1. Rearrange the following 20 functions in a decreasing order of their growth:

<table>
<thead>
<tr>
<th>Function</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log_2(n)$</td>
<td></td>
</tr>
<tr>
<td>$3^n$</td>
<td></td>
</tr>
<tr>
<td>$n^2$</td>
<td></td>
</tr>
<tr>
<td>$\sqrt{n}$</td>
<td></td>
</tr>
<tr>
<td>$n (\log_2(n))^2$</td>
<td></td>
</tr>
<tr>
<td>$n^3$</td>
<td></td>
</tr>
<tr>
<td>$n!$</td>
<td></td>
</tr>
<tr>
<td>$n$</td>
<td></td>
</tr>
<tr>
<td>$1000^n$</td>
<td></td>
</tr>
<tr>
<td>$n (\log_2(n))^2$</td>
<td></td>
</tr>
<tr>
<td>$n^{1000}$</td>
<td></td>
</tr>
<tr>
<td>$n^n$</td>
<td></td>
</tr>
<tr>
<td>$n^{1/1000}$</td>
<td></td>
</tr>
<tr>
<td>$2^n$</td>
<td></td>
</tr>
<tr>
<td>$(\log_3(n))^3$</td>
<td></td>
</tr>
<tr>
<td>$n^{1/3}$</td>
<td></td>
</tr>
<tr>
<td>$n \log_2(n)$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$(\log_2(n))^2$</td>
<td></td>
</tr>
<tr>
<td>$\log_2 \log_2(n)$</td>
<td></td>
</tr>
</tbody>
</table>

Solution:

$n^n = \omega(n!)$
$n^n = \omega(1000^n)$
$n^n = \omega(3^n)$
$n^n = \omega(2^n)$
$n^n = \omega(n^{1000})$
$n^n = \omega(n^3)$
$n^n = \omega(n^2)$
$n^n = \omega(n \log_2(n))^2$
$n^n = \omega(n \log_2(n))$
$n^n = \omega(n)$
$n^n = \omega(n / \log_2(n))$
$n^n = \omega(\sqrt{n})$
$n^n = \omega(n^{1/3})$
$n^n = \omega(n^{1/1000})$
$n^n = \omega((\log_3(n))^3)$
$n^n = \omega((\log_2(n))^2)$
$n^n = \omega(\log_2(n))$
$n^n = \omega(\log_2 \log_2(n))$
$n^n = \omega(1)$
2. (a) Prove by induction on \( n \) that for any integer \( n \geq 1 \):

\[
1 + 4 + 9 + 16 + \cdots + (n - 1)^2 + n^2 = \frac{n(n + 1)(2n + 1)}{6}.
\]

**Proof:** Define

\[
L(n) = 1 + 4 + 9 + \cdots + n^2 \\
R(n) = \frac{n(n + 1)(2n + 1)}{6}
\]

\( L(1) = R(1) \), because \( L(1) = 1 \) and \( R(1) = \frac{1 \cdot 2 \cdot 3}{6} = 1 \).

For \( n > 1 \), assume that \( L(n - 1) = R(n - 1) \) and prove that \( L(n) = R(n) \).

\[
L(n) = 1 + 4 + \cdots + (n - 1)^2 + n^2 \\
= L(n - 1) + n^2 \\
= R(n - 1) + n^2 \\
= \frac{(n - 1)n(2(n - 1) + 1)}{6} + n^2 \\
= \frac{(n - 1)n(2n - 1) + 6n^2}{6} \\
= \frac{n(2n^2 - 3n + 1 + 6n)}{6} \\
= \frac{n(2n^2 + 3n + 1)}{6} \\
= \frac{n(n + 1)(2n + 1)}{6} \\
= R(n + 1)
\]
(b) Prove by induction on $n$ that for any real number $q > 1$ and integer $n \geq 0$:

$$1 + q + q^2 + q^3 + \cdots + q^{n-1} + q^n = \frac{q^{n+1} - 1}{q - 1}.$$

**Proof:** Define

$$L(n) = 1 + q + \cdots + q^{n-1} + q^n$$

$$R(n) = \frac{q^{n+1} - 1}{q - 1}$$

$L(0) = 1$ and $R(0) = \frac{q^1 - 1}{q - 1} = 1$. Therefore, $L(0) = R(0)$.

For $n > 0$, assume that $L(n-1) = R(n-1)$ and prove that $L(n) = R(n)$.

$$L(n) = 1 + q + \cdots + q^{n-1} + q^n$$

$$= L(n-1) + q^n$$

$$= R(n-1) + q^n$$

$$= \frac{q^n - 1}{q - 1} + q^n$$

$$= \frac{(q^n - 1) + ((q - 1)q^n)}{q - 1}$$

$$= \frac{(q^n - 1) + (q^{n+1} - q^n)}{q - 1}$$

$$= \frac{q^{n+1} - 1}{q - 1}$$

$$= R(n)$$

**Another proof:**

$$L(n) = 1 + q + q^2 + \cdots + q^{n-1} + q^n$$

$$qL(n) = q + q^2 + \cdots + q^{n-1} + q^n + q^{n+1}$$

$$qL(n) - L(n) = q^{n+1} - 1$$

$$L(n)(q - 1) = q^{n+1} - 1$$

$$L(n) = \frac{q^{n+1} - 1}{q - 1}$$

$$L(n) = R(n)$$
3. (a) A bag contains socks of several different colors. You take socks out of the bag one at a time without looking until you have two socks of the same color (a matching pair).

- The bag contains $n$ white socks and $n$ black socks. How many socks do you need to take out of the bag to guarantee having one matching pair?

  **Answer 3.** After you take 2 socks, if both are of the same color then you are done. Otherwise one sock is white and one sock is black. With the third sock, you are guaranteed to have either a white pair or a black pair.

  **Pigeonhole intuition:** Think of the 2 colors as 2 holes and the socks as pigeons.

- The bag contains $n$ white socks and $n$ black socks. How many socks do you need to take out of the bag to guarantee having $k$ ($1 \leq k \leq n$) matching pairs?

  **Answer $2k + 1$.** After you take $2k$ socks, you will have $w$ white socks and $b$ black socks such that $w + b = 2k$ and $0 \leq w, b \leq n$. If both $w$ and $b$ are even then you have $w/2 + b/2 = k$ matching pairs. Otherwise both $w$ and $b$ are odd. Assume $w = 2w' + 1$ and $b = 2b' + 1$. Therefore you have $w' + b' = k - 1$ matching pairs and one white sock and one black sock. With the last sock (number $2k + 1$), you are guaranteed to have another matching pair either a white pair or a black pair for a total of $k$ matching pairs.

- The bag contains $n$ socks for each one of a $c \geq 1$ different colors. How many socks do you need to take out of the bag to guarantee having one matching pair?

  **Answer $c + 1$.** If after taking $c$ socks, you have a match than you are done. Otherwise, all the $c$ socks are of different colors. Therefore, the next sock must match one of the socks you already have.

  **Pigeonhole intuition:** Think of the $c$ colors as $c$ holes and the socks as pigeons.

- The bag contains $n$ socks for each one of a $c \geq 1$ different colors. How many socks do you need to take out of the bag to guarantee having $k$ ($1 \leq k \leq n$) matching pairs?

  **Answer $2k + c - 1$.** The maximum possible number of socks with exactly $k - 1$ matching pairs can happen when you take an odd number of socks from each color. That is, you take $2k - 2$ matched socks and $c$ non-matched socks for a total of $2k + c - 2$ socks. Now, the next sock you take must match one of the non-matched socks. Therefore, after taking $2k + c - 1$ socks, you are guaranteed to have another matching pair.

  **Remark:** Note that $c = 2$ for the case of $n$ white socks and $n$ black socks. Indeed, $2k + c - 1 = 2k + 1$.

(b) A bag contains $n$ left shoes and $n$ right shoes. You take shoes out of the bag one at a time until you have at least one left shoe and one right shoe (a matching pair).

- How many shoes do you need to take out of the bag to guarantee having one matching pair?

  **Answer $n + 1$.** The first $n$ shoes could all be left shoes or could all be right shoes. The $n + 1$st shoe guarantees one matching pair.

- How many shoes do you need to take out of the bag to guarantee having $k$ ($1 \leq k \leq n$) matching pairs?

  **Answer $n + k$.** The first $n$ shoes could all be left shoes or could all be right shoes. The $k$ extra shoes guarantee $k$ matching pairs.

  **Pigeonhole intuition:** Imagine $n$ holes each may contain at most one male pigeon and at most one female pigeon.
4. Find the exact solution to the following recursive formulas.

(a) **Definition:**

\[
T(0) = x \\
T(n) = T(n - 1) + y
\]

**Bottom-Up evaluation:**

\[
T(0) = x \\
T(1) = T(0) + y = x + y \\
T(2) = T(1) + y = x + 2y \\
T(3) = T(2) + y = x + 3y \\
T(4) = T(3) + y = x + 4y \\
\vdots \\
T(i) = T(i - 1) + y = x + iy \\
\vdots \\
T(n) = T(n - 1) + y = x + ny
\]

**Top-Down evaluation:**

\[
T(n) = T(n - 1) + y \\
= ((T(n - 2) + y) + y = T(n - 2) + 2y \\
= ((T(n - 3) + y) + 2y = T(n - 3) + 3y \\
= ((T(n - 4) + y) + 3y = T(n - 4) + 4y \\
\vdots \\
= ((T(n - i) + y) + (i - 1)y = T(n - i) + iy \\
\vdots \\
= ((T(n - n) + y) + (n - 1)y = T(0) + ny \\
= x + ny
\]

**Solution:** \( T(n) = x + ny. \)

**Proof by induction:**

\( T(0) = x = x + 0 \cdot y. \)

For \( n > 0 \), assume \( T(n - 1) = x + (n - 1)y \) and prove that \( T(n) = x + ny. \)

\[
T(n) = T(n - 1) + y \\
= x + (n - 1)y + y \\
= x + ny
\]
(b) Definition:

\[
\begin{align*}
T(0) &= s \\
T(n) &= rT(n-1)
\end{align*}
\]

Bottom-Up evaluation:

\[
\begin{align*}
T(0) &= s \\
T(1) &= rT(0) = rs \\
T(2) &= rT(1) = r(rs) = r^2s \\
T(3) &= rT(2) = r(r^2s) = r^3s \\
T(4) &= rT(3) = r(r^3s) = r^4s \\
& \vdots \\
T(i) &= r(T(i-1)) = r(r^{i-1}s) = r^is \\
& \vdots \\
T(n) &= r(T(n-1)) = r(r^{n-1}s) = r^n s
\end{align*}
\]

Top-Down evaluation:

\[
\begin{align*}
T(n) &= rT(n-1) \\
& = r(r(T(n-2))) = r^2T(n-2) \\
& = r^2(r(T(n-3))) = r^3T(n-3) \\
& = r^3(r(T(n-4))) = r^4T(n-4) \\
& \vdots \\
& = r^{i-1}(r(T(n-i))) = r^iT(n-i) \\
& \vdots \\
& = r^{n-1}(r(T(n-n))) = r^nT(0) \\
& = r^n s
\end{align*}
\]

Solution: \( T(n) = r^n s. \)

Proof by induction:

\( T(0) = s = r^0 s. \)

For \( n > 0, \) assume \( T(n-1) = r^{n-1}s \) and prove that \( T(n) = r^n s. \)

\[
\begin{align*}
T(n) &= rT(n-1) \\
& = r(r^{n-1}s) \\
& = r^n s
\end{align*}
\]
(c) Definition:

\[
T(1) = z \\
T(n) = nT(n-1)
\]

Bottom-Up evaluation:

\[
T(1) = z \\
T(2) = 2T(1) = 2z \\
T(3) = 3T(2) = 3(2z) = 6z \\
T(4) = 4T(3) = 4(6z) = 24z \\
T(5) = 5T(4) = 5(24z) = 120z \\
\vdots \\
T(i) = iT(i-1) = i((i-1)!z) = i!z \\
\vdots \\
T(n) = nT(n-1) = n((n-1)!z) = n!z
\]

Top-Down evaluation:

\[
T(n) = nT(n-1) \\
= n((n-1)T(n-2)) = n(n-1)T(n-2) \\
= n(n-1)((n-2)T(n-3)) = n(n-1)(n-2)T(n-3) \\
= n(n-1)(n-2)((n-3)T(n-4)) = n(n-1)(n-2)(n-3)T(n-4) \\
\vdots \\
= n(n-1)\cdots(n-2)((n-1)T(n-2)) = n(n-1)\cdots(n-2)(n-1)T(n-2) \\
\vdots \\
= n(n-1)\cdots(n-1)(n-2)!T(n-3) = n(n-1)\cdots3\cdot2\cdot T(1) \\
= n!z
\]

Solution: \( T(n) = n!z \).

Proof by induction:

\( T(1) = z = 1!z \).

For \( n > 1 \), assume \( T(n-1) = (n-1)!z \) and prove that \( T(n) = n!z \).

\[
T(n) = nT(n-1) \\
= n((n-1)!z) \\
= n!z
\]
(d) Definition:

\[ T(1) = 1 \]
\[ T(n) = T(n-1) + (2n - 1) \]

Bottom-Up evaluation:

\[
\begin{align*}
T(1) &= 1 = 1^2 \\
T(2) &= T(1) + (2 \cdot 2 - 1) = 1 + 3 = 4 = 2^2 \\
T(3) &= T(2) + (2 \cdot 3 - 1) = 4 + 5 = 9 = 3^2 \\
T(4) &= T(3) + (2 \cdot 4 - 1) = 9 + 7 = 16 = 4^2 \\
T(5) &= T(4) + (2 \cdot 5 - 1) = 16 + 9 = 25 = 5^2 \\
&\vdots \\
T(i) &= T(i-1) + (2i - 1) = (i-1)^2 + (2i - 1) = i^2 \\
&\vdots \\
T(n) &= T(n-1) + (2n - 1) = (n-1)^2 + (2n - 1) = n^2
\end{align*}
\]

Top-Down evaluation:

\[
\begin{align*}
T(n) &= T(n-1) + (2n - 1) \\
&= (T(n-2) + (2n - 3)) + (2n - 1) = T(n-2) + (4n - 4) \\
&= (T(n-3) + (2n - 5)) + (4n - 4) = T(n-3) + (6n - 9) \\
&= (T(n-4) + (2n - 7)) + (6n - 9) = T(n-4) + (8n - 16) \\
&\vdots \\
&= (T(n - i) + (2n - 2i + 1)) + (2(i - 1)n - (i - 1)^2) = T(n - i) + (2m - i^2) \\
&\vdots \\
&= (T(n - (n - 1)) + (2n - 2(n - 1) + 1)) + (2((n - 1) - 1)n - ((n - 1) - 1)^2) \\
&= (T(1) + (2n - (2n - 2) + 1)) + (2n(n - 2) - (n - 2)^2) \\
&= T(1) + 3 + (2n^2 - 4n) - (n^2 - 4n + 4) \\
&= 1 + 3 + n^2 - 4 \\
&= n^2
\end{align*}
\]

Solution: \( T(n) = n^2 \).

Proof by induction:

\( T(1) = 1^2 = 1 \).

For \( n > 1 \), assume \( T(n-1) = (n-1)^2 \) and prove that \( T(n) = n^2 \).

\[
\begin{align*}
T(n) &= T(n-1) + (2n - 1) \\
&= (n-1)^2 + (2n - 1) \\
&= n^2 - 2n + 1 + 2n - 1 \\
&= n^2
\end{align*}
\]
5. Solve the following recursive formulas using the master theorem. Assume that \( n = 2^k \) for some integer \( k \) for parts (a) and (c) and that \( n = (4/3)^k \) for some integer \( k \) for part (b).

(a)

\[
\begin{align*}
T(1) &= 1 \\
T(n) &= 8T(n/2) + n^2
\end{align*}
\]

- \( a = 8 \)
- \( b = 2 \)
- \( d = 2 \)
- \( \log_b a = \log_2 8 = 3 \geq 2 = d \)

\[\Rightarrow \text{Master theorem Case I: } T(n) = \Theta(n^{\log_b a}) = \Theta(n^3).\]

(b)

\[
\begin{align*}
T(1) &= 1 \\
T(n) &= T(3n/4) + 10
\end{align*}
\]

- \( a = 1 \)
- \( b = 4/3 \)
- \( d = 0 \)
- \( \log_b a = \log_{4/3} 1 = 0 = d \)

\[\Rightarrow \text{Master theorem Case II: } T(n) = \Theta(n^d \log n) = \Theta(\log n).\]

(c)

\[
\begin{align*}
T(1) &= 1 \\
T(n) &= T(n/2) + \sqrt{n}
\end{align*}
\]

- \( a = 1 \)
- \( b = 2 \)
- \( d = 1/2 \)
- \( \log_b a = \log_2 1 = 0 < 1/2 = d \)

\[\Rightarrow \text{Master theorem Case III: } T(n) = \Theta(n^d) = \Theta(\sqrt{n}).\]
6. What is the Θ running time and the exact number of iterations of the following functions.

(a) \( f(n) \) (* \( n > 0 \) is an integer number *)

\[
\begin{align*}
&c = 0 \\
&\text{for } i = 1 \text{ to } n \text{ do} \\
&\quad \text{for } j = 1 \text{ to } n \text{ do} \\
&\quad \quad \text{for } k = 1 \text{ to } n \text{ do} \\
&\quad \quad \quad c := c + 1 \\
\end{align*}
\]

**Answer:** The value of \( c \) at the end is the number of iterations as a function of \( n \). Based on the three loops, it follows that at the end

\[
c = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} 1 = \sum_{i=1}^{n} \sum_{j=1}^{n} n = \sum_{i=1}^{n} n^{2} = n^{3}
\]

Therefore, the running time is \( \Theta(n^{3}) \).

(b) \( f(n) \) (* \( n > 0 \) is an integer number *)

\[
\begin{align*}
&c = 0 \\
&\text{for } i = 1 \text{ to } n \text{ do} \\
&\quad \text{for } j = 1 \text{ to } i \text{ do} \\
&\quad \quad \text{for } k = 1 \text{ to } j \text{ do} \\
&\quad \quad \quad c := c + 1 \\
\end{align*}
\]

**Answer:** The value of \( c \) at the end is the number of iterations as a function of \( n \). Based on the three loops, it follows that at the end

\[
c = \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1
\]

Since \( \sum_{k=1}^{j} 1 = j \), it follows that

\[
c = \sum_{i=1}^{n} i \sum_{j=1}^{i} 1
\]

Since \( \sum_{j=1}^{i} j = 1 + 2 + \cdots + i = i(i+1)/2 \), it follows that

\[
c = \sum_{i=1}^{n} \frac{i(i+1)}{2}
\]

\[
= \frac{1}{2} \left( \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \right)
\]

\[
= \frac{1}{2} \left( \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right)
\]

\[
= \frac{(2n^3 + 3n^2 + n) + (3n^2 + 3n)}{12}
\]

\[
= \frac{2n^3 + 6n^2 + 4n}{12}
\]

\[
= \frac{n(n^2 + 3n + 2)}{6}
\]

\[
= \frac{n(n+1)(n+2)}{6}
\]

Therefore, the running time is \( \Theta(n^{3}) \).
(c) $g(x)$ \ (\* $x > 1$ is a real number \*)

while $x > 1$ do
\[ x := x/3 \]

Answer: The number of iterations as a function of $x$ is $\lceil \log_3 x \rceil$.

To see this, assume $3^{k-1} < x \leq 3^k$. After one iteration $3^{k-2} < x \leq 3^{k-1}$ and after two iterations $3^{k-3} < x \leq 3^{k-2}$. By induction, it can be proven that $3^{k-i-1} < x \leq 3^{k-i}$ after $i$ iterations. As a result, $3^{-1} < x \leq 3^0 = 1$ after $k$ iterations and the loop terminates. By definition, $k = \lceil \log_3 x \rceil$.

The running time of the function $g(x)$ is therefore $\Theta(\log x)$. 