1. Rearrange the following 20 functions in a decreasing order of their growth:

<table>
<thead>
<tr>
<th>Function</th>
<th>$1$</th>
<th>$3^n$</th>
<th>$n^2$</th>
<th>$\sqrt{n}$</th>
<th>$\frac{n}{\log_2(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log_2(n)$</td>
<td>$1$</td>
<td>$3^n$</td>
<td>$n^2$</td>
<td>$\sqrt{n}$</td>
<td>$\frac{n}{\log_2(n)}$</td>
</tr>
<tr>
<td>$n^3$</td>
<td></td>
<td>$n!$</td>
<td>$n$</td>
<td>$1000^n$</td>
<td>$n(\log_2(n))^2$</td>
</tr>
<tr>
<td>$n^{1000}$</td>
<td></td>
<td>$n^n$</td>
<td>$n^{1/1000}$</td>
<td>$2^n$</td>
<td>$(\log_3(n))^3$</td>
</tr>
<tr>
<td>$n^{1/3}$</td>
<td></td>
<td>$n\log_2(n)$</td>
<td>$1$</td>
<td>$(\log_2(n))^2$</td>
<td>$\log_2 \log_2(n)$</td>
</tr>
</tbody>
</table>

Solution:

\[ 1 = o(\log_2 \log_2(n)) = o(\log_2(n)) = o((\log_2(n))^2) = o((\log_3(n))^3) = o(n^{1/1000}) = o(n^{1/3}) = o(\sqrt{n}) = o(n/\log_2(n)) = o(n) = o(n \log_2(n)) = o(n(\log_2(n))^2) = o(n^2) = o(n^3) = o(n^{1000}) = o(2^n) = o(3^n) = o(1000^n) = o(n!) = o(n^n) \]
2. (a) Prove by induction on \( n \) that for any integer \( n \geq 1 \):

\[
1 + 4 + 9 + 16 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

**Proof:** Define

\[
L(n) = 1 + 4 + 9 + \cdots + n^2
\]

\[
R(n) = \frac{n(n+1)(2n+1)}{6}
\]

\( L(1) = R(1) \), because \( L(1) = 1 \) and \( R(1) = \frac{1 \cdot 2 \cdot 3}{6} = 1 \).

For \( n \geq 1 \), assume that \( L(n) = R(n) \) and prove that \( L(n+1) = R(n+1) \).

\[
L(n+1) = 1 + 4 + \cdots + n^2 + (n+1)^2
\]

\[
= L(n) + (n+1)^2
\]

\[
= R(n) + (n+1)^2
\]

\[
= \frac{n(n+1)(2n+1)}{6} + (n+1)^2
\]

\[
= (n+1)n(2n+1) + 6(n+1)^2
\]

\[
= \frac{(n+1)(2n^2 + n) + (n+1)(6n + 6)}{6}
\]

\[
= \frac{(n+1)(2n^2 + 7n + 6)}{6}
\]

\[
= \frac{(n+1)(n+2)(2n+3)}{6}
\]

\[
= \frac{(n+1)((n+1) + 1)(2(n+1) + 1)}{6}
\]

\[
= R(n+1)
\]
(b) Prove by induction on $n$ that for any real number $q > 1$ and integer $n \geq 0$:

$$1 + q + q^2 + q^3 + \cdots + q^{n-1} + q^n = \frac{q^{n+1} - 1}{q - 1}.$$ 

**Proof:** Define

$$L(n) = 1 + q + \cdots + q^{n-1} + q^n$$

$$R(n) = \frac{q^{n+1} - 1}{q - 1}$$

$L(0) = 1$ and $R(0) = \frac{q^1 - 1}{q - 1} = 1$. Therefore, $L(0) = R(0)$.

For $n \geq 0$, assume that $L(n) = R(n)$ and prove that $L(n+1) = R(n+1)$.

$$L(n+1) = 1 + q + \cdots + q^{n} + q^{n+1}$$

$$= L(n) + q^{n+1}$$

$$= R(n) + q^{n+1}$$

$$= \frac{q^{n+1} - 1}{q - 1} + q^{n+1}$$

$$= \frac{(q^{n+1} - 1) + ((q - 1)q^{n+1})}{q - 1}$$

$$= \frac{q^{n+1} - 1 + (q^{n+2} - q^{n+1})}{q - 1}$$

$$= \frac{q^{n+2} - 1}{q - 1}$$

$$= R(n+1)$$

Another proof:

$$L(n) = 1 + q + q^2 + \cdots + q^{n-1} + q^n$$

$$qL(n) = q + q^2 + \cdots + q^{n-1} + q^n + q^{n+1}$$

$$qL(n) - L(n) = q^{n+1} - 1$$

$$L(n)(q - 1) = q^{n+1} - 1$$

$$L(n) = \frac{q^{n+1} - 1}{q - 1}$$

$$L(n) = R(n)$$
3. (a) A bag contains \( n \) white socks and \( n \) black socks. You take socks out of the bag one at a time until you have two socks of the same color (a matching pair).

- How many socks do you need to take out of the bag to guarantee that you will have one matching pair?

  **Answer** 3. After you take 2 socks, if both are of the same color then you are done. Otherwise one sock is white and one sock is black. With the third sock, you are guaranteed to have either a white pair or a black pair.

- How many socks do you need to take out of the bag to guarantee that you will have \( k \) (\( 1 \leq k \leq n \)) matching pairs?

  **Answer** \( 2k + 1 \). After you take \( 2k \) socks, you will have \( w \) white socks and \( b \) black socks such that \( w + b = 2k \) and \( 0 \leq w, b \leq n \). If both \( w \) and \( b \) are even then you have \( w/2 + b/2 = k \) matching pairs. Otherwise both \( w \) and \( b \) are odd. Assume \( w = 2w' + 1 \) and \( b = 2b' + 1 \). Therefore you have \( w' + b' = k - 1 \) matching pairs and one white sock and one black sock. With the last sock (number \( 2k + 1 \)), you are guaranteed to have another matching pair either a white pair or a black pair for a total of \( k \) matching pairs.

  **Pigeonhole intuition:** Think of the colors as 2 holes and the socks as \( 2k + 1 \) pigeons.

(b) A bag contains \( n \) left shoes and \( n \) right shoes. You take shoes out of the bag one at a time until you have at least one left shoe and one right shoe (a matching pair).

- How many shoes do you need to take out of the bag to guarantee that you will have one matching pair?

  **Answer** \( n + 1 \). The first \( n \) shoes could all be left shoes or could all be right shoes. The \( n + 1 \)st show must guarantee a matching pair.

- How many shoes do you need to take out of the bag to guarantee that you will have \( k \) (\( 1 \leq k \leq n \)) matching pairs?

  **Answer** \( n + k \). The first \( n \) shoes could all be left shoes or could all be right shoes. The \( k \) shoes must guarantee \( k \) matching pairs.

  **Pigeonhole intuition:** Imagine \( n \) holes each may contain at most one right shoe and one left shoe where the shows are the pigeons.
4. Find the exact solution to the following recursive formulas.

(a) **Definition:**

\[
T(1) = 1 \\
T(n) = T(n-1) + 3
\]

**Small values:**

\[
T(1) = 1 \\
T(2) = T(1) + 3 = 4 \\
T(3) = T(2) + 3 = 7 \\
T(4) = T(3) + 3 = 10 \\
T(5) = T(4) + 3 = 13
\]

**Guess:** \( T(n) = 3n - 2. \)

**Proof by induction:**

\[
T(1) = 3 \cdot 1 - 2 = 1.
\]

For \( n \geq 1 \), assume \( T(n) = 3n - 2 \) and prove that \( T(n + 1) = 3(n + 1) - 2. \)

\[
T(n + 1) = T(n) + 3 \\
= (3n - 2) + 3 \\
= (3n + 3) - 2 \\
= 3(n + 1) - 2
\]

(b) **Definition:**

\[
T(1) = 1 \\
T(n) = T(n-1) + (2n - 1)
\]

**Small values:**

\[
T(1) = 1 \\
T(2) = T(1) + (2 \cdot 2 - 1) = 1 + 3 = 4 \\
T(3) = T(2) + (2 \cdot 3 - 1) = 4 + 5 = 9 \\
T(4) = T(3) + (2 \cdot 4 - 1) = 9 + 7 = 16 \\
T(5) = T(4) + (2 \cdot 5 - 1) = 16 + 9 = 25
\]

**Guess:** \( T(n) = n^2. \)

**Proof by induction:**

\[
T(1) = 1^2 = 1.
\]

For \( n \geq 1 \), assume \( T(n) = n^2 \) and prove that \( T(n + 1) = (n + 1)^2. \)

\[
T(n + 1) = T(n) + (2(n + 1) - 1) \\
= T(n) + (2n - 1) \\
= n^2 + 2n + 1 \\
= (n + 1)^2
\]
(c) Definition:

\[ T(1) = 3 \]
\[ T(n) = 3T(n-1) \]

Small values:

\[ T(1) = 3 \]
\[ T(2) = 3T(1) = 3 \cdot 3 = 9 \]
\[ T(3) = 3T(2) = 3 \cdot 9 = 27 \]
\[ T(4) = 3T(3) = 3 \cdot 27 = 81 \]
\[ T(5) = 3T(4) = 3 \cdot 81 = 243 \]

Guess: \( T(n) = 3^n \).
Proof by induction:
\( T(1) = 3^1 = 3 \).
For \( n \geq 1 \), assume \( T(n) = 3^n \) and prove that \( T(n+1) = 3^{n+1} \).

\[
T(n+1) = 3T(n)
= 3 \cdot 3^n
= 3^{n+1}
\]

(d) Definition:

\[ T(1) = 1 \]
\[ T(n) = nT(n-1) \]

Small values:

\[ T(1) = 1 \]
\[ T(2) = 2T(1) = 2 \cdot 1 = 2 \]
\[ T(3) = 3T(2) = 3 \cdot 2 = 6 \]
\[ T(4) = 4T(3) = 4 \cdot 6 = 24 \]
\[ T(5) = 5T(4) = 5 \cdot 24 = 120 \]

Guess: \( T(n) = n! \).
Proof by induction:
\( T(1) = 1! = 1 \).
For \( n \geq 1 \), assume \( T(n) = n! \) and prove that \( T(n+1) = (n+1)! \).

\[
T(n+1) = (n+1)T(n)
= (n+1)n!
= (n+1)!
\]
5. Solve the following recursive formulas using the master theorem. Assume that \( n = 2^k \) for some integer \( k \) for parts (a) and (c) and that \( n = (4/3)^k \) for some integer \( k \) for part (b).

(a)

\[
\begin{align*}
T(1) &= 1 \\
T(n) &= 8T(n/2) + n^2
\end{align*}
\]

- \( a = 8 \)
- \( b = 2 \)
- \( d = 2 \)
- \( \log_b a = \log_2 8 = 3 > 2 = d \)

\[\Rightarrow \text{Master theorem Case I: } T(n) = \Theta(n^{\log_b a}) = \Theta(n^3).\]

(b)

\[
\begin{align*}
T(1) &= 1 \\
T(n) &= T(3n/4) + 10
\end{align*}
\]

- \( a = 1 \)
- \( b = 4/3 \)
- \( d = 0 \)
- \( \log_b a = \log_{4/3} 1 = 0 = d \)

\[\Rightarrow \text{Master theorem Case II: } T(n) = \Theta(n^d \log n) = \Theta(\log n).\]

(c)

\[
\begin{align*}
T(1) &= 1 \\
T(n) &= T(n/2) + \sqrt{n}
\end{align*}
\]

- \( a = 1 \)
- \( b = 2 \)
- \( d = 1/2 \)
- \( \log_b a = \log_2 1 = 0 < 1/2 = d \)

\[\Rightarrow \text{Master theorem Case III: } T(n) = \Theta(n^d) = \Theta(\sqrt{n}).\]
6. What is the $\Theta$ running time and the exact number of iterations of the following functions.

(a) $f(n)$ (* $n > 0$ is an integer number *)

\[
\begin{align*}
c & = 0 \\
\text{for } i = 1 \text{ to } n \text{ do} \\
& \quad \text{for } j = 1 \text{ to } n \text{ do} \\
& \quad \quad \text{for } k = 1 \text{ to } n \text{ do} \\
& \quad \quad \quad c := c + 1
\end{align*}
\]

**Answer:** The value of $c$ at the end is the number of iterations as a function of $n$. Based on the three loops, it follows that at the end

\[
c = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} 1 = n^3
\]

Therefore, the running time is $\Theta(n^3)$.

(b) $f(n)$ (* $n > 0$ is an integer number *)

\[
\begin{align*}
c & = 0 \\
\text{for } i = 1 \text{ to } n \text{ do} \\
& \quad \text{for } j = 1 \text{ to } i \text{ do} \\
& \quad \quad \text{for } k = 1 \text{ to } j \text{ do} \\
& \quad \quad \quad c := c + 1
\end{align*}
\]

**Answer:** The value of $c$ at the end is the number of iterations as a function of $n$. Based on the three loops, it follows that at the end

\[
c = \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1
\]

Since $\sum_{k=1}^{j} 1 = j$, it follows that

\[
c = \sum_{i=1}^{n} \sum_{j=1}^{i} j
\]

Since $\sum_{j=1}^{i} j = 1 + 2 + \cdots + i = i(i + 1)/2$, it follows that

\[
c = \sum_{i=1}^{n} \frac{i(i + 1)}{2}
\]

\[
= \frac{1}{2} \left( \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \right)
\]

\[
= \frac{1}{2} \left( \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right)
\]

\[
= \frac{(2n^3 + 3n^2 + n) + (3n^2 + 3n)}{12}
\]

\[
= \frac{2n(n^2 + 3n + 2)}{12}
\]

\[
= \frac{n(n+1)(n+2)}{6}
\]

Therefore, the running time is $\Theta(n^3)$.

(c) $g(x)$ (* $x > 1$ is a real number *)

\[
\text{while } x > 1 \text{ do} \\
\quad x := x/3
\]

**Answer:** The number of iterations as a function of $x$ is $\lfloor \log_3 x \rfloor$. Therefore, the running time is $\Theta(\log n)$. 

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