Algorithms

Assignment: Matrix Multiplication

Complexity

- The standard matrix-multiplication algorithm multiplies two $n \times n$ matrices using exactly n^3 multiplications and $n^2(n-1)$ additions.
- Strassen's algorithm multiplies two 2×2 matrices using exactly 7 multiplications and a total of 18 additions and subtractions.
- For $n = 2^k$, where k is a nonnegative integer, the total number of arithmetic operations performed by Strassen's algorithm is

  $$\Theta(n^{\log_2 7}) = \Theta(n^{2.807354922\ldots}) = O(n^{2.81}).$$

  For other values of n, padding both matrices to dimensions equal to the smallest power of two greater than or equal to n preserves this asymptotic bound.

Problems

1. Apply Strassen's algorithm recursively, using the 2×2 procedure as the base case. For each of $n = 4, 8, 16$, compute the exact number of multiplications and the exact total number of additions and subtractions required to multiply two $n \times n$ matrices. Include operations performed at every recursion level. Explain how you obtained each count.
2. Let $M(n)$ be the number of multiplications performed by Strassen's algorithm on two $n \times n$ matrices, where $n = 2^k$ and $k \geq 0$. Establish the recursive formula for $M(n)$, including its base case, and derive an exact closed-form expression for $M(n)$. Ignore additions and subtractions in this problem.
3. Let $A(n)$ be the total number of additions and subtractions performed by Strassen's algorithm on two $n \times n$ matrices, where $n = 2^k$ and $k \geq 0$. Establish the recursive formula for $A(n)$, including its base case, and derive an exact closed-form expression for $A(n)$. Ignore multiplications in this problem.
4. Assume that $n = 3^k$ for some integer $k \geq 1$. Suppose a method for multiplying two 3×3 matrices uses 25 multiplications and a total of 50 additions and subtractions. Consider the recursive algorithm that applies this method to $(n/3) \times (n/3)$ matrix blocks. Let $T(n)$ denote the total number of arithmetic operations performed. Establish the recursive formula for $T(n)$, including its base case, and use the Master Theorem to solve it.
5. Suppose a method for multiplying two 3×3 matrices uses r multiplications and a constant number of additions and subtractions. Recursively apply this method to $n \times n$ matrices, where n is a power of 3. What is the largest integer value of r for which the resulting algorithm has asymptotically lower arithmetic complexity than Strassen's algorithm? Justify your answer.
6. Assume that $n = 70^k$ for some integer $k \geq 1$. Suppose a fixed method for multiplying two 70×70 matrices uses 143,640 multiplications and a total of α additions and subtractions, where α is a constant independent of n. Consider the recursive algorithm that applies this method to $(n/70) \times (n/70)$ matrix blocks. Let $T(n)$ denote the total number of arithmetic operations performed. Establish the recursive formula for $T(n)$, including its base case, and use the Master Theorem to solve it.