B, E, F) \\
B & \rightarrow (A, C, G) \\
C & \rightarrow (B, D, H) \\
D & \rightarrow (C, E, I) \\
E & \rightarrow (A, D, J) \\
F & \rightarrow (A, H, I) \\
G & \rightarrow (B, I, J) \\
H & \rightarrow (C, F, J) \\
I & \rightarrow (D, F, G) \\
J & \rightarrow (E, G, H)
\end{align*}
2. Special sets in a graph.

(a) An **independent set** is a set of vertices that have no edges among them. What is the size of the **largest** independent set in the Petersen graph?

**Answer:** 4. See below two examples. The black vertices are the independent set.

(b) Prove that a larger set does not exist.

**Proof:** The largest independent set in the cycle $C_5$ is 2. To see this, let $v$ be one of the vertices in an independent set of $C_5$. Then $v$’s two neighbors cannot be in this independent set and since the remaining two vertices are neighbors only one of them can be in this independent set. The Petersen graph has an outer cycle $C_5 = (A, B, C, D, E)$ and an inner cycle $C_5 = (F, H, J, G, I)$. Each cycle may “contribute” at most two vertices to the independent set even without the additional restrictions imposed by the edges that connect both cycles. Therefore, the size of the maximum independent set in the Petersen graph is at most 4.

**Remark:** Up to “symmetry” there is only one independent set of size 4 in the unlabeled Petersen graph. Such an independent set must contain two non-adjacent vertices in the outer cycle and two non-adjacent vertices in the inner cycle. Moreover, a pair of non-adjacent vertices in any of the cycle can be combined with only one pair of non-adjacent vertices in the other cycle. This implies that there are exactly five different independent sets of size 4 in the labeled Petersen graph: $(A, C, I, J)$ (the left example above), $(A, D, G, H)$, $(B, D, F, J)$ (the right example above), $(B, E, H, I)$, and $(C, E, F, G)$.
A clique is a set of vertices that contains all the possible edges among them. What is the size of the largest clique in the Petersen graph?

**Answer:** 2. See below two examples. The black vertices are the clique.

![Diagrams of the Petersen graph with cliques highlighted.](attachment:image.png)

(d) Prove that a larger set does not exist.

**Proof:** By inspecting all possible triplets of vertices, it can be shown that the Petersen graph does not have a triangle graph ($K_3$). Therefore, $K_2$ is the largest clique in the Petersen graph.

**Remark:** Up to “symmetry” there is only one clique of size 2 in the unlabeled Petersen graph. Such a clique is an edge. Since the Petersen graph has ten edges, this implies that there are exactly ten different cliques of size 2 in the labeled Petersen graph.
(e) A **vertex cover** is a set of vertices for which each edge has at least one of its vertices in the set. What is the size of the **smallest** vertex cover in the Petersen graph?

**Answer:** 6. See below two examples. The black vertices are the vertex cover.

![Example of vertex cover](image)

(f) Prove that a smaller set does not exist.

**Proof:** The smallest vertex cover in the cycle $C_5$ is 3 since each vertex can cover at most two edges and the cycle has five edges. The Petersen graph has an outer cycle $C_5 = (A, B, C, D, E)$ and an inner cycle $C_5 = (F, H, J, G, I)$. Each cycle must “contribute” at least three vertices to the vertex cover even without the additional restrictions imposed by the edges that connect both cycles. Therefore, the size of the minimum vertex cover in the Petersen graph is at least 6.

**Remark:** Up to “symmetry” there is only one vertex cover of size 6 in the unlabeled Petersen graph. Such a vertex cover must contain three vertices in the outer cycle such that one is not adjacent to the other two which are neighbors and three vertices in the inner cycle such that one is not adjacent to the other two which are neighbors. Moreover, in order to cover the five edges that connect the two cycles, any such triplet of vertices in any of the cycle can be combined with only one such triplet of vertices in the other cycle. This implies that there are exactly five different independent sets of size 6 in the labeled Petersen graph: $(B, D, E, F, G, H)$ (the left example above), $(A, C, D, F, G, J)$, $(A, C, E, G, H, I)$ (the right example above), $(A, B, D, H, I, J)$, and $(B, C, E, F, I, J)$.
(g) A dominating set is a set of vertices for which all other vertices have at least one neighbor in this set. What is the size of the smallest dominating set in the Petersen graph?

**Answer:** 3. See below two examples. The black vertices are the dominating set.

![Example Diagram](image-url)

(h) Prove that a smaller set does not exist.

**Proof:** Each vertex can dominate at most three other vertices since the degree of each vertex is 3. Therefore, a dominating set with two vertices can dominate at most eight vertices including themselves. As a result, a dominating set in the Petersen graph must contain at least 3 vertices.

**Remark:** Up to “symmetry” there is only one dominating of size 3 in the unlabeled Petersen graph. Such a dominating set must contain two non-adjacent vertices either in the outer cycle or in the inner cycle and one vertex in the other cycle. Moreover, a pair of non-adjacent vertices in any of the cycles can be combined with only one vertex in the other cycle. This implies that there are exactly 10 = 5·2 different dominating sets of size 3 in the labeled Petersen graph: \((A, C, G)\) (the left example above), \((B, D, H)\), 
\((C, E, I)\), \((A, D, J)\), \((B, F, E)\), \((A, H, I)\), \((B, I, J)\) (the right example above), \((C, F, J)\), 
\((D, F, G)\), and \((E, G, H)\).

The maximum distance between any two vertices in the Petersen graph is 2. That is, starting with any vertex, it is possible to reach any other vertex with two adjacent edges (hops). As a result, any set that includes all the three neighbors of a particular vertex is a dominating set. Indeed, each one of the above ten different dominating sets in the labeled Petersen graph is a set that includes exactly the three neighbors of one of the vertices in the graph.
3. Paths and cycles in a graph.

Definitions:

- A path \( P = \langle v_0, v_1, \ldots, v_k \rangle \) of length \( k \) (number of edges) is an ordered list of \( k + 1 \) vertices such that the edge \((v_i, v_{i+1})\) exists for \( 0 \leq i \leq k - 1 \) and all the edges are different. In a simple path all the vertices are different.

- A cycle \( C = \langle v_0, v_1, \ldots, v_{k-1}, v_0 \rangle \) of length \( k \) is a path that starts and ends with the same vertex. In a simple cycle all the vertices except \( v_0 = v_k \) are different.

- An Euler path is a path that contains all the edges in the graph. An Euler cycle is a cycle that contains all the edges in the graph.

- A Hamiltonian path is a simple path that contains all the vertices in the graph. A Hamiltonian cycle is a simple cycle that contains all the vertices in the graph.

Theorem: The Petersen graph has neither an Euler cycle nor an Euler path. It has a Hamiltonian path but does not have a Hamiltonian cycle.

Proof: The first three parts of this problem prove this theorem except the claim that the Petersen graph does not have a Hamiltonian cycle. This claim can be verified by inspecting all possible simple cycles. Since the Petersen graphs is “very symmetric” there are many short-cuts that significantly reduce the number of inspected cycles. See https://www.youtube.com/watch?v=AVe-Qy-VcVQ for an elegant proof that the Petersen graph does not have a Hamiltonian cycle.

Observation: The following drawing of the Petersen graph is sometimes more useful than the drawing from the first page in demonstrating the answers to the four parts of this question. Both drawings are labelled-isomorphic.
(a) Find one of the longest paths (does not have to be simple) in the Petersen graph.

**Answer:** $(F - A - B - C - D - E - J - H - F - I - G - J)$ is a path that contains 11 edges. The four edges that are not in the path are $(A, E), (C, H), (B, G),$ and $(D, I)$.

**Proposition:** The length of the longest path in the Petersen graphs is 11.

**Proof:** In any path, the path-degree of each vertex is even except the end vertices whose path-degree is odd. Therefore, in any path in the Petersen graph, for which the degree of each vertex is 3, the path-degree of two vertices is 1 or 3 while the path-degree of the rest of the vertices is 2. Observe that the number of edges in any path is the sum of the path-degrees of the vertices divided by 2 since each edge is counted twice. Therefore, the longest possible path in the Petersen graph would contain eight vertices with one appearance in the path and two vertices (the end vertices) with two appearances in the path. Such a path would have $(2 \cdot 3 + 8 \cdot 2)/2 = 11$ edges.

(b) Find one of the longest cycles (does not have to be simple) in the Petersen graph.

**Answer:** In any cycle the degree of each vertex is even. Therefore, in any cycle in the Petersen graph, the cycle-degree of each vertex is exactly 2. As a result, any cycle in the Petersen graph is also a simple cycle in which each vertex appears at most once in the cycle.
See Part (d) for the construction of a simple cycle with 9 edges and 9 vertices.
(c) Find one of the Hamiltonian paths in the Petersen graph.

**Answer:** \( (A - B - C - D - E - J - H - F - I - G) \) is one of the Hamiltonian paths of the Petersen graph. This path starts at \( A \) and ends at \( H \).

Remark: The Petersen graph is a “symmetric” graph in the sense that by relabeling the vertices every vertex can be in the center of the above drawing. By inspecting all the possible Hamiltonian paths, it can be verified that for given two non-neighbors vertices \( u \) and \( v \) there are four different Hamiltonian paths that start with \( u \) and end with \( v \). Note, that since the Petersen graph does not have a Hamiltonian cycle, there cannot be a Hamiltonian path that start with \( u \) and ends with \( v \) for two neighbors \( u \) and \( v \). Since any vertex has six non-neighbor vertices, it follows that there are 24 Hamiltonian paths that start with a given vertex \( v \). This implies that the Petersen graph has a total of 240 different labeled Hamiltonian paths. There are only 120 different labeled Hamiltonian paths, if the Hamiltonian path \( v_0, v_1, \ldots, v_9 \) is equivalent to the reversed path \( v_9, v_8, \ldots, v_0 \) where the labels of the Peterson graph are \( v_i \) for \( 0 \leq i \leq 9 \).
(d) Find one of the longest simple cycles in the Petersen graph.

**Answer:** \((A - B - C - D - E - J - G - I - F - A)\) is a simple cycle that contains 9 vertices. The only vertex that is not in the cycle is \(H\).

**Remark:** The Petersen graph does not have a Hamiltonian cycle (a cycle of length 10). Therefore, a cycle of length 9 is the longest simple cycle in the Petersen graph.
4. A proper coloring of a graph is an assignment of “colors” to the vertices such that the colors assigned to the two vertices of any edge are different.

**Notations:** Let the vertices of the graph be \( v_0, v_1, \ldots, v_{n-1} \). Let the colors be the positive integers 1, 2, \ldots. Let the color assigned to \( v_i \) be \( c(v_i) \) or \( c_i \). For a given graph \( G \), let \( \chi(G) \) be the minimum number of colors that are required to properly color all the vertices of \( G \).

(a) How many colors are needed to color the null graph \( (N_n) \) with \( n \geq 1 \) vertices?
**Answer:** \( \chi(N_n) = 1 \). One color that is assigned to all the \( n \) vertices is enough. There are no conflicts because there are no edges. Therefore, \( c_i = 1 \) for all \( 1 \leq i \leq n \) is an optimal proper coloring of \( N_n \).

(b) How many colors are needed to color the complete graph \( (K_n) \) with \( n \) vertices?
**Answer:** \( \chi(K_n) = n \). Each vertex must have a different color since all possible edges exist. Therefore, \( c_i = i \) for all \( 1 \leq i \leq n \) is an optimal proper coloring of \( K_n \).

(c) How many colors are needed to color a tree \( (T_n) \) with \( n \) vertices?
**Answer:** Trivially, \( \chi(T_1) = 1 \) for a tree with one vertex. For \( n \geq 2 \), \( \chi(T_n) = 2 \). Color an arbitrary vertex with the color 1. Then color its neighbors with the color 2. Repeat the following until all the vertices of the graph are colored: color with 2 all the neighbors of a newly colored by 1 vertex and color with 1 all the neighbors of a newly colored by 2 vertex.
Since there are no cycles in a tree, it follows that each edge is reached once by the above process. The first vertex in the edge is colored by 1 or 2 while the second vertex is colored by 2 or 1 respectively.
A graph with at least one edge requires at least two colors. Therefore, the above proper coloring of the tree with two colors is optimal.

(d) How many colors are needed to color a bipartite \( (B_n) \) graph?
**Answer:** \( \chi(B_n) = 2 \) for a non-null bipartite graph. By definition, the vertices of a bipartite graph are composed of two sets \( X \) and \( Y \) such that all the edges in the graph are between a vertex from \( X \) and a vertex from \( Y \). As a result, coloring all the vertices of \( X \) with 1 and all the vertices of \( Y \) with 2 is a proper coloring.
If the bipartite graph has at least one edge, this edge already requires two colors. Therefore, the above coloring is optimal for all non-null bipartite graphs.

(e) How many colors are needed to color the cycle graph \( (C_n) \) with \( n \) vertices?
**Answer:** \( \chi(C_n) = 2 \) for an even \( n \geq 2 \) and \( \chi(C_n) = 3 \) for an odd \( n \geq 3 \). Let the edges of the cycle be \((v_i, v_{i+1})\) for \( 0 \leq i \leq n-1 \) where \((n-1)+1 = 0\). Consider the path graph \( P_n = (v_0, v_1, \ldots, v_{n-1}) \) (the cycle \( C_n \) without the edge \((v_{n-1}, v_0)\)). \( P_n \) is a tree and therefore can be colored with the colors 1 and 2 as described in part (c).
Observe that if the tree-coloring process starts with \( v_0 \), then \( c_0 = 1 \) and \( c_{n-1} = 2 \) if \( n \) is even while \( c_{n-1} = 1 \) if \( n \) is odd.
Assume \( n \) is even. Then by adding back the edge \((v_{n-1}, v_0)\) to the path \( P_n \) and maintaining the colors, a proper coloring of the cycle \( C_n \) with two colors is obtained since \( c_0 \neq c_{n-1} \).
Assume \( n \) is odd. By changing the color of \( c_{n-1} \) to 3 and maintaining the colors of the other vertices from the path coloring, a proper coloring of the cycle \( C_n \) with three colors is obtained since \( c_0 \neq c_{n-1} \).
Finally, assume to the contrary that \( C_n \) can be colored with two colors for an odd \( n \geq 3 \). It follows that the colors of all the even indexed vertices \( v_0, v_2, \ldots, v_{n-1} \) must be the same. This is a contradiction since \( c_0 \) must be different than \( c_{n-1} \).
(f) Color the Petersen graph with as few colors as possible.

**Answer:** See below a proper coloring of the Petersen graph with three colors.

In part (e), it was proven that an odd-length cycle cannot be colored with two colors. The Petersen graph contains two cycle of length 5: \((A, B, C, D, E)\) and \((F, H, J, G, I)\). Each of these cycles requires three colors. Therefore, even without the five connecting edges \((A, F), (B, G), (C, H), (D, I), \) and \((E, J)\), a proper coloring of the Petersen graph requires at least three colors.

\[\text{Diagram of Petersen graph with three colors.}\]

**Remark I:** Assume without loss of generality that the coloring of the outer cycle uses the colors 1 and 2 twice and the color 3 only once. It can be shown that in the inner cycle, the color 3 must appear twice. Assume without loss of generality that the color 1 appears once in the inner cycle. Then it can be shown that there is only one way to assign the colors to the inner cycle. As a result, up to “symmetry”, there is only one proper coloring of the Petersen graph with three colors.

**Remark II:** A careful counting implies that in the labeled Petersen graph, there are fifteen ways to color the outer cycle with three colors. Moreover, there are only two options for the color that appears once in the inner cycle because it cannot be the same as the color that appears once in the outer cycle. Finally, after fixing this color, there is only one way to assign the colors to the inner cycle to guarantee that all the five connecting edges \((A, F), (B, G), (C, H), (D, I), \) and \((E, J)\) are colored properly. As a result, there are \(30 = 15 \cdot 2\) ways to color the labeled Petersen graph with three colors.
5. Trees and bipartite graphs.

**Notations:** Define the following sets of graphs:

- $\mathcal{G}$: the set of all graphs.
- $\mathcal{B}$: the set of all bipartite graphs.
- $\mathcal{T}$: the set of all tree graphs.

**Observation:** $\mathcal{T} \subset \mathcal{G}$ and $\mathcal{B} \subset \mathcal{G}$: There are graphs that are not trees and there are graphs that are not bipartite.

**Proof:** Consider the triangle graph $C_3$ with the vertices $u$, $v$, and $w$ and the edges $(u, v)$, $(u, w)$, and $(v, w)$.

- $C_3$ has a cycle and therefore it is not a tree.
- Assume $C_3$ is a bipartite graph $H = (X, Y)$ in which all the edges are between vertices from $X$ and vertices from $Y$. The existence of the edge $(u, v)$ implies that $u$ and $v$ must belong one to $X$ and one to $Y$. But then the edges $(u, w)$ and $(v, w)$ imply that $w$ can belong to neither $X$ nor $Y$. A contradiction.

In general, the complete graph $K_n$ for $n \geq 3$ is not a bipartite graph and is not a tree.

**Proposition:** $\mathcal{T} \subseteq \mathcal{B}$: Every tree is a bipartite graph.

**Proof:** A tree can be colored with two colors (see Part (c) of Problem 4). Call the colors 1 and 2. Consider the following partition of the vertices of the tree: $X$ contains all the vertices whose color is 1 and $Y$ contains all the vertices whose color is 2. The tree is bipartite since by the definition of coloring there are no edges between vertices with the same color and as a result there are no edges between two vertices from $X$ or between two vertices from $Y$.

**Proposition:** $\mathcal{B} \nsubseteq \mathcal{T}$: There exists a bipartite graph that is not a tree.

**Proof:** $C_4$ the cycle of size 4 is a bipartite graph that is not a tree because trees do not contain cycles.

In general, any complete bipartite graph $K_{r,s}$ in which $r, s \geq 2$ is not a tree because it contains at least one cycle.

**Corollary:** $\mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$. 
6. Prove that for every $n \geq 5$ there exists a graph with $n$ vertices, all of which have degree 4.

**Proof I:** By induction. The base case is $n = 5$. The degree of each vertex in the complete graph $K_5$ is 4. See the figure below.

![Diagram of a complete graph K5](image)

Assume that for $n \geq 6$ there exists a graph $G$ with $n - 1$ vertices in which the degrees of all its vertices $v_0, v_1, \ldots, v_{n-2}$ are 4. Therefore, the sum of the degrees of all the vertices of $G$ is $4(n - 1)$. By the *handshaking lemma*, this sum is twice the number of edges in $G$. Hence, $G$ has exactly $2n - 2$ edges.

Observe that the only graphs in which any pair of edges intersect are the star graphs $S_n$ for $n \geq 3$ and $C_3$ the cycle of 3. Since $G$ has $2n - 2$ edges it cannot be a star which has only $n - 1$ edges and since $n - 1 > 3$ $G$ cannot be $C_3$. As a result, $G$ must have two disjoint edges $(x, y)$ and $(z, w)$ for which the four vertices $0 \leq x, y, z, w \leq n - 2$ are different.

Let $H$ be the graph $G$ without the edges $(x, y)$ and $(z, w)$ and with a new vertex $v = n - 1$ that is connected to the vertices $x, y, z, w$. By definition the degree of $v$ in $H$ is 4. Also, the degree in $H$ of any vertex that is not $x, y, z, w$ remains 4 as it was in $G$. Finally, the degrees in $H$ of the four vertices $x, y, z, w$ remain 4 because each one of them lost an edge and gained a new edge with $v$.

It follows that $H$ has $n$ vertices and that the degrees of all of them is 4 as required.

See below a figure that shows the changes from $G$ (the left side of the figure) to $H$ (the right side of the figure) for the four vertices $x, y, z, w$ and the new vertex $v$. Note that there could be edges in $G$ and therefore also in $H$ between other pairs of vertices among the four vertices $x, y, z, w$.

![Diagram showing changes from G to H](image)
Proof II: For $n \geq 5$, construct the following graph with the vertices $0, 1, \ldots, n-1$. The four neighbors of vertex $0 \leq i \leq n-1$ are: $i - 2, i - 1, i + 1, i + 2$ where the additions and subtractions are done modulo $n$ (e.g., $(n - 1) + 1 = 0$ and $1 - 2 = n - 1$).

For example, for $n \geq 11$, the four neighbors of vertex 8 are vertices $\{6, 7, 9, 10\}$ and the four neighbors of vertex 1 are $\{n - 1, 0, 2, 3\}$.

Observe, that the edge definitions are consistent since if $j = i + 1$ then $i = j - 1$ and if $j = i + 2$ then $i = j - 2$.

By definition, the degree of each vertex is exactly 4 as required.

See the figure below for the case $n = 8$ in the left side and the case $n = 9$ in the right side.
7. Prove that no graph has all the degrees different; that is, prove that in a degree sequence there is at least one repeated number.

**Proof:** Assume a graph with \( n \geq 2 \) vertices. There are only \( n \) possible values for a degree of a vertex in a simple graph: 0, 1, \ldots, \( n-1 \). However if the degree of a vertex is 0 (no neighbors) then no other vertex may have degree \( n-1 \) (being a neighbor off all the other vertices) and if the degree of a vertex is \( n-1 \) (being a neighbor off all the other vertices) then no other vertex may have degree 0 (no neighbors). As a result there are only two options for values of a degree of a vertex in a simple graph: either 0, 1, \ldots, \( n-2 \) or 1, \ldots, \( n-1 \). In both cases there are only \( n-1 \) possible values for the degree. By the pigeonhole principle, two out of the \( n \) degrees must be equal.

8. A *bridge* in a connected graph is an edge \((u, v)\) whose removal from the graph disconnects it. Prove that if \( G \) is a connected graph for which all the vertices have an even degree, then \( G \) does not have a bridge.

**Proof:** Let \( G \) be a connected graph for which all the vertices have an even degree. Assume to the contrary that the edge \( e = (u, v) \) is a bridge in \( G \). Remove the \( e \) from \( G \). Since \( e \) is a bridge, \( G \) is partitioned to two disjoint connected components \( G_u \) and \( G_v \) such that \( u \in G_u \) and \( v \in G_v \).

The degree of any vertex in \( G_u \) except \( u \) is even since the removal of \( e \) changes only the degrees of \( u \) and \( v \). The degree of \( u \) in \( G_u \) is odd because it is reduced by one after the removal of \( e \). As a result, the sum of the degrees of all the vertices in \( G_u \) is odd. A contradiction, since by the **handshaking lemma** the sum of the degrees in any graph is even (\( 2m \) for a graph with \( m \) edges).