Algorithms

Assignment: Mathematical Background and Growth of Functions

Name: .................................................................
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Grade

Good Luck!
1. Rearrange the following 20 functions in a decreasing order of their growth:

\[
\begin{array}{cccccc}
\log_2(n) & 3^n & n^2 & \sqrt{n} & \frac{n}{\log_2(n)} \\
n^3 & n! & n & 1000^n & n(\log_2(n))^2 \\
n^{1000} & n^n & n^{1/1000} & 2^n & (\log_3(n))^3 \\
n^{1/3} & n \log_2(n) & 1 & (\log_2(n))^2 & \log_2 \log_2(n)
\end{array}
\]
2. (a) Prove by induction on \( n \) that for any integer \( n \geq 1 \):

\[
1 + 4 + 9 + 16 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

(b) Prove by induction on \( n \) that for any real number \( q > 1 \) and integer \( n \geq 0 \):

\[
1 + q + q^2 + q^3 + \cdots + q^{n-1} + q^n = \frac{q^{n+1} - 1}{q - 1}.
\]
3. (a) A bag contains socks of several different colors. You take socks out of the bag one at a time without looking until you have two socks of the same color (a matching pair).

- The bag contains $n$ white socks and $n$ black socks. How many socks do you need to take out of the bag to guarantee having one matching pair?
- The bag contains $n$ white socks and $n$ black socks. How many socks do you need to take out of the bag to guarantee having $k$ ($1 \leq k \leq n$) matching pairs?
- The bag contains $n$ socks for each one of $c \geq 1$ different colors. How many socks do you need to take out of the bag to guarantee having one matching pair?
- The bag contains $n$ socks for each one of $c \geq 1$ different colors. How many socks do you need to take out of the bag to guarantee having $k$ ($1 \leq k \leq n$) matching pairs?

**Hint:** Think of the Pigeonhole Principle.
(b) A bag contains $n$ left shoes and $n$ right shoes. You take shoes out of the bag one at a time until you have at least one left shoe and one right shoe (a matching pair).

- How many shoes do you need to take out of the bag to guarantee having one matching pair?
- How many shoes do you need to take out of the bag to guarantee having $k$ ($1 \leq k \leq n$) matching pairs?

**Hint:** Think of the Pigeonhole Principle.
4. Find the exact solution to the following recursive formulas. You may guess the solution and then prove it by induction.

(a) $T(0) = x$ and $T(n) = T(n - 1) + y$ for two real numbers $x$ and $y$.

(b) $T(0) = s$ and $T(n) = rT(n - 1)$ for two real numbers $s$ and $r$. 
(c) \( T(1) = z \) and \( T(n) = nT(n-1) \) for a real number \( z \).

(d) \( T(1) = 1 \) and \( T(n) = T(n-1) + (2n - 1) \).
5. Solve the following recursive formulas using the master theorem. Assume that \( n = 2^k \) for some integer \( k \) for parts (a) and (c) and that \( n = (4/3)^k \) for some integer \( k \) for part (b).

(a) \( T(1) = 1 \) and \( T(n) = 8T(n/2) + n^2 \).

(b) \( T(1) = 1 \) and \( T(n) = T(3n/4) + 10 \).

(c) \( T(1) = 1 \) and \( T(n) = T(n/2) + \sqrt{n} \).
6. What is the $\Theta$ running time and the exact number of iterations of the following functions.

(a) $f(n)$ (* $n > 0$ is an integer number *)

\[
\begin{align*}
  & c = 0 \\
  & \text{for } i = 1 \text{ to } n \text{ do} \\
  & \quad \text{for } j = 1 \text{ to } n \text{ do} \\
  & \quad \quad \text{for } k = 1 \text{ to } n \text{ do} \\
  & \quad \quad \quad \quad c := c + 1
\end{align*}
\]

(b) $f(n)$ (* $n > 0$ is an integer number *)

\[
\begin{align*}
  & c = 0 \\
  & \text{for } i = 1 \text{ to } n \text{ do} \\
  & \quad \text{for } j = 1 \text{ to } i \text{ do} \\
  & \quad \quad \text{for } k = 1 \text{ to } j \text{ do} \\
  & \quad \quad \quad \quad c := c + 1
\end{align*}
\]

(c) $g(x)$ (* $x > 1$ is a real number *)

\[
\begin{align*}
  & \text{while } x > 1 \text{ do} \\
  & \quad x := x/3
\end{align*}
\]