First Steps in Graph Theory

50 Problems, Solutions, and Beyond

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Abstract

This booklet is a collection of 50 problems, with detailed solutions, for students who are taking their first steps in graph theory, for example, in a first course on discrete mathematics or on algorithms. The problems are organized by topic into seven sections, and each section can be studied on its own. Most problems start with a definition and an example, continue with concrete questions about specific graphs, and end with a general question that asks for a proof. Section 1 deals with graph isomorphism: deciding whether two graphs are the same up to the names of their vertices, relabeling graphs, and complements. Section 2 introduces families of graphs, such as paths, cycles, trees, forests, regular graphs, bipartite graphs, triangle-free graphs, and line graphs, together with two families defined for this collection, the balloon-graphs and the eight-graphs. Section 3 represents graphs by adjacency matrices and adjacency lists, both for undirected and for directed graphs, and uses these representations in simple algorithms for common neighbors, triangles, line graphs, subgraphs, unions, intersections, and complements. Section 4 deals with paths, cycles, distances, and connectivity: longest paths, Euler paths and cycles, Hamiltonian paths and cycles, the diameter, the radius, and the center of a graph, bridges, and articulation points. Section 5 studies degrees and degree sequences, including the handshake lemma, graphic sequences and the Havel–Hakimi algorithm, and regular graphs of degrees 1, 2, 3, and 4. Section 6 is about coloring the vertices and the edges of graphs, from basic families and complete graphs to line graphs. Section 7 studies special sets of vertices and edges: cliques, independent sets, vertex covers, dominating sets, and matchings, and the relations among them.

Three graphs accompany the reader throughout the booklet: the Petersen graph, the Grötzsch graph, and the FourTriangles graph. The Petersen graph has 10 vertices and 15 edges; every vertex has degree 3, it has no triangle, and it has no Hamiltonian cycle. The Grötzsch graph has 11 vertices and 20 edges; it has no triangle, yet it needs 4 colors. The FourTriangles graph has 12 vertices and 18 edges; every vertex has degree 3, and its edges form four triangles joined by six more edges. They appear in every section, so that each new concept can be explored on familiar examples, and the answers for the three graphs can be compared across sections. Detailed solutions to all the problems are given in the second part of the booklet.

How to use this booklet. Each problem is self-contained, assuming only basic concepts such as graph isomorphism, so the problems can be read in any order. Each solution starts by restating its problem, so it can also be read on its own. Many solutions go beyond the question, with alternative proofs, remarks, and extra topics such as the relations among different concepts and the symmetry of the three graphs. References between problems appear only in footnotes and in the list of concepts on page iii. In the PDF file, the number of a problem links to its solution and vice versa.

A personal note. The choice of topics, problems, and solutions reflects the personal view of the first author, who teaches undergraduate courses in discrete mathematics and graduate courses in algorithms. This booklet does not presume to set a standard for other educators who teach graph theory, nor to replace the many excellent textbooks on the subject; it is offered as one possible path for students who are taking their first steps.

The role of Claude. The problems originate in the courses of the first author. Claude, an AI model, worked with the first author on every part of the booklet: it organized the problems into topical sections, wrote new solutions and expanded existing ones, proposed additional problems, parts, proofs, and remarks, drew most of the figures, checked the solutions for correctness and consistency, verified many of the claims by computer, and typeset the booklet. Every part of the booklet was reviewed, corrected, and approved by the first author.

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Contents

		Problem	Solution
Concepts across the problems, and notation		**iii**	
1	**Graph Isomorphism**	**1**	**27**
1.1	Symmetric drawings of the Petersen graph and a look-alike	1	27
1.2	Relabeling the Grötzsch graph	1	29
1.3	Relabeling the FourTriangles graph	2	30
1.4	Always, never, or sometimes isomorphic	2	31
1.5	Non-isomorphic graphs with six vertices	2	33
1.6	Pairs of non-isomorphic graphs	3	34
1.7	The molecules C_4H_y	3	35
1.8	Complements and self-complementary graphs	3	36
2	**Families of Graphs**	**4**	**38**
2.1	The Petersen, Grötzsch, and FourTriangles graphs	4	38
2.2	Trees with six vertices	4	41
2.3	All graphs with four vertices and their complements	5	43
2.4	Removing edges from the prism graph	5	45
2.5	Trees, forests, bipartite graphs, and triangle-free graphs	6	47
2.6	Line graphs of some families of graphs	6	49
2.7	Balloon-graphs	6	50
2.8	Eight-graphs	7	52
3	**Representing Graphs**	**8**	**54**
3.1	The Petersen graph: adjacency matrix, lists, and common neighbors	8	54
3.2	A directed Petersen graph: adjacency matrix and lists	9	57
3.3	The Grötzsch graph: adjacency matrix and lists	10	59
3.4	The FourTriangles graph: adjacency matrix, lists, and triangles	11	61
3.5	Line graphs: the prism graph and algorithms	12	64
3.6	Labeled subgraphs	13	67
3.7	The union and the intersection of two graphs	13	69
3.8	Computing the complement of a graph	13	71
4	**Paths, Cycles, Distances, and Connectivity**	**14**	**73**
4.1	Longest paths and cycles in the three graphs	14	73
4.2	Hamiltonian paths and cycles in the three graphs	14	76
4.3	Distances in basic graphs	15	78
4.4	Distances in the three graphs	15	80
4.5	The center of a tree	16	82
4.6	Bridges	16	84
4.7	Articulation points	16	86
4.8	Bridges versus articulation points	17	88
5	**Degrees and Degree Sequences**	**18**	**90**
5.1	The handshake lemma	18	90
5.2	Graphic sequences	18	91
5.3	The Havel–Hakimi algorithm	19	93
5.4	1-regular and 2-regular graphs	19	95
5.5	3-regular graphs	19	97
5.6	4-regular graphs	19	100
5.7	Two vertices with the same degree	19	103
6	**Graph Coloring**	**20**	**105**
6.1	Vertex coloring of basic graphs	20	105
6.2	Edge coloring of basic graphs	20	107
6.3	Coloring complete graphs	20	108
6.4	Vertex coloring of the three graphs	21	109
6.5	Edge coloring of the three graphs	21	110
6.6	Line graphs and colorings	21	112
7	**Special Sets of Vertices and Edges**	**22**	**114**
7.1	Cliques	22	114
7.2	Independent sets	23	118
7.3	Vertex covers	24	121
7.4	Dominating sets	25	124
7.5	Matchings	26	127

Concepts Across the Problems

Many concepts appear in several problems, often in different sections. For each concept below, the numbers are the problems that use it, and "solutions of" lists solutions that use it although their problems do not mention it. In the PDF file, clicking a number leads to the problem or to the solution.

The three graphs

- **The Petersen graph:** 1.1, 2.1, 3.1, 3.2 (directed), 4.1, 4.2, 4.4(a), 5.1(b), 6.4, 6.5, 7.1–7.5.
- **The Grötzsch graph:** 1.2, 2.1, 3.3, 4.1, 4.2, 4.4(b), 5.1(b), 6.4, 6.5, 7.1–7.5.
- **The FourTriangles graph:** 1.3, 2.1, 3.4, 4.1, 4.2, 4.4(c), 5.1(b), 6.4, 6.5, 7.1–7.5; solution of 6.6(d).
- **No cycles of length 3 or 4 in the Petersen graph:** 2.1(a), 3.1(e); solutions of 1.1(b), 5.5(d), and 6.5.
- **The cycles of length 5 in the Petersen graph:** solution of 3.1(e).
- **Diameter 2 of the Petersen graph:** 3.1(d), 4.4(a),(d); solutions of 1.1(a),(b) and 7.4.
- **The five spokes of the Petersen graph:** solutions of 1.1(a), 3.1(e), 4.2(b), 6.5, and 7.5(d).
- **Shadows in the Grötzsch graph:** solutions of 3.3(d) and 6.4.
- **Symmetries:** 1.1–1.3; solutions of 2.4 and 7.1–7.5 (Symmetry and counting).

Relations and operations

- **Isomorphism:** Section 1, 2.2(b), 2.3(b), 2.7, 2.8, 5.2–5.5.
- **Complements:** 1.8, 2.3(b)–(d), 3.8, 7.1(e); solutions of 5.5(c) and 5.6(b).
- **Self-complementary graphs:** 1.8(d), 2.3(b).
- **Line graphs:** 2.6, 3.5, 6.6; solution of 7.1.
- **Labeled subgraphs, unions, and intersections:** 3.6, 3.7.

Families of graphs

- **Null graphs (no edges):** 6.1(a), 7.1–7.5; solutions of 2.6(a),(e).
- **Paths P_n:** 1.8(d), 2.6(b), 2.7, 4.3(a), 6.2(c).
- **Cycles C_n:** 1.8(a),(d), 2.6(c), 2.7, 2.8, 4.3(b), 5.4, 6.1(d),(e), 6.2(d).
- **Stars S_n:** 1.8, 2.6(d), 4.3(d), 6.2(b), 6.6(d).
- **Wheels W_n:** 6.1(e).
- **Complete graphs K_n:** 1.4(d), 4.3(c), 6.3, 7.1–7.5; solutions of 2.6(e), 5.5, 5.6, and 6.6(d).
- **Complete bipartite graphs $K_{r,s}$:** 4.3(e), 6.6(b)–(d); solutions of 5.2(b), 5.5(c), 5.6(b), and 6.1(c).
- **Trees and forests:** 1.4(e), 1.6(d), 2.1(a), 2.2, 2.3(a), 2.4(a), 2.5, 4.5, 4.6(a), 4.7(a), 6.1(b), 6.2(e).
- **Bipartite graphs:** 1.6(b), 2.1(a), 2.3(a), 2.4(b), 2.5(b),(c), 6.1(c); solutions of 1.6(c), 5.5(c), and 6.5.
- **Triangle-free graphs:** 1.1(b), 2.1(a), 2.5; solution of 6.6(d).
- **Regular graphs:** 1.1(b), 2.1(a), 2.3(a), 2.6(e), 5.4–5.6.
- **Planar graphs:** solutions of 3.5(a) and 5.5(c).
- **Balloon-graphs and eight-graphs:** 2.7, 2.8.
- **The prism graph:** 1.8(b), 2.4, 3.5; solutions of 1.1(b), 3.8, and 5.5(c).
- **The cube, the Frucht graph, the Wagner graph, and the Heawood graph:** solution of 5.5(d).

Representations and algorithms

- **Adjacency matrices:** 3.1–3.8.
- **Adjacency lists:** 3.1–3.8.
- **Directed graphs, in-degrees, and out-degrees:** 3.2.
- **Algorithms and their complexity:** 3.1(c), 3.2(d), 3.3(c), 3.4(c), 3.5(f),(g), 3.6–3.8, 5.3.
- **Lower bounds for every algorithm:** solutions of 3.5(f) and 3.6(a).
- **The Havel–Hakimi algorithm:** 5.3.

Paths, cycles, distances, and connectivity

- **Triangles:** 1.3, 3.4(c); solutions of 2.4, 4.8(d), 6.4, and 7.1.
- **Cycles of odd length:** 6.1(d); solutions of 1.6(c), 2.1(a), 2.4(b), 2.5(c), 6.1, and 6.4.
- **Common neighbors:** 3.1(c),(d); solutions of 1.1(b), 3.4, 4.3(e), 5.2(c), and 7.4.

- **Euler paths and cycles:** 4.1.
- **Hamiltonian paths and cycles:** 4.2; solutions of 1.1(a), 6.5, and 7.5.
- **Distances, eccentricity, diameter, radius, and center:** 2.2(a), 4.3–4.5; solutions of 1.1(a),(b), 2.7(b), 3.1(d), 4.7(c), and 7.4.
- **Connected graphs:** 1.7, 2.3(c),(d), 2.4(c), 4.3–4.8.
- **Connected components:** solutions of 1.5(c), 1.8, 2.1, 2.3(c), 2.5(b), 4.6(b), 4.7(a), 5.2(d), 5.3(c), and 5.4(b).
- **Bridges:** 4.6, 4.8; solution of 2.7(b).
- **Articulation points:** 4.7, 4.8.

Degrees

- **Maximum degree Δ:** 2.2(a),(c), 2.6(e), 6.2(a),(e); solutions of 1.6(b), 6.3(b), and 6.5.
- **Leaves (vertices of degree 1):** 1.6(d), 1.8, 2.2(a),(c); solutions of 2.7(b), 4.5(b), 4.7(a), 4.8(b), and 6.6(d).
- **Vertices of even and odd degree:** 4.6(b), 4.7(b), 5.1(c), 5.5(a); solutions of 4.1 and 5.1(d).
- **Degree sequences:** 1.6(c), 2.2(a), 5.1(b), 5.2, 5.3, 5.7; solution of 2.3(b).
- **Graphic sequences and realizations:** 5.2, 5.3; solutions of 1.6(c) and 5.7.
- **The handshake lemma:** 5.1; solutions of 1.7, 2.1, 3.3(c), 4.6(b), 5.2, 5.5(a), and 5.6(c).
- **Partitions of integers:** 5.4(c),(d).

Coloring

- **Vertex coloring:** 6.1, 6.3(a), 6.4, 6.6, 7.1(d), 7.2(d); solution of 2.1(a).
- **Edge coloring:** 6.2, 6.3(b), 6.5, 6.6; solution of 7.5(d).

Special sets of vertices and edges

- **Cliques:** 7.1.
- **Independent sets:** 7.1(e), 7.2, 7.3(d).
- **Vertex covers:** 7.3, 7.4(d), 7.5(f).
- **Dominating sets:** 7.4.
- **Matchings:** 2.6(a), 7.5; solution of 5.4(a).
- **Perfect matchings:** solution of 7.5(d).

Proof techniques

- **Proofs by induction:** solutions of 4.8(d), 5.5(d), and 5.6(c).
- **The pigeonhole principle:** solution of 5.7.
- **Counting by inclusion–exclusion:** solution of 7.2.
- **Exhaustive search:** solutions of 4.7(b) and 5.3.

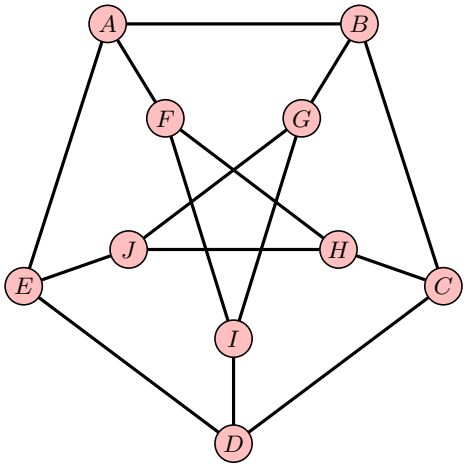
Notation

P_n	the path with n vertices	(u, v)	the edge between u and v
C_n	the cycle with n vertices	$u \to v$	the directed edge from u to v
K_n	the complete graph with n vertices	$A - B - C - D$	a path, written by its vertices
S_n	the star with n vertices	$\mathcal{P} = \langle v_0, v_1, \ldots, v_k \rangle$	a path with k edges
$K_{r,s}$	the complete bipartite graph with sides of sizes r and s	$d(u, v)$	the distance between u and v
W_n	the wheel with n vertices	d_i	the degree of the vertex v_i
M_k	the matching with k edges	Δ	the maximum degree
$\overline{G}$	the complement of G	D_G	the degree sequence of G
$L(G)$	the line graph of G	$\chi(G)$	the minimum number of colors needed to color G
α	the size of a largest independent set	$\lfloor x \rfloor$, $\lceil x \rceil$	x rounded down and up
n, m	the numbers of vertices and edges	$p(n)$	the number of ways to write n as a sum of integers that are at least 3
O, Ω, Θ	upper, lower, and tight bounds on the complexity	CL, IS, VC, DS, M	a clique, an independent set, a vertex cover, a dominating set, and a matching

Part I: Problems

1 Graph Isomorphism

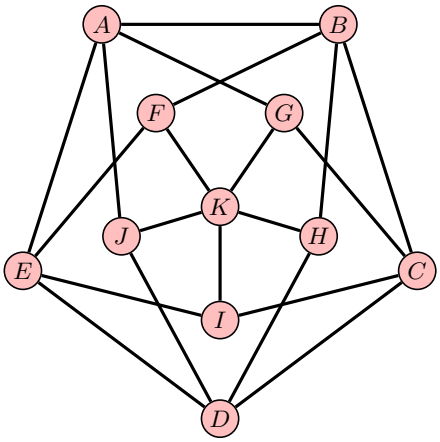
1.1. Consider the drawing of the Petersen graph below.

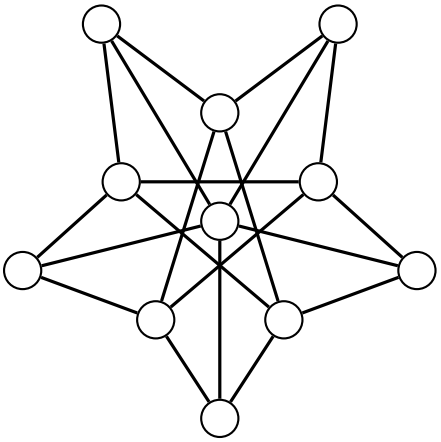


(a) Find at least two additional drawings of the Petersen graph with a symmetry, different from the drawing above. Label their vertices with $A, B, \ldots, J$ so that the same pairs of labels are joined by edges as in the drawing above, and explain why this shows that your drawings depict the Petersen graph.

(b) Draw a 3-regular graph (every vertex has degree 3) with 10 vertices, and therefore 15 edges, that has no triangle and is not isomorphic to the Petersen graph. Explain why your graph is not isomorphic to the Petersen graph.

1.2. Consider the Grötzsch graph on the left below, with the 11 vertices $A, B, C, D, E, F, G, H, I, J, K$.

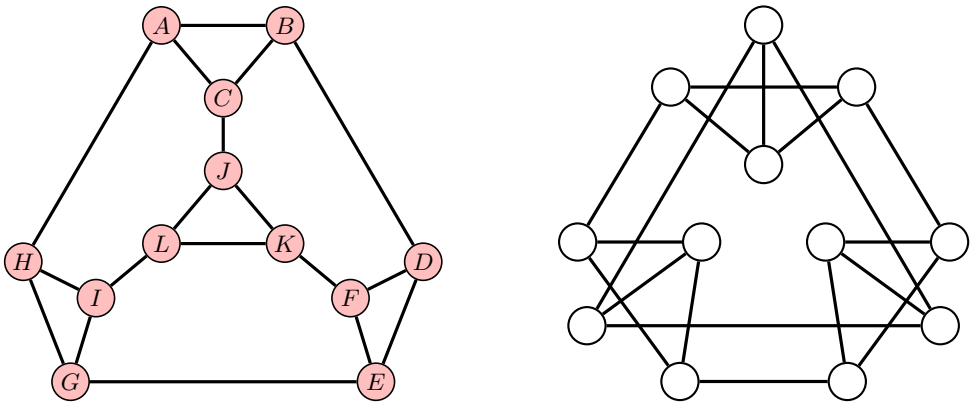




(a) Assign the 11 labels $A, B, \ldots, K$ to the 11 vertices in the right drawing to show that the graph it depicts is isomorphic to the Grötzsch graph. That is, for any pair of vertices $u, v \in \{A, B, \ldots, K\}$, the edge (u, v) exists in the left drawing if and only if it exists in the right drawing with your label assignment. Explain your strategy for finding the labels.

(b) How many different assignments of the 11 labels to the vertices of the right drawing show that it depicts the Grötzsch graph? Explain your answer.

1.3. Consider the FourTriangles graph on the left below, with the 12 vertices $A, B, C, D, E, F, G, H, I, J, K, L$. It consists of four triangles, and every two of them are joined by one edge (the graph is also known as the truncated tetrahedron graph).



(a) Assign the 12 labels $A, B, \ldots, L$ to the 12 vertices in the right drawing to show that the graph it depicts is isomorphic to the FourTriangles graph. That is, for any pair of vertices $u, v \in \{A, B, \ldots, L\}$, the edge (u, v) exists in the left drawing if and only if it exists in the right drawing with your label assignment. Explain your strategy for finding the labels.

(b) How many different assignments of the 12 labels to the vertices of the right drawing show that it depicts the FourTriangles graph? Explain your answer.

1.4. Two simple (no self-loops and no parallel edges) undirected graphs G and H are **isomorphic** if there exists a bijection f from the vertices of G to the vertices of H such that an edge (u, v) exists in G **if and only if** the edge $(f(u), f(v))$ exists in H.

In each of the following parts, determine whether G and H are (i) always isomorphic, or (ii) never isomorphic, or (iii) sometimes isomorphic and sometimes not isomorphic.

Justify your answers.

(a) For $n \geq 1$, G has n vertices, H has $n + 1$ vertices, and both have the same number of edges.

(b) Both graphs have the same number $n \geq 2$ of vertices and exactly one edge.

(c) Both graphs have the same number $n \geq 4$ of vertices and exactly two edges.

(d) Both graphs have the same number $n \geq 2$ of vertices and all possible edges.

(e) Both graphs are trees (connected graphs without cycles) with the same number $n \geq 4$ of vertices.

1.5. Let G and H be two simple undirected graphs, each with 6 vertices and the **same** number of edges.

In each of the following parts, if it is possible to find two non-isomorphic graphs G and H that satisfy the conditions, draw examples for G and H and explain why G and H are not isomorphic. If it is not possible, explain why.

(a) No additional condition.

(b) Every vertex of G and of H has degree 5.

(c) Every vertex of G and of H has degree 2.

(d) Every vertex of G and of H has degree 1.

1.6. In each of the following parts, draw two non-isomorphic graphs that satisfy the conditions, and explain why they are not isomorphic.

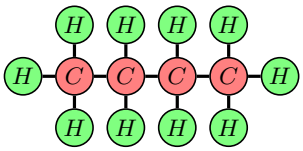
(a) Simple graphs with 4 vertices and 2 edges.

(b) Simple bipartite graphs (graphs whose vertices can be split into two sides so that every edge joins the two sides) with 4 vertices and 3 edges.

(c) Simple graphs with 6 vertices and the degree sequence $(3, 3, 2, 2, 2, 2)$, that is, two vertices of degree 3 and four vertices of degree 2.

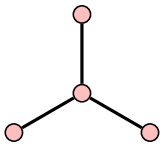
(d) Trees (connected graphs without cycles) with 6 vertices and 3 leaves (vertices of degree 1).

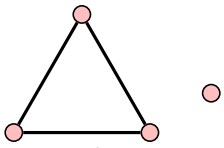
1.7. There are six non-isomorphic connected molecules of the type C_4H_y for some positive integer y in which each C-atom has degree 4, each H-atom has degree 1, any two C-atoms are joined by at most one edge, and every H-atom is connected to a C-atom. One of them, the molecule C_4H_{10} in which the four C-atoms form a path, is shown below. Illustrate the other five.



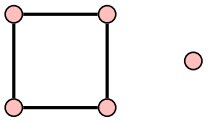
Hint: The number y of H-atoms is uniquely determined by the interconnection structure among the four C-atoms.

1.8. The **complement** of a simple graph G, denoted by $\overline{G}$, is the graph with the same vertices as G in which two different vertices are adjacent if and only if they are **not** adjacent in G. For example, the complement of the star S_4 with 4 vertices (left) is a triangle plus a vertex with no neighbors (right): the middle vertex of the star, which is adjacent to all the other vertices, has no neighbors in the complement, while the three leaves, which are not adjacent to each other, form a triangle.

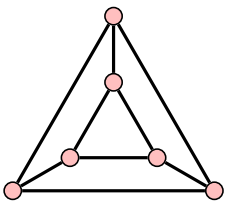

star


its complement

(a) Find the complement of the graph below: a cycle with 4 vertices plus a vertex with no neighbors. Draw it so that its shape is easy to recognize.



(b) Find the complement of the prism graph below. Which well-known graph is it?

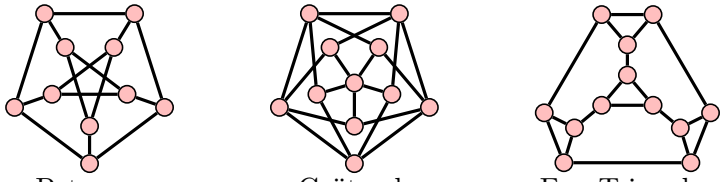


(c) Let G and H be two simple graphs. Prove that G and H are isomorphic if and only if $\overline{G}$ and $\overline{H}$ are isomorphic.

(d) Let P_4 be the path with 4 vertices and let C_5 be the cycle with 5 vertices. Show that $\overline{P_4}$ is isomorphic to P_4 and that $\overline{C_5}$ is isomorphic to C_5. Such graphs are called **self-complementary**.

2 Families of Graphs

2.1. Below are the Petersen graph, the Grötzsch graph, and the FourTriangles graph.



(a) For each graph, mark Yes (Y) or No (N) in the table: is it connected, a tree, regular (all degrees are equal), triangle-free (no cycle of length 3), bipartite (the vertices can be split into two sides so that every edge joins the two sides)? Justify briefly.

	Petersen	Grötzsch	FourTriangles
connected			
a tree			
regular			
triangle-free			
bipartite			

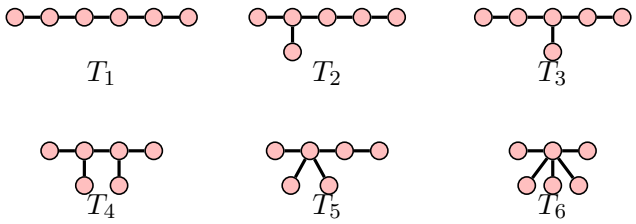
In (b)–(d), find a disconnected graph with as many vertices as the given graph and the same answers in the other four rows.

(b) The Petersen graph.

(c) The Grötzsch graph.

(d) The FourTriangles graph.

2.2. Below are all 6 trees with 6 vertices, up to isomorphism.



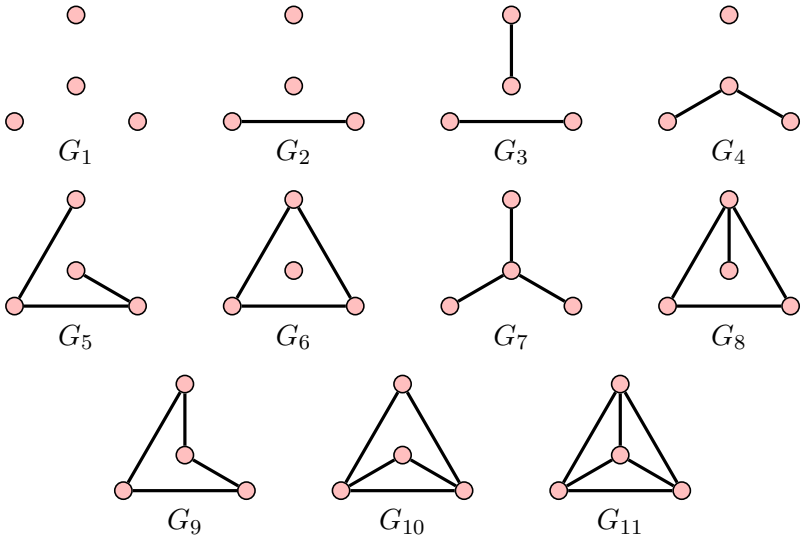
(a) For each tree, write in the table its **diameter** (in a tree, the number of edges on a longest path), its number of leaves (vertices of degree 1), its maximum degree, and its **sorted degree sequence** (its degrees in non-increasing order).

	diameter	number of leaves	maximum degree	sorted degree sequence
T_1				(___ ≥ ___ ≥ ___ ≥ ___ ≥ ___ ≥ ___)
T_2				(___ ≥ ___ ≥ ___ ≥ ___ ≥ ___ ≥ ___)
T_3				(___ ≥ ___ ≥ ___ ≥ ___ ≥ ___ ≥ ___)
T_4				(___ ≥ ___ ≥ ___ ≥ ___ ≥ ___ ≥ ___)
T_5				(___ ≥ ___ ≥ ___ ≥ ___ ≥ ___ ≥ ___)
T_6				(___ ≥ ___ ≥ ___ ≥ ___ ≥ ___ ≥ ___)

(b) Which two of the trees have the same values in all four columns of the table? Explain why they are not isomorphic.

(c) In the table, the number of leaves of each tree is at least its maximum degree. Prove that this holds for every tree with at least 2 vertices.

2.3. Below are all 11 simple graphs with 4 vertices, up to isomorphism.



(a) For each graph, mark Yes (Y) or No (N) in the table to indicate whether it is connected, a forest (no cycles), a tree (a connected forest), regular (all degrees are equal), or bipartite (the vertices can be split into two sides so that every edge joins the two sides).

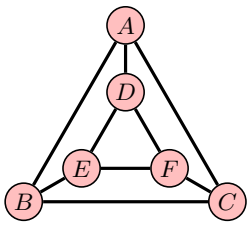
	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}
connected graphs											
forests											
trees											
regular graphs											
bipartite graphs											

(b) The **complement** of a simple graph G has the same vertices as G, and two different vertices are adjacent in it if and only if they are not adjacent in G. Pair each of the 11 graphs with its complement, up to isomorphism. Which of the graphs is isomorphic to its own complement?

(c) Prove that the complement of a disconnected graph is connected.

(d) Is the converse true, that is, is the complement of a connected graph always disconnected?

2.4. Consider the prism graph below. In each of the following parts, find the minimum number of edges whose removal leaves a graph with the required property (all the 6 vertices remain), mark such a set of edges, and explain why fewer edges do not suffice.



(a) The remaining graph is a tree.

(b) The remaining graph is bipartite.

(c) The remaining graph is disconnected.

2.5. A **forest** is a graph with no cycles, and a **tree** is a connected forest. A graph is **bipartite** if its vertices can be split into two sides so that every edge joins the two sides, and it is **triangle-free** if it has no cycle of length 3, that is, no three pairwise adjacent vertices.

(a) Prove or disprove: (i) every tree is a forest; (ii) every forest is a tree.
(b) Prove or disprove: (i) every forest is bipartite; (ii) every bipartite graph is a forest.
(c) Prove or disprove: (i) every bipartite graph is triangle-free; (ii) every triangle-free graph is bipartite.

2.6. The **line graph** $L(G)$ of a graph G has a vertex for every edge of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share an endpoint. Each vertex of $L(G)$ is named by the two endpoints of its edge. For example, below are a graph G (left) and its line graph $L(G)$ (right).

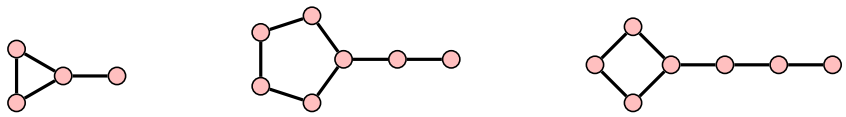


For each of the following graphs G, find the number of vertices of $L(G)$, the degrees of its vertices, and its number of edges. If $L(G)$ is a well-known graph, name it.

(a) The matching M_k with $k \geq 1$ edges: $2k$ vertices and k edges, no two sharing an endpoint.
(b) The path P_n with $n \geq 2$ vertices.
(c) The cycle C_n with $n \geq 3$ vertices.
(d) The star S_n with $n \geq 2$ vertices: a middle vertex adjacent to the other $n-1$ vertices, and no other edges.
(e) A Δ-regular graph G with n vertices, for some $\Delta \geq 1$: every vertex of G has degree Δ.

2.7. A **balloon-graph** is a graph that is composed of a cycle and a path that share exactly 1 vertex.

- All the edges of the graph are either part of the cycle or part of the path.
- Exactly one vertex is shared by the cycle and the path, and it is an end of the path.
- The cycle has at least 3 vertices and therefore at least 2 vertices belong only to the cycle.
- The path has at least 2 vertices and therefore at least 1 vertex belongs only to the path.

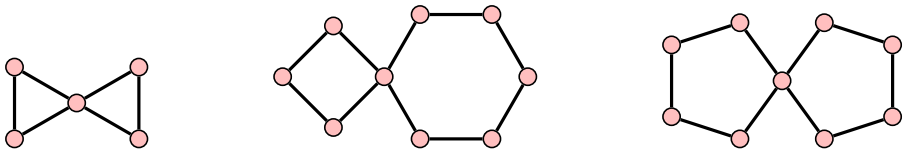


A balloon-graph must contain at least 4 vertices. Shown above are the only balloon-graph with 4 vertices and two of the four non-isomorphic balloon-graphs with 7 vertices. Of these two, one has a cycle with 5 vertices and a path with 3 vertices, and the other has a cycle with 4 vertices and a path with 4 vertices.

(a) Illustrate all non-isomorphic balloon-graphs with 4, 5, 6, and 7 vertices.
(b) For $n \geq 4$, how many non-isomorphic balloon-graphs with n vertices exist? Give your answer as a function of n. Justify your answer.
(c) For $n \geq 4$, all balloon-graphs with n vertices have the same number of edges. What is this number? Prove your answer.
(d) For $n \geq 4$, all balloon-graphs with n vertices have the same number of vertices of each degree. Let G be a balloon-graph with n vertices. How many vertices in G have degree 0? How many vertices in G have degree 1? How many vertices in G have degree 2? How many vertices in G have degree 3? How many vertices in G have degree greater than 3? Justify your answers.

2.8. An **eight-graph** is a graph that is composed of two cycles that share exactly 1 vertex.

- All the edges of the graph are part of one of the two cycles.
- Exactly one vertex is shared by both cycles.
- Every other vertex belongs to exactly one of the cycles.
- Each cycle has at least 3 vertices, including the shared vertex.



An eight-graph must contain at least 5 vertices. Shown above are the only eight-graph with 5 vertices and two of the three non-isomorphic eight-graphs with 9 vertices. Of these two, one has a cycle with 4 vertices and a cycle with 6 vertices, and the other has two cycles with 5 vertices each.

(a) Illustrate all non-isomorphic eight-graphs with 5, 6, 7, 8, and 9 vertices.

(b) For $n \geq 5$, how many non-isomorphic eight-graphs with n vertices exist? Give your answer as a function of n. Justify your answer.

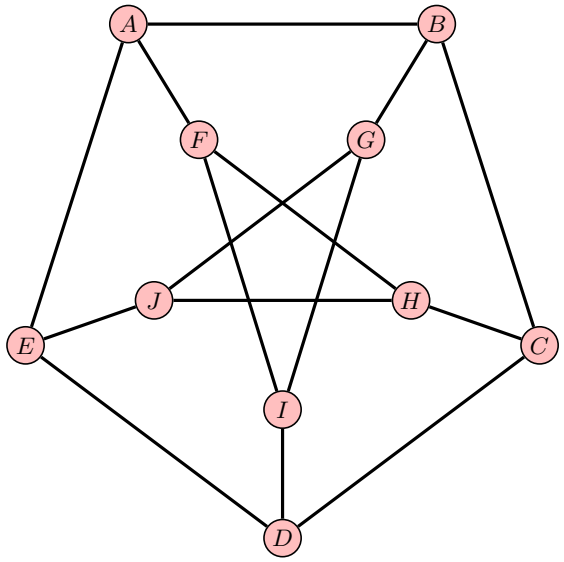
Hint: The number is the same for the odd number $n = 2k - 1$ and the even number $n = 2k$.

(c) For $n \geq 5$, all eight-graphs with n vertices have the same number of edges. What is this number? Prove your answer.

(d) For $n \geq 5$, all eight-graphs with n vertices have the same number of vertices of each degree. Let G be an eight-graph with n vertices. How many vertices in G have degree 0? How many vertices in G have degree 1? How many vertices in G have degree 2? How many vertices in G have degree 3? How many vertices in G have degree 4? How many vertices in G have degree greater than 4? Justify your answers.

3 Representing Graphs

3.1. Consider the Petersen graph below. Use the vertex order $A, B, \ldots, J$ for the rows and columns of the matrix and for the lists.



(a) Construct the adjacency matrix of this graph.

(b) Construct the adjacency lists of this graph.

	A	B	C	D	E	F	G	H	I	J
A										
B										
C										
D										
E										
F										
G										
H										
I										
J										

Adjacency lists:

A: ( )

B: ( )

C: ( )

D: ( )

E: ( )

F: ( )

G: ( )

H: ( )

I: ( )

J: ( )

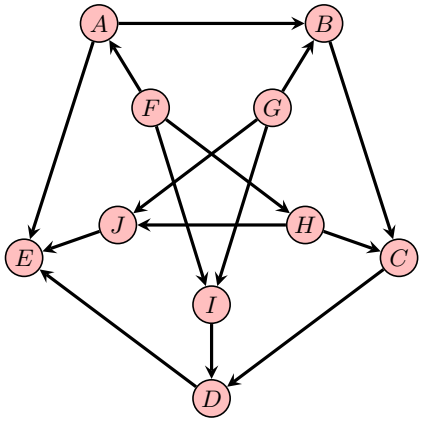
(c) Explain how to find the common neighbors of two vertices u and v using the adjacency matrix, and using the adjacency lists. What is the complexity of each method?

(d) Show that in the Petersen graph, every two adjacent vertices have no common neighbor, and every two non-adjacent vertices have exactly one common neighbor.

(e) Use part (d) to show that the Petersen graph has no cycle of length 3 or 4, and conclude that its shortest cycle has length 5.

3.2. Consider the directed graph below, obtained by orienting the edges of the Petersen graph. Use the vertex order $A, B, \ldots, J$ for the rows and columns of the matrix and for the lists.

In the adjacency matrix, the entry in row u and column v is 1 if and only if the edge $u \to v$ exists. The outgoing list of a vertex v contains the vertices u with an edge $v \to u$, and the incoming list of v contains the vertices u with an edge $u \to v$.



(a) Construct the adjacency matrix of this directed graph.

(b) Construct the outgoing and incoming adjacency lists of this directed graph.

	A	B	C	D	E	F	G	H	I	J
A										
B										
C										
D										
E										
F										
G										
H										
I										
J										

Outgoing:

A: ( )
B: ( )
C: ( )
D: ( )
E: ( )
F: ( )
G: ( )
H: ( )
I: ( )
J: ( )

Incoming:

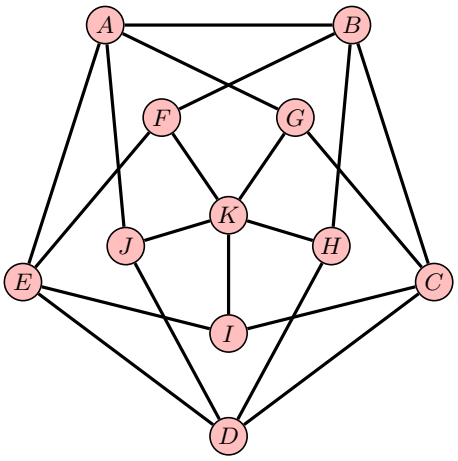
A: ( )
B: ( )
C: ( )
D: ( )
E: ( )
F: ( )
G: ( )
H: ( )
I: ( )
J: ( )

(c) Show that the sum of each row of the adjacency matrix is the **out-degree** of its vertex (the number of edges that leave it), and the sum of each column is the **in-degree** of its vertex (the number of edges that enter it). Conclude that the sum of the out-degrees equals the sum of the in-degrees, and find both sums for this directed graph.

(d) Design an algorithm that constructs the incoming lists of a directed graph with n vertices and m edges from its outgoing lists in $O(n + m)$ time.

(e) The **reverse** of a directed graph is obtained by reversing the direction of every edge. Explain why the adjacency matrix of the reverse is the transpose of the adjacency matrix (rows and columns exchanged). How can the outgoing and incoming lists of the reverse be obtained from the outgoing and incoming lists of the original directed graph?

3.3. Consider the Grötzsch graph below, with the 11 vertices $A, B, C, D, E, F, G, H, I, J, K$. Use the vertex order $A, B, \ldots, K$ for the rows and columns of the matrix and for the lists.



(a) Construct the adjacency matrix of the Grötzsch graph.

(b) Construct the adjacency lists of the Grötzsch graph.

	A	B	C	D	E	F	G	H	I	J	K
A											
B											
C											
D											
E											
F											
G											
H											
I											
J											
K											

Adjacency lists:

A: ( )

B: ( )

C: ( )

D: ( )

E: ( )

F: ( )

G: ( )

H: ( )

I: ( )

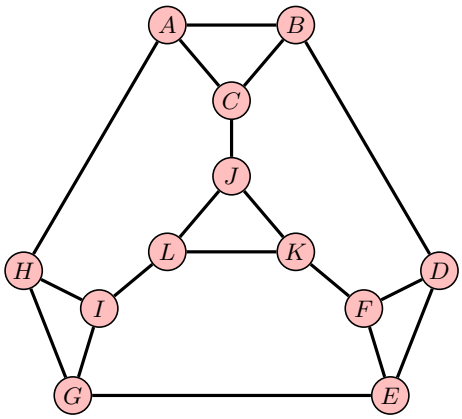
J: ( )

K: ( )

(c) Compare the two representations of the Grötzsch graph. How many entries does each of them use? How much time does each of them need to decide whether two given vertices are adjacent, and to list all the neighbors of a given vertex?

(d) In the adjacency matrix, compare the row of F with the row of A in the columns A, B, C, D, E. For each of G, H, I, J, find a vertex among A, B, C, D, E with the same property. What does this say about the structure of the Grötzsch graph?

3.4. Consider the FourTriangles graph below, with the 12 vertices $A, B, C, D, E, F, G, H, I, J, K, L$. Use the vertex order $A, B, \ldots, L$ for the rows and columns of the matrix and for the lists.



(a) Construct the adjacency matrix of the FourTriangles graph.

(b) Construct the adjacency lists of the FourTriangles graph.

	A	B	C	D	E	F	G	H	I	J	K	L
A												
B												
C												
D												
E												
F												
G												
H												
I												
J												
K												
L												

Adjacency lists:

A: ( )

B: ( )

C: ( )

D: ( )

E: ( )

F: ( )

G: ( )

H: ( )

I: ( )

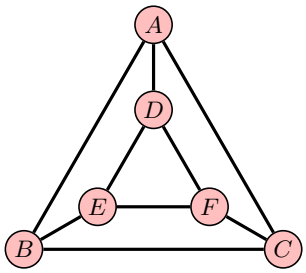
J: ( )

K: ( )

L: ( )

(c) Explain how the four triangles of the FourTriangles graph can be seen in its adjacency matrix. Design an algorithm that counts the triangles in a graph represented by an adjacency matrix. What is the complexity of your algorithm? How many triangles does the FourTriangles graph have?

3.5. The **line graph** $L(G)$ of a graph G has a vertex for every edge of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share an endpoint. Each vertex of $L(G)$ is named by the two endpoints of its edge, for example, AB for the edge (A, B). Consider the prism graph below.



(a) Draw the line graph of the prism graph.

(b) How many vertices and how many edges does the line graph of the prism graph have? What are the degrees of its vertices?

(c) Construct the adjacency matrix of the line graph of the prism graph. Use the vertex order $AB, AC, BC, DE, DF, EF, AD, BE, CF$ for the rows and columns.

(d) Construct the adjacency lists of the line graph of the prism graph, using the same vertex order.

	AB	AC	BC	DE	DF	EF	AD	BE	CF
AB									
AC									
BC									
DE									
DF									
EF									
AD									
BE									
CF									

Adjacency lists:

AB: ( )

AC: ( )

BC: ( )

DE: ( )

DF: ( )

EF: ( )

AD: ( )

BE: ( )

CF: ( )

(e) Let G be a simple graph with n vertices of degrees $d_1, d_2, \ldots, d_n$ and with m edges. Prove that $L(G)$ has m vertices and $\sum_{i=1}^{n} d_i(d_i - 1)/2$ edges. Check this formula against part (b).

(f) Design an algorithm that constructs the adjacency matrix of $L(G)$ from the adjacency matrix of G. What is the complexity of your algorithm?

(g) Design an algorithm that constructs the adjacency lists of $L(G)$ from the adjacency lists of G. What is the complexity of your algorithm?

3.6. For $n \geq 1$, let X and Y be two simple undirected labeled graphs on the vertices $\{1, 2, \ldots, n\}$.

The graph Y is a **labeled subgraph** of X if every edge of Y is also an edge of X (however, X may have edges that are not edges of Y).

(a) Assume that both graphs are represented by adjacency matrices. Design an algorithm that determines whether Y is a labeled subgraph of X. What is the complexity of your algorithm as a function of n? Justify the correctness of your algorithm and your complexity claim.

(b) Assume that X and Y are represented by adjacency lists, where X has m_X edges and Y has m_Y edges. Design an algorithm that determines whether Y is a labeled subgraph of X in $O(n + m_X + m_Y)$ time.

(c) Show an example of two labeled graphs X and Y on the same vertices, such that Y is not a labeled subgraph of X, but Y becomes a labeled subgraph of X after renaming its vertices.

3.7. For $n \geq 1$, let X and Y be two simple undirected labeled graphs on the vertices $\{1, 2, \ldots, n\}$. The **union** U of X and Y is the graph on $\{1, 2, \ldots, n\}$ whose edges are exactly the edges that belong to at least one of X and Y. The **intersection** W of X and Y is the graph on $\{1, 2, \ldots, n\}$ whose edges are exactly the edges that belong to both X and Y.

(a) Assume that both graphs are represented by adjacency matrices. Design an algorithm that constructs the adjacency matrices of U and of W. What is the complexity of your algorithm as a function of n? Justify the correctness of your algorithm and your complexity claim.

(b) Assume that X and Y are represented by adjacency lists, and that every list is sorted in increasing order. Design an algorithm that constructs the sorted adjacency lists of U and of W in $O(n + m_X + m_Y)$ time, where X has m_X edges and Y has m_Y edges.

(c) Let m_X, m_Y, m_U, and m_W be the numbers of edges of X, Y, U, and W, respectively. Prove that $m_X + m_Y = m_U + m_W$.

3.8. For $n \geq 1$, let X be a simple undirected labeled graph on the vertices $\{1, 2, \ldots, n\}$. The **complement** $\overline{X}$ of X is the graph on $\{1, 2, \ldots, n\}$ in which two different vertices are adjacent if and only if they are not adjacent in X.

(a) If X has m edges, how many edges does $\overline{X}$ have?

(b) Assume that X is represented by an adjacency matrix. Design an algorithm that constructs the adjacency matrix of $\overline{X}$. What is the complexity of your algorithm as a function of n? Justify the correctness of your algorithm and your complexity claim.

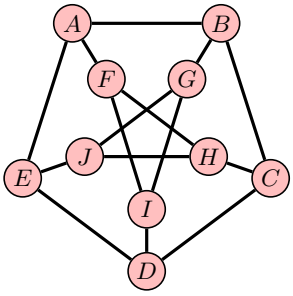
(c) Assume that X is represented by adjacency lists. Design an algorithm that constructs the adjacency lists of $\overline{X}$. What is the complexity of your algorithm? Explain why no algorithm that reads all of its input and writes all of its output can be asymptotically faster.

4 Paths, Cycles, Distances, and Connectivity

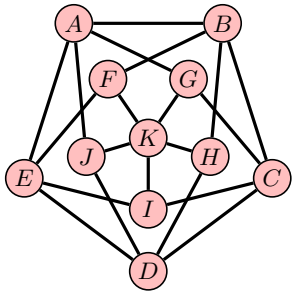
4.1. A **path** $\mathcal{P} = \langle v_0, v_1, \ldots, v_k \rangle$ of length k (number of edges) is an ordered list of $k+1$ vertices such that the edge (v_i, v_{i+1}) exists for $0 \leq i \leq k-1$ and all the edges are different. Note that a vertex may appear more than once in a path. A **cycle** $\mathcal{C} = \langle v_0, v_1, \ldots, v_{k-1}, v_0 \rangle$ of length k is a path that starts and ends at the same vertex.

An **Euler path** is a path that traverses all the edges of the graph exactly once. An **Euler cycle** is a cycle that traverses all the edges of the graph exactly once.

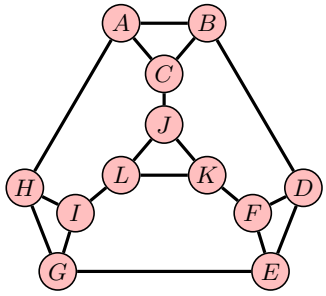
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

None of the three graphs has an Euler path or an Euler cycle.

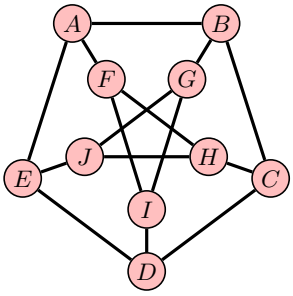
(a) For each of the three graphs, find one of its longest paths, and explain why no longer path exists.

(b) For each of the three graphs, find one of its longest cycles, and explain why no longer cycle exists.

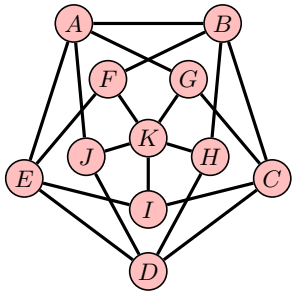
4.2. A **simple path** $\mathcal{P} = \langle v_0, v_1, \ldots, v_k \rangle$ of length k (number of edges) is an ordered list of $k+1$ distinct vertices such that the edge (v_i, v_{i+1}) exists for $0 \leq i \leq k-1$. A **simple cycle** $\mathcal{C} = \langle v_0, v_1, \ldots, v_{k-1}, v_0 \rangle$ of length $k \geq 3$ is an ordered list of k distinct vertices $v_0, v_1, \ldots, v_{k-1}$ such that the edge (v_i, v_{i+1}) exists for $0 \leq i \leq k-2$ and the edge (v_{k-1}, v_0) exists.

A **Hamiltonian path** is a simple path that visits all the vertices of the graph. A **Hamiltonian cycle** is a simple cycle that visits all the vertices of the graph.

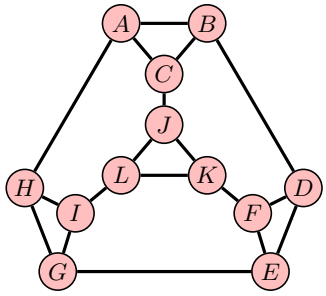
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

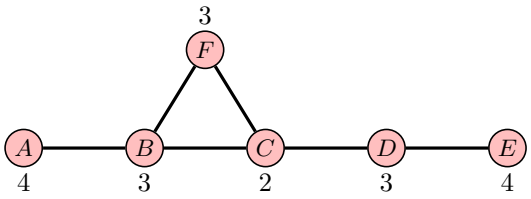
All three graphs have Hamiltonian paths. The Grötzsch graph and the FourTriangles graph have Hamiltonian cycles, but the Petersen graph does not.

(a) For each of the three graphs, find one of its Hamiltonian paths.

(b) For each of the three graphs, find one of its longest simple cycles, and explain why no longer simple cycle exists.

4.3. Let G be a connected graph, that is, every two of its vertices are joined by a path. The **distance** $d(u, v)$ between two vertices u and v is the number of edges of a shortest path from u to v; in particular, $d(u, u) = 0$. The **eccentricity** of a vertex v is the largest distance from v to a vertex of G. The **diameter** of G is the largest eccentricity of its vertices, the **radius** of G is the smallest eccentricity of its vertices, and the **center** of G is the set of vertices whose eccentricity equals the radius.

For example, in the graph below, $d(A, E) = 4$ and $d(A, F) = 2$. The number next to each vertex is its eccentricity. Hence, the diameter is 4, the radius is 2, and the center is $\{C\}$.



(a) Find the diameter, the radius, and the center of the path P_n with $n \geq 2$ vertices $v_1, \ldots, v_n$ and the $n-1$ edges (v_i, v_{i+1}) for $1 \leq i \leq n-1$.

(b) Find the diameter, the radius, and the center of the cycle C_n with $n \geq 3$ vertices $v_1, \ldots, v_n$ and the n edges (v_i, v_{i+1}) for $1 \leq i \leq n-1$ and (v_n, v_1).

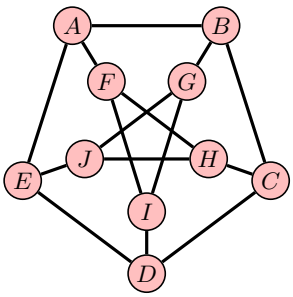
(c) Find the diameter, the radius, and the center of the complete graph K_n with $n \geq 2$ vertices, in which every two vertices are adjacent.

(d) Find the diameter, the radius, and the center of the star S_n with $n \geq 3$ vertices: one middle vertex that is adjacent to the other $n-1$ vertices, and no other edges.

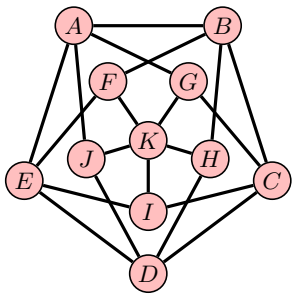
(e) Find the diameter, the radius, and the center of the complete bipartite graph $K_{r,s}$ with $r, s \geq 2$: there are r vertices on one side and s vertices on the other side, and two vertices are adjacent if and only if they are on different sides.

(f) Prove that in every connected graph, the radius is at most the diameter, and the diameter is at most twice the radius.

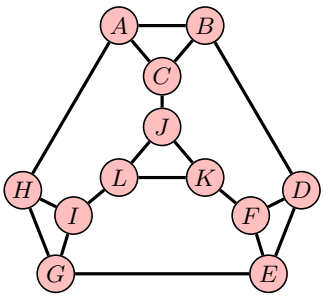
4.4. Let G be a connected graph, that is, every two of its vertices are joined by a path. The **distance** $d(u, v)$ between two vertices u and v is the number of edges of a shortest path from u to v; in particular, $d(u, u) = 0$. The **eccentricity** of a vertex v is the largest distance from v to a vertex of G. The **diameter** of G is the largest eccentricity of its vertices, the **radius** of G is the smallest eccentricity of its vertices, and the **center** of G is the set of vertices whose eccentricity equals the radius. Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

(a) Find the diameter, the radius, and the center of the Petersen graph.

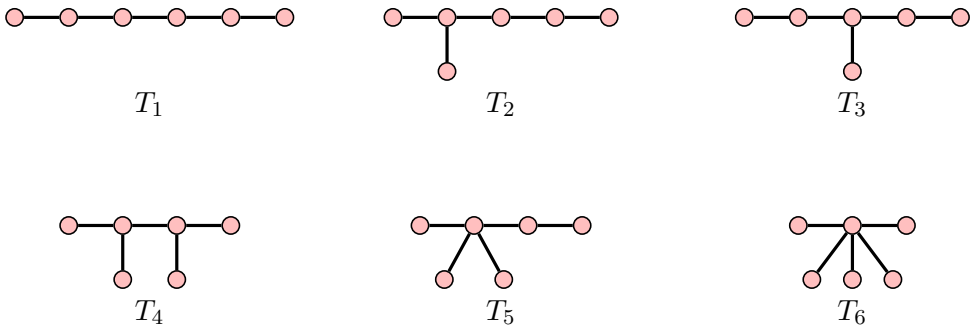
(b) Find the diameter, the radius, and the center of the Grötzsch graph.

(c) Find the diameter, the radius, and the center of the FourTriangles graph.

(d) Prove that a connected graph in which every vertex has degree at most 3 and whose diameter is 2 has at most 10 vertices.

4.5. Let G be a connected graph, that is, every two of its vertices are joined by a path. The **distance** $d(u, v)$ between two vertices u and v is the number of edges of a shortest path from u to v; in particular, $d(u, u) = 0$. The **eccentricity** of a vertex v is the largest distance from v to a vertex of G. The **diameter** of G is the largest eccentricity of its vertices, the **radius** of G is the smallest eccentricity of its vertices, and the **center** of G is the set of vertices whose eccentricity equals the radius.

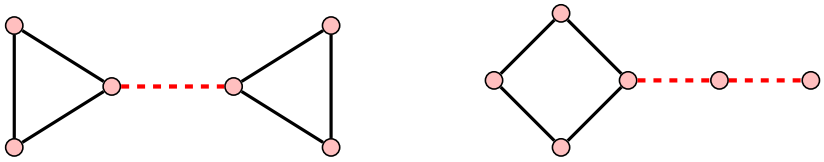
(a) Below are the 6 trees with 6 vertices, up to isomorphism. Find the diameter, the radius, and the center of each tree.



(b) Prove that the center of every tree consists of either a single vertex or two adjacent vertices.

4.6. A graph is connected if every two of its vertices are joined by a path. A **bridge** in a connected graph G is an edge whose removal from G disconnects it.

In the examples below, the bridges are dashed: the left graph has one bridge and the right graph has two bridges.



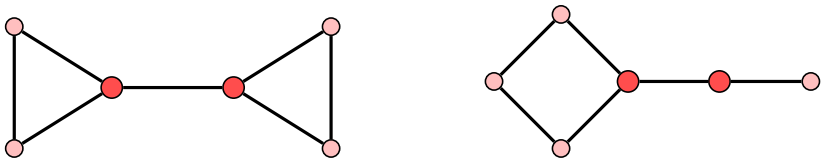
(a) Let T be a tree with $n \geq 2$ vertices. Which edges of T are bridges? Justify your answer.

(b) Prove that if G is a connected graph in which every vertex has an even degree, then G does not have a bridge.

(c) Prove that an edge of a connected graph is a bridge if and only if it does not belong to any cycle (a path that starts and ends at the same vertex and does not repeat edges).

4.7. A graph is connected if every two of its vertices are joined by a path. An **articulation point** in a connected graph G is a vertex whose removal from G, together with all the edges that touch it, disconnects it.

In the examples below, the articulation points are red: each of the two graphs has two articulation points.



(a) Let T be a tree with $n \geq 3$ vertices. Which vertices of T are articulation points? Justify your answer.

(b) A connected graph in which every vertex has an even degree may have an articulation point. Find such a graph with the fewest vertices, and explain why no such graph with fewer vertices exists.

(c) Prove that every connected graph with at least 2 vertices has at least two vertices that are not articulation points.

4.8. A graph is connected if every two of its vertices are joined by a path. A **bridge** in a connected graph G is an edge whose removal from G disconnects it. An **articulation point** in a connected graph G is a vertex whose removal from G, together with all the edges that touch it, disconnects it.

(a) Let G be a connected graph with at least 3 vertices, and let (u, v) be a bridge in G. Prove that at least one of u and v is an articulation point.

(b) Show an example of a bridge such that one of its endpoints is not an articulation point.

(c) Show that an edge whose two endpoints are articulation points is not necessarily a bridge.

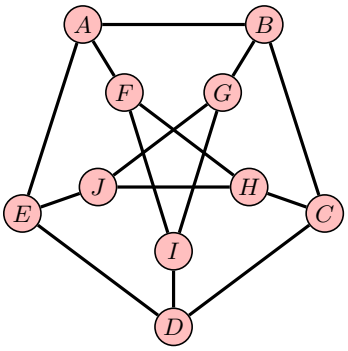
(d) Show that a connected graph without bridges may have articulation points: for every $k \geq 2$, find a connected graph with $2k + 1$ vertices and no bridges, in which $k - 1$ vertices are articulation points.

5 Degrees and Degree Sequences

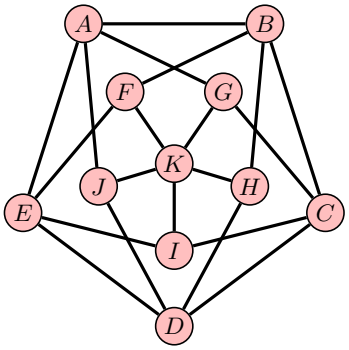
5.1. The **handshake lemma** states that in every graph, the sum of the degrees of all the vertices is twice the number of edges.

(a) Explain why the handshake lemma holds.

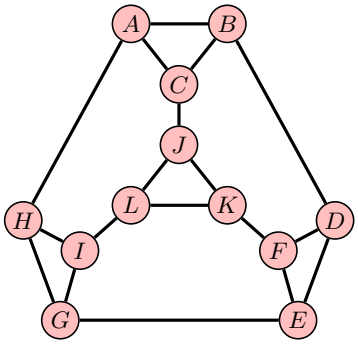
(b) Write the degree sequences (the lists of the degrees of their vertices) of the Petersen graph, the Grötzsch graph, and the FourTriangles graph drawn below. Use the handshake lemma to compute the number of edges of each graph.



Petersen



Grötzsch



FourTriangles

(c) Conclude that in every graph, the number of vertices with an odd degree is even.

(d) Can each of 9 people shake hands with exactly 3 of the others? Explain.

5.2. Let G be a simple (no self-loops and no parallel edges) undirected graph on the $n \geq 1$ vertices $\{v_1, v_2, \ldots, v_n\}$. If vertex v_i has d_i neighbors (equivalently, the degree of v_i is d_i), then the **degree sequence** of G is $D_G = (d_1, d_2, \ldots, d_n)$. A sequence D is **graphic** if there is a simple graph G whose degree sequence is D; such a graph G is called a **realization** of D.

Among the following four sequences, one is not graphic, one is graphic with exactly one realization up to isomorphism, one is graphic with exactly two non-isomorphic realizations, and one is graphic with exactly three non-isomorphic realizations. Which is which? If a sequence is not graphic, explain why. If a sequence is graphic, illustrate all of its non-isomorphic realizations.

(a) $D = (4, 4, 2, 2, 2)$

(b) $D = (3, 3, 2, 2, 2)$

(c) $D = (4, 4, 1, 1, 1, 1)$

(d) $D = (2, 2, 2, 2, 1, 1)$

5.3. A sequence is **graphic** if it is the degree sequence of a simple graph, called a **realization** of the sequence. The **Havel–Hakimi algorithm** constructs a realization in rounds. Every vertex has a **residual degree**, which is initially its degree in the sequence. In each round, a **pivot** vertex with a positive residual degree d becomes adjacent to d other vertices with the highest residual degrees. Then the residual degree of the pivot becomes 0, and the residual degree of each of its d new neighbors decreases by 1.

The algorithm stops either when all the residual degrees are 0, and then the new edges of all the rounds form a realization, or when a pivot with residual degree d cannot find d other vertices with positive residual degrees, and then the sequence is not graphic.

The sequence $D = (5, 4, 3, 3, 2, 1, 1, 1)$ is graphic. Let $A, B, \ldots, H$ be its eight vertices, with initial residual degrees $5, 4, 3, 3, 2, 1, 1, 1$, respectively. In parts (a), (b), and (d), for each round, show the sequence before the round, the new edges, and the sequence after the round, until the sequence becomes $(0, 0, 0, 0, 0, 0, 0, 0)$, and draw only the final realization.

(a) Construct a realization of the sequence D by running the Havel–Hakimi algorithm, in which the pivot in each round is a vertex with the highest residual degree. Tie-breaker rule: select the pivot and its neighbors in alphabetical order from A to H.

(b) Construct a realization of the sequence D by running the Havel–Hakimi algorithm, in which the pivot in each round is a vertex with the smallest positive residual degree. Tie-breaker rules: select the pivot in alphabetical order from H to A, and select its neighbors in alphabetical order from A to H.

(c) Prove that the realizations from parts (a) and (b) are not isomorphic.

(d) Find another sequence of pivots (in each round, any vertex with a positive residual degree may be the pivot) that generates a realization of D that is not isomorphic to the realizations from parts (a) and (b).

(e) Prove that the third realization is not isomorphic to the realizations from parts (a) and (b).

5.4. A simple graph is 1**-regular** if every vertex has degree 1, and it is 2**-regular** if every vertex has degree 2. For example, the cycle C_n with $n \geq 3$ vertices is 2-regular. For $n \geq 3$, let $p(n)$ be the number of ways to write n as a sum of integers that are at least 3, where the order of the summands does not matter. For example, $p(7) = 2$, since $7 = 7$ and $7 = 4 + 3$.

(a) Find all n for which there exists a 1-regular graph with n vertices. For each such n, find a 1-regular graph with n vertices, and prove that it is the only one up to isomorphism.

(b) Prove that every 2-regular graph is a disjoint union of cycles, each with at least 3 vertices.

(c) Prove that for every $n \geq 3$, the number of non-isomorphic 2-regular graphs with n vertices is $p(n)$.

(d) Find $p(n)$ for $3 \leq n \leq 10$, and draw all the non-isomorphic 2-regular graphs with 10 vertices.

5.5. A simple graph is 3**-regular**, also called **cubic**, if every vertex has degree 3.

(a) Explain why there is no 3-regular graph with an odd number of vertices.

(b) Draw the only 3-regular graph with 4 vertices.

(c) Draw two non-isomorphic 3-regular graphs with 6 vertices, and explain why they are not isomorphic. Prove that there are no others, up to isomorphism.

(d) Prove that for every even $n \geq 4$ there exists a 3-regular graph with n vertices.

5.6. A simple graph is 4**-regular** if every vertex has degree 4.

(a) Explain why there is no 4-regular graph with fewer than 5 vertices.

(b) Draw a 4-regular graph with n vertices for each $n = 5, 6, 7, 8, 9$.

(c) Prove that for every $n \geq 5$ there exists a 4-regular graph with n vertices.

5.7. Prove that in every simple graph with at least 2 vertices, two vertices have the same degree. That is, prove that the degree sequence of such a graph contains at least one repeated value.

6 Graph Coloring

6.1. A **vertex coloring** (or simply a **coloring**) of a graph assigns colors to its vertices so that every two adjacent vertices get different colors.

(a) What is the minimum number of colors needed to color a null graph (a graph with no edges) with $n \geq 1$ vertices? Explain.

(b) What is the minimum number of colors needed to color a tree with $n \geq 2$ vertices? Explain.

(c) What is the minimum number of colors needed to color a bipartite graph (a graph whose vertices can be split into two sides so that every edge joins the two sides) with at least one edge? Explain.

(d) What is the minimum number of colors needed to color the cycle C_n with $n \geq 3$ vertices? Explain. Note that the answer is different when n is even and when n is odd.

(e) The **wheel** W_n with $n \geq 4$ vertices consists of a cycle with $n-1$ vertices and one more vertex, the **hub**, that is adjacent to all the vertices of the cycle. What is the minimum number of colors needed to color W_n? Explain.

6.2. An **edge coloring** of a graph assigns colors to its edges so that every two edges that share an endpoint get different colors.

(a) Explain why an edge coloring of any graph uses at least as many colors as the maximum degree of the graph.

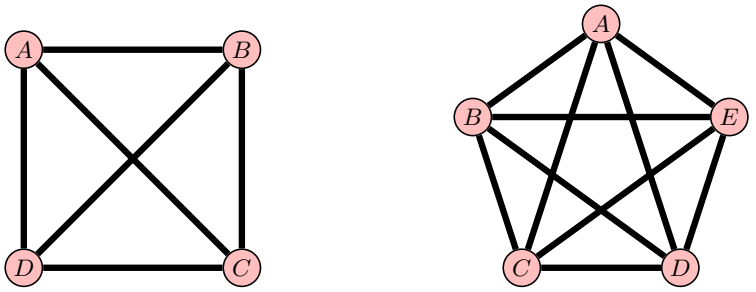
(b) What is the minimum number of colors needed to color the edges of the star S_n with $n \geq 2$ vertices (one middle vertex adjacent to the other $n-1$ vertices, and no other edges)? Explain.

(c) What is the minimum number of colors needed to color the edges of the path P_n with $n \geq 3$ vertices? Explain.

(d) What is the minimum number of colors needed to color the edges of the cycle C_n with $n \geq 3$ vertices? Explain. Note that the answer is different when n is even and when n is odd.

(e) Prove that the edges of any tree can be colored with as many colors as its maximum degree.

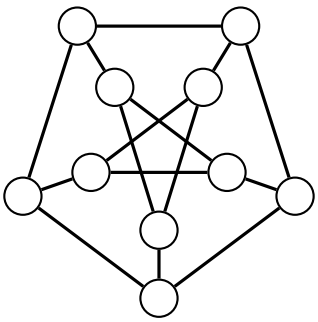
6.3. A **vertex coloring** of a graph assigns colors to its vertices so that every two adjacent vertices get different colors, and an **edge coloring** of a graph assigns colors to its edges so that every two edges that share an endpoint get different colors. In the complete graph K_n with n vertices, every two vertices are adjacent. Below are K_4 (left) and K_5 (right).



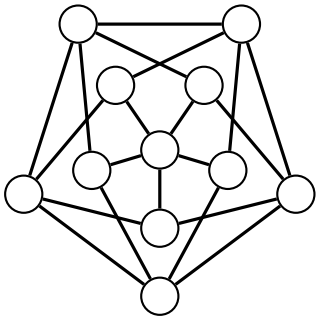
(a) What is the minimum number of colors needed to color the **vertices** of the complete graph K_n with $n \geq 1$ vertices? Explain.

(b) What is the minimum number of colors needed to color the **edges** of the complete graph K_4? And of K_5? Explain. **Hint:** How many edges can have the same color?

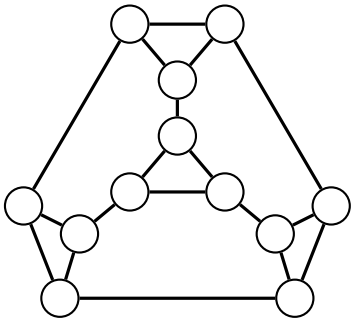
6.4. A **vertex coloring** (or simply a **coloring**) of a graph assigns colors to its vertices so that every two adjacent vertices get different colors. Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



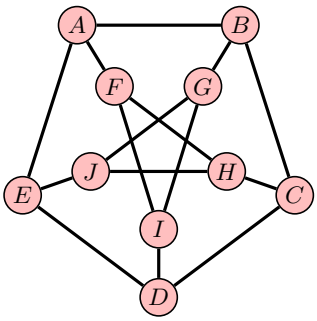
Grötzsch



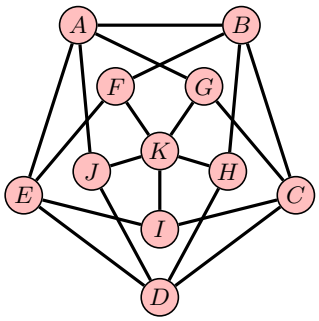
FourTriangles

For each of the three graphs, color its vertices with as few colors as possible, and explain why it is impossible to color its vertices with fewer colors.

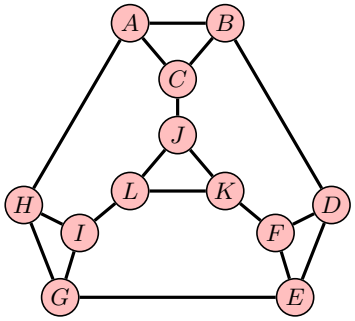
6.5. An **edge coloring** of a graph assigns colors to its edges so that every two edges that share an endpoint get different colors. Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

For each of the three graphs, color its edges with as few colors as possible, and explain why it is impossible to color its edges with fewer colors.

6.6. The **line graph** $L(G)$ of a graph G has a vertex for every edge of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share an endpoint. Each vertex of $L(G)$ is named by the two endpoints of its edge. A **vertex coloring** of a graph assigns colors to its vertices so that every two adjacent vertices get different colors, and an **edge coloring** of a graph assigns colors to its edges so that every two edges that share an endpoint get different colors.

(a) Prove that for every graph G, the minimum number of colors needed to color the edges of G equals the minimum number of colors needed to color the vertices of $L(G)$.

(b) Let $K_{3,3}$ be the complete bipartite graph with 3 vertices on each side, in which two vertices are adjacent if and only if they are on different sides. Draw $K_{3,3}$ and its line graph $L(K_{3,3})$. Color the edges of $K_{3,3}$ with 3 colors, and color the vertices of $L(K_{3,3})$ accordingly.

(c) For $m \geq 2$, let $K_{m,m}$ be the complete bipartite graph with m vertices on each side, in which two vertices are adjacent if and only if they are on different sides. Find the minimum number of colors needed to color its vertices and the minimum number of colors needed to color its edges. Conclude that the vertices of a graph and the vertices of its line graph may require very different numbers of colors.

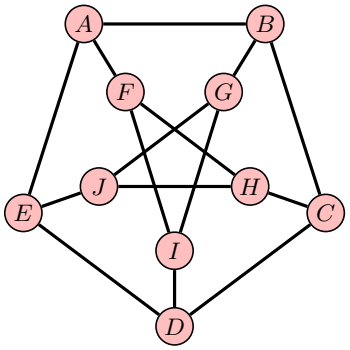
(d) Prove that the star $K_{1,3}$ (the star S_4), with one middle vertex adjacent to the other 3 vertices and no other edges, is not the line graph of any graph. Conclude that every edge coloring problem is a vertex coloring problem, but not every vertex coloring problem is an edge coloring problem.

7 Special Sets of Vertices and Edges

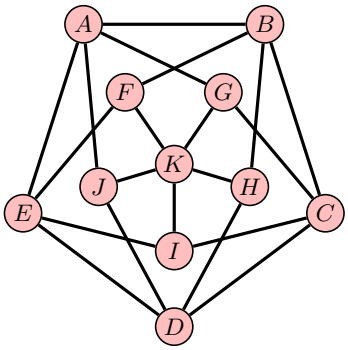
7.1. Let G be a simple undirected graph with $n \geq 1$ vertices. A **clique** in G is a set CL of vertices, every two of which are adjacent. That is, for any two different vertices $u, v \in CL$, the graph G includes the edge (u, v).

For example, the set of all the vertices of the complete graph K_n (every two vertices are adjacent) is a clique of size n. In contrast, in the null graph (no edges) a clique contains at most one vertex.

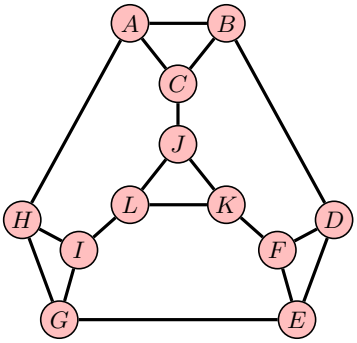
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

(a) What is the size of the **largest** clique in the Petersen graph? Give an example of such a clique and prove that no larger clique exists.

(b) What is the size of the **largest** clique in the Grötzsch graph? Give an example of such a clique and prove that no larger clique exists.

(c) What is the size of the **largest** clique in the FourTriangles graph? Give an example of such a clique and prove that no larger clique exists.

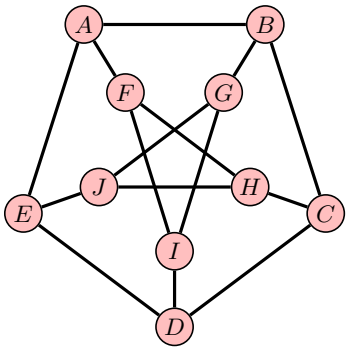
(d) A **vertex coloring** (or simply a **coloring**) of a graph assigns colors to its vertices so that every two adjacent vertices get different colors. Prove that every coloring of a graph uses at least as many colors as the size of its largest clique. What lower bound does this give for each of the three graphs above?

(e) The **complement** $\overline{G}$ of a graph G has the same vertices as G, and two different vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. An **independent set** is a set of vertices, no two of which are adjacent. Prove that a set of vertices is a clique in G if and only if it is an independent set in $\overline{G}$. What is the size of the largest independent set in the complement of the Petersen graph?

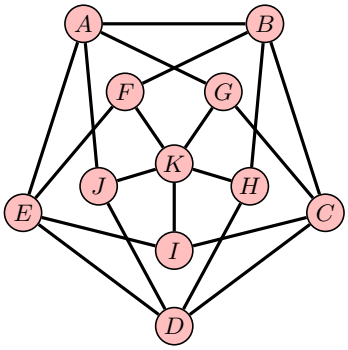
7.2. Let G be a simple undirected graph with $n \geq 1$ vertices. An **independent set** in G is a set IS of vertices, no two of which are adjacent. That is, for any two different vertices $u, v \in IS$, the graph G does not include the edge (u, v).

For example, the set of all the vertices of the null graph (no edges) is an independent set of size n. In contrast, in the complete graph K_n (every two vertices are adjacent) an independent set contains at most one vertex.

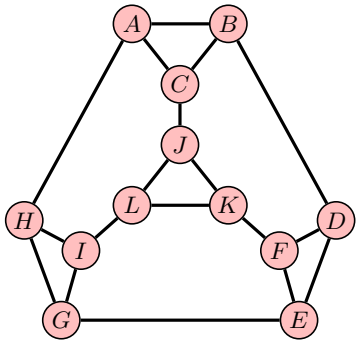
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

(a) What is the size of the **largest** independent set in the Petersen graph? Give an example of such a set and prove that no larger independent set exists.

(b) What is the size of the **largest** independent set in the Grötzsch graph? Give an example of such a set and prove that no larger independent set exists.

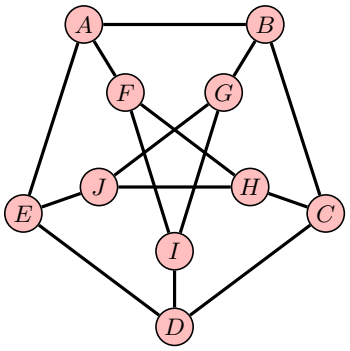
(c) What is the size of the **largest** independent set in the FourTriangles graph? Give an example of such a set and prove that no larger independent set exists.

(d) A **vertex coloring** (or simply a **coloring**) of a graph assigns colors to its vertices so that every two adjacent vertices get different colors. Prove that if a graph has n vertices and its largest independent set has size α, then every coloring of the graph uses at least n/α colors. What lower bound does this give for each of the three graphs above?

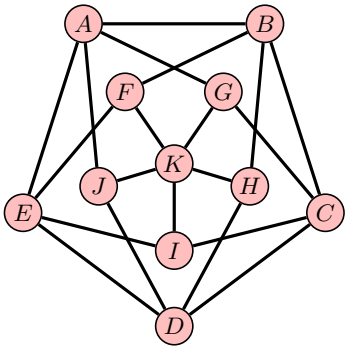
7.3. Let G be a simple undirected graph with $n \geq 1$ vertices. A **vertex cover** in G is a set VC of vertices that contains at least one endpoint of every edge. That is, for any edge (u, v) of G, $u \in VC$ or $v \in VC$ (or both).

For example, the empty set is a vertex cover of the null graph (no edges), because there are no edges to cover. In contrast, in the complete graph K_n (every two vertices are adjacent) any vertex cover must contain at least $n - 1$ vertices, because if a set misses two vertices, then no vertex in the set covers the edge between them.

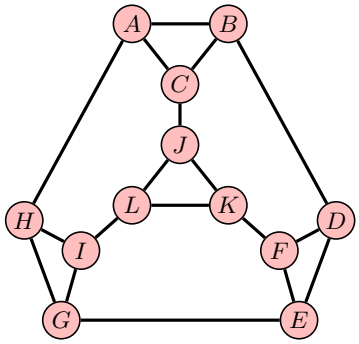
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

(a) What is the size of the **smallest** vertex cover in the Petersen graph? Give an example of such a vertex cover and prove that no smaller vertex cover exists.

(b) What is the size of the **smallest** vertex cover in the Grötzsch graph? Give an example of such a vertex cover and prove that no smaller vertex cover exists.

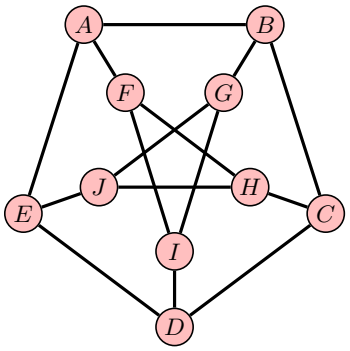
(c) What is the size of the **smallest** vertex cover in the FourTriangles graph? Give an example of such a vertex cover and prove that no smaller vertex cover exists.

(d) An **independent set** is a set of vertices, no two of which are adjacent. Let G be a graph with n vertices, and let V be the set of its vertices. Prove that a set IS of vertices is an independent set in G if and only if $V \setminus IS$ is a vertex cover in G. Conclude that the size of the largest independent set plus the size of the smallest vertex cover is n.

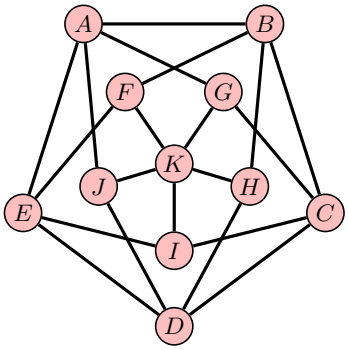
7.4. Let G be a simple undirected graph with $n \geq 1$ vertices. A **dominating set** in G is a set DS of vertices for which every vertex not in DS has at least one neighbor in DS. That is, for any vertex v of G, $v \in DS$ or v has a neighbor in DS.

For example, in the null graph (no edges) any dominating set must contain all the vertices, and therefore this is the only possible dominating set. In contrast, in the complete graph K_n (every two vertices are adjacent) any single vertex forms a dominating set, because it is adjacent to all the other vertices.

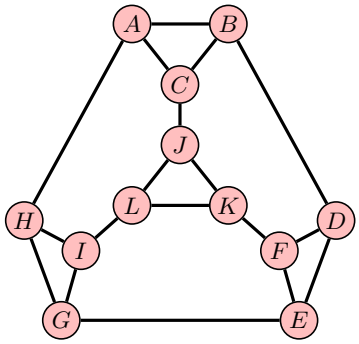
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

(a) What is the size of the **smallest** dominating set in the Petersen graph? Give an example of such a dominating set and prove that no smaller dominating set exists.

(b) What is the size of the **smallest** dominating set in the Grötzsch graph? Give an example of such a dominating set and prove that no smaller dominating set exists.

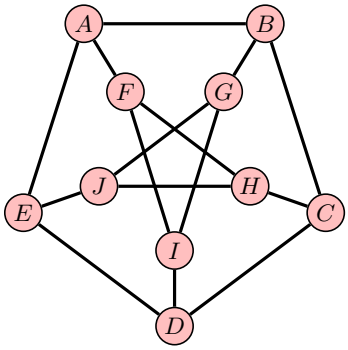
(c) What is the size of the **smallest** dominating set in the FourTriangles graph? Give an example of such a dominating set and prove that no smaller dominating set exists.

(d) A **vertex cover** is a set of vertices that contains at least one endpoint of every edge. Prove that in a graph without isolated vertices (vertices with no neighbors), every vertex cover is a dominating set. Show that a dominating set is not necessarily a vertex cover. Moreover, show that for every $n \geq 2$, there is a graph with n vertices whose smallest vertex cover is $n - 1$ times as large as its smallest dominating set.

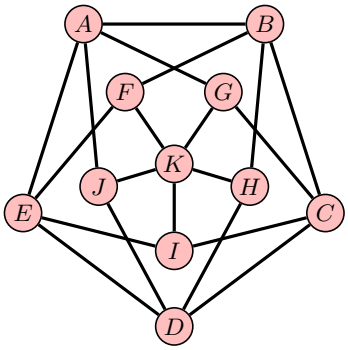
7.5. Let G be a simple undirected graph with $n \geq 1$ vertices. A **matching** in G is a set M of edges, no two of which share an endpoint. That is, every vertex of G is an endpoint of at most one edge of M.

For example, the only matching in the null graph (no edges) is the empty set. In contrast, the largest matching in the complete graph K_n (every two vertices are adjacent) has size $\lfloor n/2 \rfloor$. On the one hand, a matching with k edges has $2k$ different endpoints, so $2k \leq n$. On the other hand, since there is an edge between every pair of vertices, the vertices can be split into $\lfloor n/2 \rfloor$ pairs, and the edges between the two vertices of each pair form a matching (if n is odd, one vertex is left out).

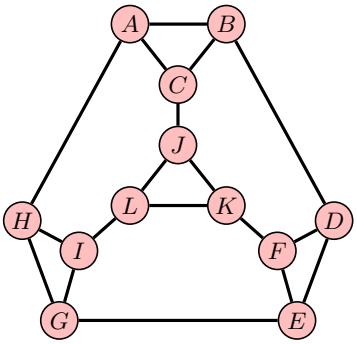
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

(a) What is the size of the **largest** matching in the Petersen graph? Give an example of such a matching and prove that no larger matching exists.

(b) What is the size of the **largest** matching in the Grötzsch graph? Give an example of such a matching and prove that no larger matching exists.

(c) What is the size of the **largest** matching in the FourTriangles graph? Give an example of such a matching and prove that no larger matching exists.

(d) The Petersen graph has exactly six matchings of size 5. Draw them, prove that every two of them share an edge, and conclude that the edges of the Petersen graph cannot be split into three matchings of size 5.

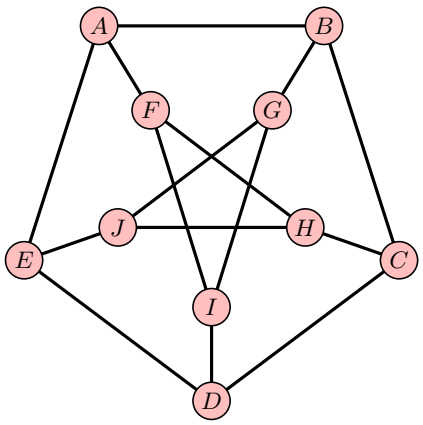
(e) A matching is **maximal** if no edge can be added to it so that it remains a matching, and **maximum** if it has the largest possible size. Show that every maximum matching is maximal. Find maximal matchings of sizes 3 and 4 in the Petersen graph, and prove that the Petersen graph has no maximal matching with fewer than 3 edges.

(f) A **vertex cover** is a set of vertices that contains at least one endpoint of every edge. Prove that in every graph, every vertex cover is at least as large as every matching, and that the endpoints of a maximal matching form a vertex cover. Conclude that these endpoints form a vertex cover that is at most twice as large as the smallest vertex cover.

Part II: Solutions

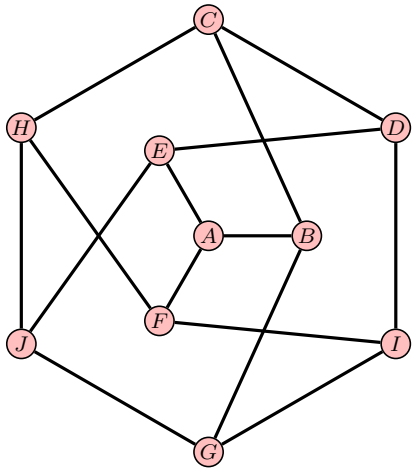
1 Graph Isomorphism

1.1. Consider the drawing of the Petersen graph below.



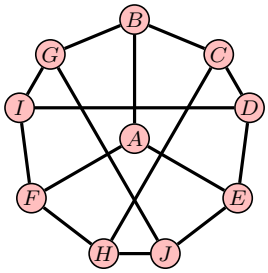
(a) Find at least two additional drawings of the Petersen graph with a symmetry, different from the drawing above. Label their vertices with $A, B, \ldots, J$ so that the same pairs of labels are joined by edges as in the drawing above, and explain why this shows that your drawings depict the Petersen graph.

Answer: The drawing above has 15 edges: the outer cycle $A-B-C-D-E-A$ with the edges (A,B), (B,C), (C,D), (D,E), (E,A); the inner cycle $F-H-J-G-I-F$ with the edges (F,H), (H,J), (J,G), (G,I), (I,F); and the five spokes (A,F), (B,G), (C,H), (D,I), (E,J). Below are three additional symmetric drawings of the Petersen graph, with the vertices placed in different arrangements: $1+3+6$, $1+9$, and 10 on a circle. In each of them, the vertices are labeled so that the edges are exactly these 15 edges. Hence, each drawing depicts the Petersen graph: the identity map on the labels is an isomorphism.

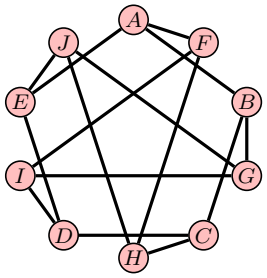


Middle vertex, triangle, and hexagon $(1+3+6)$.[1] The vertex A is in the middle, its three neighbors B, E, F are around it, and the six vertices at distance 2 from A form the hexagon $C-D-I-G-J-H-C$. Each of B, E, F is adjacent to two opposite vertices of this cycle: B to C and G, E to D and J, F to H and I. Rotating by one third of a full turn maps the drawing onto itself. Together, these are the $3+6+6=15$ edges of the drawing above.

[1] Every vertex of the Petersen graph is at distance at most 2 from A; see also the solution of Problem 4.4(a) and (d) on page 80.



Middle vertex and a 9-cycle $(1+9)$**.** The nine vertices on the circle form the cycle $B-C-D-E-J-H-F-I-G-B$. The vertex A in the middle is adjacent to B, E, and F, every third vertex of the cycle, and the three chords (C,H), (D,I), (G,J) each jump over three vertices. Rotating by one third of a full turn maps the drawing onto itself. This drawing makes long paths and cycles easy to see;[2] different drawings serve different problems. Together, these are the $9+3+3=15$ edges of the drawing above.

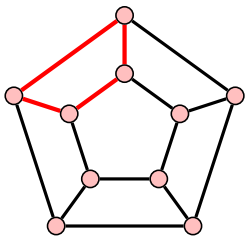


All ten vertices on a circle (10)**.** The outer vertices A,B,C,D,E alternate with the inner vertices F,G,H,I,J around the circle. Each spoke joins two neighbors on the circle, each edge of the outer cycle jumps over one vertex, and each edge of the inner cycle jumps over three vertices. Rotating by one fifth of a full turn maps the drawing onto itself. Together, these are the $5+5+5=15$ edges of the drawing above.

Additional drawings: `https://mathworld.wolfram.com/PetersenGraph.html`

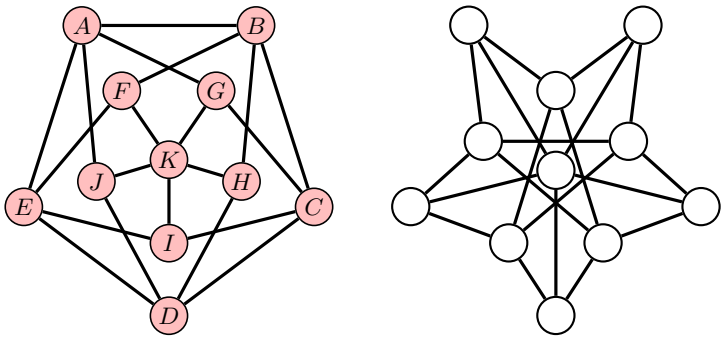
(b) Draw a 3-regular graph (every vertex has degree 3) with 10 vertices, and therefore 15 edges, that has no triangle and is not isomorphic to the Petersen graph. Explain why your graph is not isomorphic to the Petersen graph.

Answer: The **pentagonal prism** below consists of an outer cycle of length 5 and an inner cycle of length 5, with each outer vertex joined to the corresponding inner vertex. Every vertex has degree 3, and there are 10 vertices and 15 edges. There is no triangle: neither cycle of length 5 contains a triangle, and a triangle with vertices on both cycles would contain a vertex with two neighbors on the other cycle, while every vertex has only one. The prism is not isomorphic to the Petersen graph, since it has a cycle of length 4 (red), while the Petersen graph has none. Indeed, two opposite vertices of a cycle of length 4 have two common neighbors, while in the Petersen graph, every two vertices have at most one common neighbor, as can be checked in the labeled drawing above.[3] An isomorphism maps a cycle of length 4 to a cycle of length 4. Moreover, the prism has exactly five cycles of length 4 and diameter 3, while the Petersen graph has diameter 2.[4]



[2] See the solution of Problem 4.2 on page 76, where $A-B-C-D-E-J-H-F-I-G$ is a Hamiltonian path.
[3] See also the solutions of Problem 3.1(d) and (e) on pages 55 and 56.
[4] See the solution of Problem 4.4(a) on page 80.

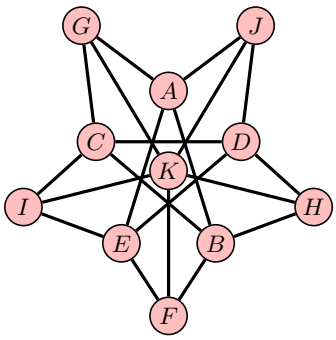
1.2. Consider the Grötzsch graph on the left below, with the 11 vertices $A, B, C, D, E, F, G, H, I, J, K$.



(a) Assign the 11 labels $A, B, \ldots, K$ to the 11 vertices in the right drawing to show that the graph it depicts is isomorphic to the Grötzsch graph. That is, for any pair of vertices $u, v \in \{A, B, \ldots, K\}$, the edge (u, v) exists in the left drawing if and only if it exists in the right drawing with your label assignment. Explain your strategy for finding the labels.

Answer: The following assignment of the 11 labels $A, B, \ldots, K$ to the 11 vertices of the right drawing (below) shows that the graph it depicts is isomorphic to the Grötzsch graph. That is, for any pair of vertices $u, v \in \{A, B, \ldots, K\}$, the edge (u, v) exists in the left drawing if and only if it exists in the right drawing with this label assignment.

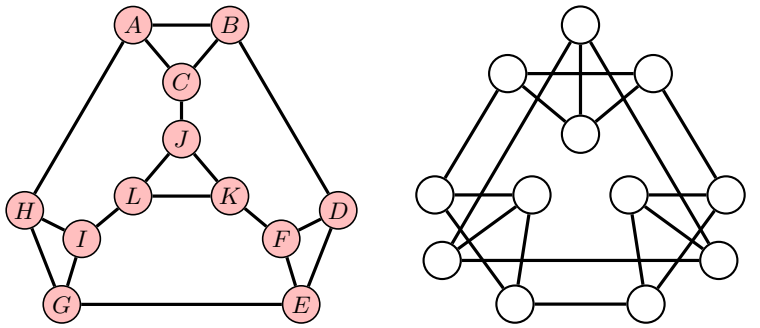
Strategy for finding the labels. An isomorphism maps every vertex to a vertex of the same degree. In the Grötzsch graph, K is the only vertex of degree 5, its five neighbors F, G, H, I, J have degree 3, and the vertices A, B, C, D, E have degree 4 and form the cycle $A-B-C-D-E-A$. Hence, label the vertex of degree 5 in the right drawing by K, and label the cycle formed by the five vertices of degree 4 by A, B, C, D, E in order. Each vertex of degree 3 is then determined by its two neighbors of degree 4: F is adjacent to B and E, G to A and C, H to B and D, I to C and E, and J to A and D.



(b) How many different assignments of the 11 labels to the vertices of the right drawing show that it depicts the Grötzsch graph? Explain your answer.

Answer: 10. By part (a), a correct assignment gives the label K to the only vertex of degree 5, and the labels A, B, C, D, E, in this order, to the cycle formed by the five vertices of degree 4; then the labels F, G, H, I, J are determined. The label A can be given to any of the 5 vertices of this cycle, and then B to either of its 2 neighbors on the cycle, and the labels C, D, E follow along the cycle. Hence, there are at most $5 \cdot 2 = 10$ correct assignments. All 10 of them are correct: each of the 5 rotations (including the rotation by a full turn) and 5 reflections of the left drawing maps the Grötzsch graph onto itself, so combining any of them with the assignment of part (a) gives a correct assignment. In other words, the Grötzsch graph can be mapped onto itself in exactly 10 ways. For example, the rotation by one fifth of a full turn maps A, B, C, D, E to B, C, D, E, A and F, G, H, I, J to G, H, I, J, F, and it keeps K in place. The reflection that keeps A in place swaps B with E, C with D, G with J, and H with I, and it keeps A, F, and K in place.

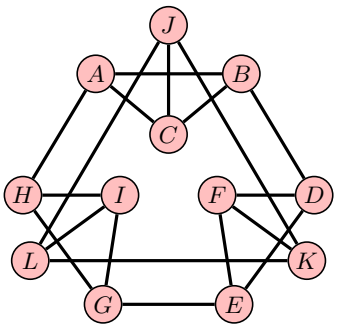
1.3. Consider the FourTriangles graph on the left below, with the 12 vertices $A, B, C, D, E, F, G, H, I, J, K, L$. It consists of four triangles, and every two of them are joined by one edge (the graph is also known as the truncated tetrahedron graph).



(a) Assign the 12 labels $A, B, \ldots, L$ to the 12 vertices in the right drawing to show that the graph it depicts is isomorphic to the FourTriangles graph. That is, for any pair of vertices $u, v \in \{A, B, \ldots, L\}$, the edge (u, v) exists in the left drawing if and only if it exists in the right drawing with your label assignment. Explain your strategy for finding the labels.

Answer: The following assignment of the 12 labels $A, B, \ldots, L$ to the 12 vertices of the right drawing (below) shows that the graph it depicts is isomorphic to the FourTriangles graph. That is, for any pair of vertices $u, v \in \{A, B, \ldots, L\}$, the edge (u, v) exists in the left drawing if and only if it exists in the right drawing with this label assignment.

Strategy for finding the labels. An isomorphism maps triangles to triangles. In the left drawing, the four triangles are A, B, C and D, E, F and G, H, I and J, K, L, and the six edges that join them are (A, H), (B, D), (C, J), (E, G), (F, K), and (I, L). Hence, first find the four triangles in the right drawing. Each vertex is then determined by its triangle and by the triangle that its joining edge leads to. For example, A is the vertex of the triangle A, B, C whose joining edge leads to the triangle G, H, I. Any naming of the four triangles works, since every two triangles are joined by one edge.



(b) How many different assignments of the 12 labels to the vertices of the right drawing show that it depicts the FourTriangles graph? Explain your answer.

Answer: 24. By part (a), a correct assignment maps triangles to triangles. The label A can be given to any of the 12 vertices, and then the labels B and C to the other two vertices of its triangle in one of 2 orders. All the other labels are then determined. The labels H, D, and J go to the vertices joined to the vertices labeled A, B, and C, respectively, by the edges that leave their triangle. This determines the triangles of H and of J, and hence the labels E and F in the triangle of D, since E is joined to the triangle of H and F to the triangle of J. In the same way, E determines G, F determines K, and the last vertex of each triangle gets the remaining label. Hence, there are at most $12 \cdot 2 = 24$ correct assignments. All 24 of them are correct: by part (a), each of the $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to name the four triangles gives a correct assignment, and different namings give different assignments. In other words, the FourTriangles graph can be mapped onto itself in exactly 24 ways.

1.4. Two simple (no self-loops and no parallel edges) undirected graphs G and H are **isomorphic** if there exists a bijection f from the vertices of G to the vertices of H such that an edge (u, v) exists in G **if and only if** the edge $(f(u), f(v))$ exists in H.

In each of the following parts, determine whether G and H are (i) always isomorphic, or (ii) never isomorphic, or (iii) sometimes isomorphic and sometimes not isomorphic.

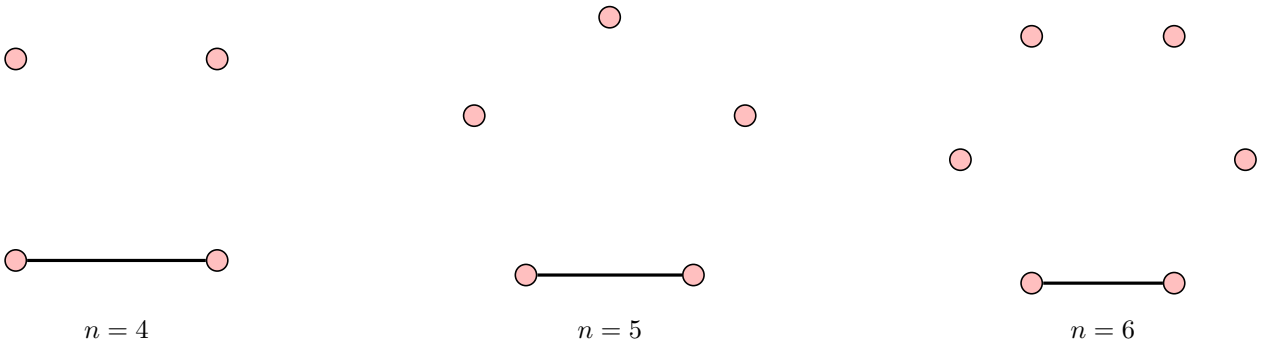
Justify your answers.

(a) For $n \geq 1$, G has n vertices, H has $n + 1$ vertices, and both have the same number of edges.

Answer: Never isomorphic. A bijection between the vertices of G and the vertices of H exists only if both graphs have the same number of vertices. This holds regardless of the edges.

(b) Both graphs have the same number $n \geq 2$ of vertices and exactly one edge.

Answer: Always isomorphic. An isomorphism maps the two endpoints of the edge of G to the two endpoints of the edge of H, and the other $n - 2$ vertices in any one-to-one way. These graphs for $n = 4, 5, 6$ are shown below.



$n = 4$ $n = 5$ $n = 6$

(c) Both graphs have the same number $n \geq 4$ of vertices and exactly two edges.

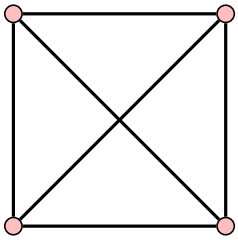
Answer: Sometimes isomorphic and sometimes not isomorphic. The two edges either share a vertex or are disjoint (possible since $n \geq 4$). Two graphs of the same kind are isomorphic: if the edges share a vertex, map the common vertex to the common vertex and the other two endpoints to the other two endpoints; if the edges are disjoint, map the two endpoints of each edge of G to the two endpoints of a different edge of H; in both cases, map the remaining vertices in any one-to-one way. Graphs of different kinds are not isomorphic, since only one of them has a vertex of degree 2. The two graphs for $n = 4$ are shown below. For $n = 3$, the two edges must share a vertex, and thus G and H are always isomorphic. This is why $n \geq 4$ in this part.



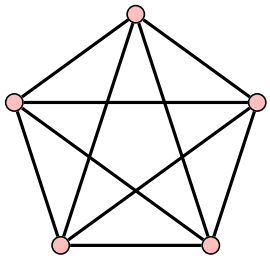
disjoint edges a shared vertex

(d) Both graphs have the same number $n \geq 2$ of vertices and all possible edges.

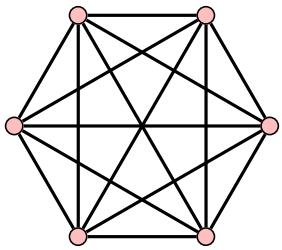
Answer: Always isomorphic. Any bijection f from the vertices of G to the vertices of H is an isomorphism, since every pair of vertices is an edge in both graphs. Hence, the complete graph K_n is the only graph, up to isomorphism, with n vertices and all the $n(n-1)/2$ possible edges. These graphs for $n = 4, 5, 6$ are shown below.



$n = 4$



$n = 5$



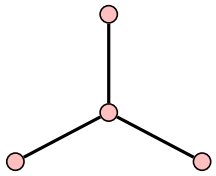
$n = 6$

(e) Both graphs are trees (connected graphs without cycles) with the same number $n \geq 4$ of vertices.

Answer: Sometimes isomorphic and sometimes not isomorphic. Two copies of the same tree are isomorphic. However, the path P_n and the star S_n with n vertices are both trees, and they are not isomorphic: the middle vertex of the star has degree $n-1 \geq 3$, while every vertex of the path has degree at most 2. The two trees with 4 vertices are shown below. For $n \leq 3$, there is only one tree with n vertices, which is why $n \geq 4$ is required.



P_4



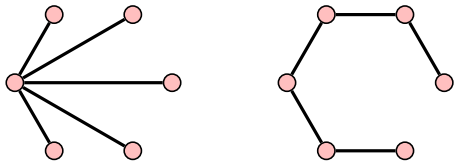
S_4

1.5. Let G and H be two simple undirected graphs, each with 6 vertices and the **same** number of edges.

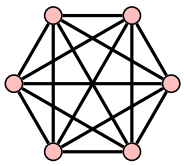
In each of the following parts, if it is possible to find two non-isomorphic graphs G and H that satisfy the conditions, draw examples for G and H and explain why G and H are not isomorphic. If it is not possible, explain why.

Remark: In all the figures below, the six vertices are placed at the same positions, the corners of a regular hexagon. Hence, the graphs differ only in their edges, which makes them easier to compare.

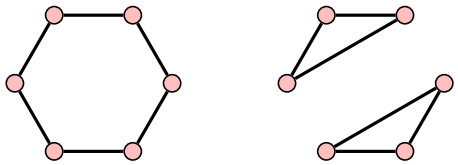
(a) No additional condition.
Answer: Possible. The following two graphs both have 6 vertices and 5 edges, and both are trees: one is a star and the other is a path. They are not isomorphic because the star has a vertex of degree 5, while every vertex of the path has degree at most 2.



(b) Every vertex of G and of H has degree 5.
Answer: Not possible. A vertex of degree 5 is adjacent to all the other five vertices. Hence, if every vertex has degree 5, then all the 15 possible edges exist, and both G and H are the complete graph K_6 shown below.

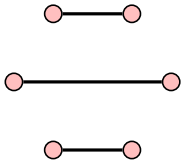


(c) Every vertex of G and of H has degree 2.
Answer: Possible. The following two graphs have 6 vertices and 6 edges, and every vertex has degree 2. One of them is a cycle of length 6, while the other consists of two triangles (cycles of length 3). They are not isomorphic because the cycle is connected, while the two triangles form a disconnected graph.



Moreover, these are the only two such graphs, up to isomorphism.[5] In a graph in which every vertex has degree 2, every connected component is a cycle. Indeed, start at any vertex and keep moving along unused edges. Since every degree is 2, every vertex other than the start is entered at most once and then left, so the route can only stop at its start, closing a cycle. Every vertex of this cycle already uses both of its edges, so the cycle is a whole connected component; repeat with the remaining vertices. Since every cycle has at least 3 vertices, the 6 vertices form either one cycle of length 6 or two cycles of length 3.

(d) Every vertex of G and of H has degree 1.
Answer: Not possible. If every vertex has degree 1,[6] then each vertex has exactly one neighbor, so the 6 vertices are paired by 3 disjoint edges. Hence, G and H are both isomorphic to the graph shown below (map the two endpoints of each edge of G to the two endpoints of a different edge of H).



[5] More generally, every 2-regular graph is a disjoint union of cycles; see the solution of Problem 5.4(b) on page 95.
[6] See also the solution of Problem 5.4(a) on page 95.

1.6. In each of the following parts, draw two non-isomorphic graphs that satisfy the conditions, and explain why they are not isomorphic.

(a) Simple graphs with 4 vertices and 2 edges.

Answer:



Both graphs have four vertices and two edges. The two graphs are not isomorphic because in one of them the two edges share a vertex and in the other the two edges are disjoint.

(b) Simple bipartite graphs (graphs whose vertices can be split into two sides so that every edge joins the two sides) with 4 vertices and 3 edges.

Answer:



Both graphs have four vertices and three edges, and they are bipartite: the left one is a star (the middle vertex on one side and the three leaves on the other side) and the right one is a path (whose vertices alternate between the two sides). The two graphs are not isomorphic because one of them has a vertex of degree 3, while in the other the maximum degree is 2.

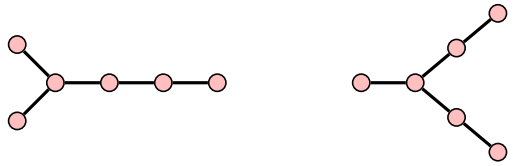
(c) Simple graphs with 6 vertices and the degree sequence $(3, 3, 2, 2, 2, 2)$, that is, two vertices of degree 3 and four vertices of degree 2.

Answer: Both graphs below have six vertices, seven edges, and the degree sequence $(3, 3, 2, 2, 2, 2)$; the two vertices of degree 3 are red. The two graphs are not isomorphic: an isomorphism maps the two vertices of degree 3 of one graph to the two vertices of degree 3 of the other, and it keeps them adjacent or non-adjacent. In the left graph, the two vertices of degree 3 are adjacent, while in the right graph they are not. Hence, two graphs with the same degree sequence are not necessarily isomorphic. The two graphs also differ in another way: the left graph is bipartite, while the right graph has a cycle of length 5 and hence is not bipartite, since the sides would alternate around this odd cycle. In fact, exactly four non-isomorphic graphs have this degree sequence.[7]



(d) Trees (connected graphs without cycles) with 6 vertices and 3 leaves (vertices of degree 1).

Answer:



Both graphs are trees with six vertices, five edges, and three leaves, and both have the degree sequence $(3, 2, 2, 1, 1, 1)$. These two trees are not isomorphic because in one of them the vertex of degree 3 is adjacent to two leaves and in the other the vertex of degree 3 is adjacent to only one leaf.[8]

[7] Graphic sequences and their realizations are the subject of Problems 5.2 and 5.3 on pages 18 and 19.

[8] These two trees also appear in the solution of Problem 2.2(b) on page 41.

1.7. There are six non-isomorphic connected molecules of the type C_4H_y for some positive integer y in which each C-atom has degree 4, each H-atom has degree 1, any two C-atoms are joined by at most one edge, and every H-atom is connected to a C-atom. One of them, the molecule C_4H_{10} in which the four C-atoms form a path, is shown below. Illustrate the other five.

Hint: The number y of H-atoms is uniquely determined by the interconnection structure among the four C-atoms.

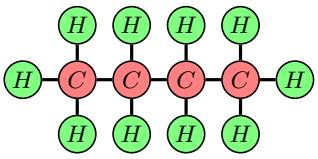
Answer: All six molecules are shown below; the first one is given in the problem. The molecules are explained by the following four observations.

1. The C-subgraph. Call the graph formed by the four C-atoms and the edges among them the **C-subgraph**, and let x be its number of edges. Every H-atom has degree 1, so its only neighbor is a C-atom, and it is never in the middle of a path between two other atoms. Hence, the molecule is connected if and only if its C-subgraph is connected. A connected graph with 4 vertices has at least 3 edges, and since any two C-atoms are joined by at most one edge, $x \le \binom{4}{2} = 6$. Thus, $3 \le x \le 6$.

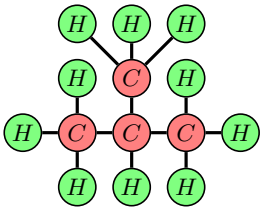
2. The number of H-atoms. The molecule has $x + y$ edges: x edges among the C-atoms and one edge for each H-atom. Every edge has two endpoints, so the sum of the degrees, $4 \cdot 4 + y \cdot 1 = 16 + y$, is twice the number of edges.[9] Hence, $16 + y = 2(x + y)$, that is, $y = 16 - 2x$: there are 10, 8, 6, or 4 H-atoms when $x = 3$, 4, 5, or 6, respectively.

3. The C-subgraph determines the molecule. A C-atom with d neighbors among the C-atoms is attached to exactly $4 - d$ H-atoms, so two molecules with isomorphic C-subgraphs are isomorphic. Conversely, an isomorphism maps C-atoms (the vertices of degree 4) to C-atoms, so molecules with non-isomorphic C-subgraphs are not isomorphic. Hence, the molecules correspond to the connected graphs with 4 vertices.[10]

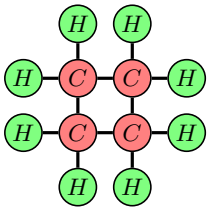
4. There are exactly six connected graphs with 4 vertices, up to isomorphism. If $x = 3$, the graph has no cycle: removing an edge of a cycle would leave a connected graph with 4 vertices and only 2 edges. Hence, it is a tree, which is a star if some vertex has degree 3, and a path otherwise. If $x = 4$, the graph is not a tree (a tree with 4 vertices is a star or a path, with 3 edges), so it has a cycle: a cycle with 4 vertices uses all the edges, and a triangle leaves one edge that joins the fourth vertex to the triangle. If $x = 5$, exactly one pair of C-atoms is not joined, and all such graphs are isomorphic (the complete graph with 4 vertices minus an edge). If $x = 6$, the graph is the complete graph with 4 vertices.



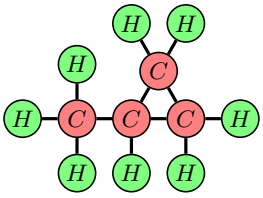
C_4H_{10}: a path



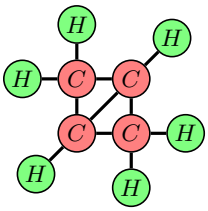
C_4H_{10}: a star



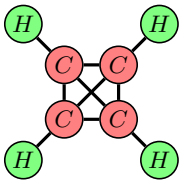
C_4H_8: a cycle



C_4H_8: a triangle plus an edge



C_4H_6: the complete graph minus an edge

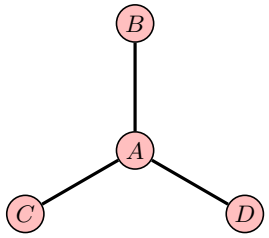


C_4H_4: the complete graph

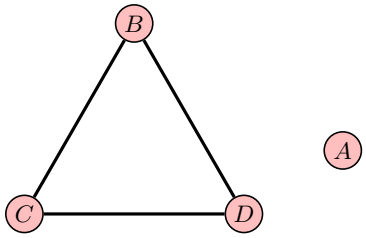
[9] The handshake lemma is the subject of Problem 5.1 on page 18.

[10] All 11 graphs with 4 vertices, 6 of them connected, are drawn in Problem 2.3 on page 5.

1.8. The **complement** of a simple graph G, denoted by $\overline{G}$, is the graph with the same vertices as G in which two different vertices are adjacent if and only if they are **not** adjacent in G. For example, the complement of the star S_4 with 4 vertices (left) is a triangle plus a vertex with no neighbors (right): the middle vertex of the star, which is adjacent to all the other vertices, has no neighbors in the complement, while the three leaves, which are not adjacent to each other, form a triangle.



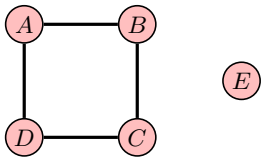
star

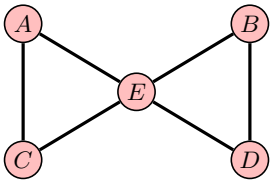


its complement

(a) Find the complement of the graph below: a cycle with 4 vertices plus a vertex with no neighbors. Draw it so that its shape is easy to recognize.

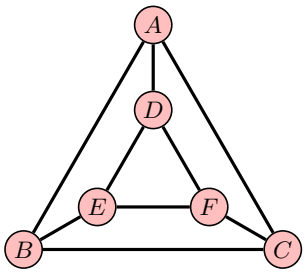
Answer: The **bow-tie** graph. The vertex E is adjacent to no vertex,[11] so in the complement it is adjacent to all the other four vertices. Among A, B, C, D, the only non-adjacent pairs in the cycle are the two diagonals (A, C) and (B, D). Hence, the complement has the 6 edges (E, A), (E, B), (E, C), (E, D), (A, C), and (B, D): two triangles E, A, C and E, B, D that share the vertex E.

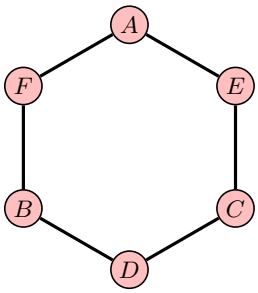




(b) Find the complement of the prism graph below. Which well-known graph is it?

Answer: The cycle C_6. In the prism graph, every vertex is adjacent to 3 of the other 5 vertices, so in the complement every vertex is adjacent to exactly 2 vertices: A to E, F; B to D, F; C to D, E; D to B, C; E to A, C; and F to A, B. These 6 edges form the single cycle $A - E - C - D - B - F - A$. Hence, the complement of the prism graph is the cycle C_6.





(c) Let G and H be two simple graphs. Prove that G and H are isomorphic if and only if $\overline{G}$ and $\overline{H}$ are isomorphic.

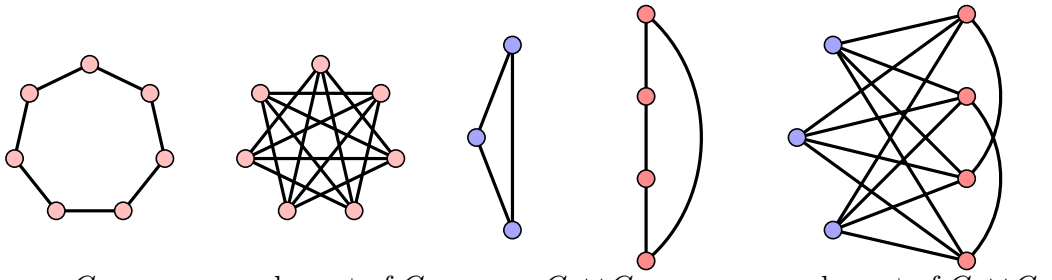
Proof: Let f be a bijection from the vertices of G to the vertices of H, and let u and v be two different vertices of G. By definition, (u, v) is an edge of $\overline{G}$ if and only if (u, v) is not an edge of G, and $(f(u), f(v))$ is an edge of $\overline{H}$ if and only if $(f(u), f(v))$ is not an edge of H. Hence, the statement "(u, v) is an edge of G if and only if $(f(u), f(v))$ is an edge of H" is equivalent to the statement "(u, v) is an edge of $\overline{G}$ if and only if $(f(u), f(v))$ is an edge of $\overline{H}$", since negating both sides of an "if and only if" statement keeps it true. Therefore, f is an isomorphism from G to H if and only if f is an isomorphism from $\overline{G}$ to $\overline{H}$, and both directions of the claim follow.

[11] Computing the complement is the subject of Problem 3.8 on page 13.

Example: This fact helps to decide whether graphs with many edges are isomorphic, by looking at their complements, which have few edges. For example, how many non-isomorphic simple graphs with 7 vertices have all their degrees equal to 4? Such a graph has 14 edges, and comparing such graphs directly is hard. However, every vertex is non-adjacent to exactly $6 - 4 = 2$ other vertices, so the complement is a graph with 7 vertices in which every vertex has degree 2.

Every connected component of such a graph is a cycle (start at any vertex and keep moving along unused edges: since every degree is 2, the route can only stop by returning to its start), and every cycle has at least 3 vertices.[12] Hence, up to isomorphism, there are exactly two such graphs: the cycle C_7 and the disjoint union $C_3 \cup C_4$ of a triangle and a cycle with 4 vertices.

By the claim just proven, there are exactly two non-isomorphic graphs with 7 vertices in which every vertex has degree 4: the complements of these two graphs. Below, each of the two graphs is shown next to its complement. In the complement of $C_3 \cup C_4$, every vertex of the triangle (blue) is adjacent to every vertex of the cycle with 4 vertices (red), and the only other edges are the two diagonals of this cycle.



C_7 complement of C_7 $C_3 \cup C_4$ complement of $C_3 \cup C_4$

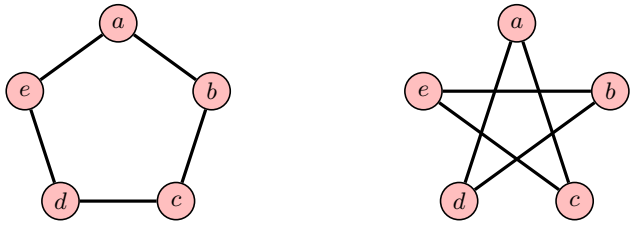
(d) Let P_4 be the path with 4 vertices and let C_5 be the cycle with 5 vertices. Show that $\overline{P_4}$ is isomorphic to P_4 and that $\overline{C_5}$ is isomorphic to C_5. Such graphs are called **self-complementary**.

Answer: Let the path P_4 be $a - b - c - d$. Its complement has the three edges (a, c), (a, d), and (b, d), which form the path $c - a - d - b$. The bijection $f(a) = c$, $f(b) = a$, $f(c) = d$, $f(d) = b$ maps the edges (a, b), (b, c), (c, d) of P_4 to the edges (c, a), (a, d), (d, b), which are exactly the edges of $\overline{P_4}$. Hence, $\overline{P_4}$ is isomorphic to P_4.



P_4 $\overline{P_4}$

Let the cycle C_5 be $a - b - c - d - e - a$. Its complement has the five edges (a, c), (a, d), (b, d), (b, e), and (c, e), which form the cycle $a - c - e - b - d - a$ (the pentagram below). The bijection $f(a) = a$, $f(b) = c$, $f(c) = e$, $f(d) = b$, $f(e) = d$ maps the edges (a, b), (b, c), (c, d), (d, e), (e, a) of C_5 to the edges (a, c), (c, e), (e, b), (b, d), (d, a), which are exactly the edges of $\overline{C_5}$. Hence, $\overline{C_5}$ is isomorphic to C_5.

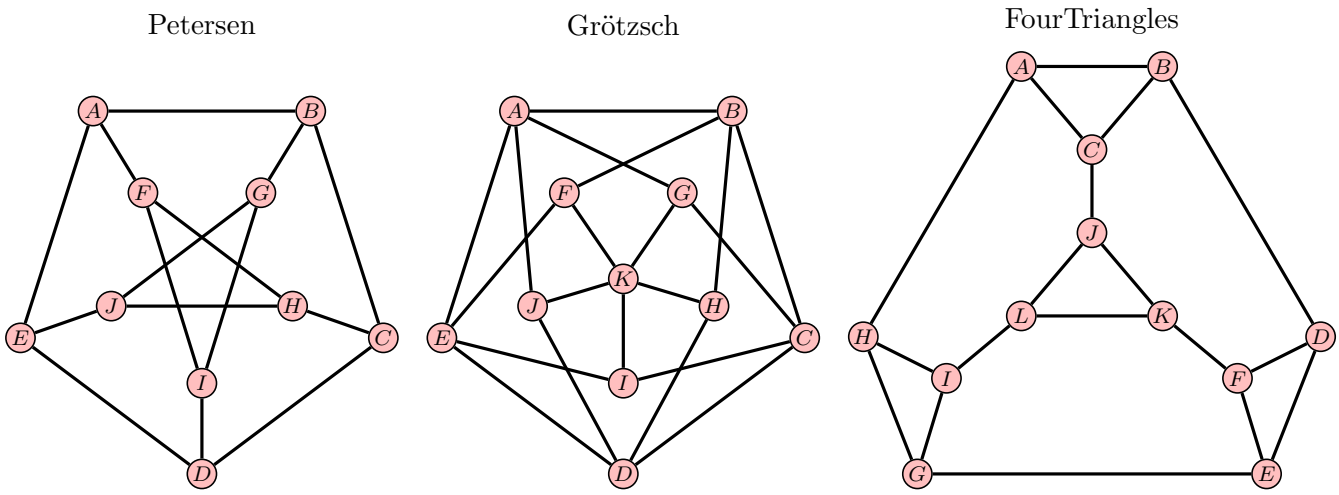


C_5 $\overline{C_5}$

[12] See also the solution of Problem 5.4 on page 95; the complement of C_7 also appears in the solution of Problem 5.6(b) on page 100.

2 Families of Graphs

2.1. Below are the Petersen graph, the Grötzsch graph, and the FourTriangles graph.



The vertices are labeled only to make the reasoning easier to follow.

(a) For each graph, mark Yes (Y) or No (N) in the table: is it connected, a tree, regular (all degrees are equal), triangle-free (no cycle of length 3), bipartite (the vertices can be split into two sides so that every edge joins the two sides)? Justify briefly.

Answer:

	Petersen	Grötzsch	FourTriangles
connected	Y	Y	Y
a tree	N	N	N
regular	Y	N	Y
triangle-free	Y	Y	N
bipartite	N	N	N

Connected. All three graphs are connected. In the Petersen graph, the outer cycle $A-B-C-D-E-A$ joins all the outer vertices, and every inner vertex is a neighbor of an outer vertex: F of A, G of B, H of C, I of D, and J of E. In the Grötzsch graph, each of F, G, H, I, J is a neighbor of K, and each of A, B, C, D, E is a neighbor of one of F, G, H, I, J. Hence, every vertex is joined to K by a path, and any two vertices are joined by a path through K. In the FourTriangles graph, any two vertices of the same triangle are joined by an edge, and every two triangles are joined by an edge.

A tree. None of the three graphs is a tree. A tree has no cycle, but the Petersen graph and the Grötzsch graph contain the cycle $A-B-C-D-E-A$, and the FourTriangles graph contains the triangle $A-B-C-A$. Alternatively, a tree with n vertices has exactly $n-1$ edges, while the three graphs have 10, 11, and 12 vertices and 15, 20, and 18 edges, respectively.

Regular. The Petersen graph is regular: every outer vertex has two neighbors on the outer cycle and one inner neighbor, and every inner vertex has two neighbors on the inner cycle $F-H-J-G-I-F$ and one outer neighbor, so all the degrees are 3. The FourTriangles graph is regular: every vertex has two neighbors in its own triangle, and it is an endpoint of exactly one of the six edges (A, H), (B, D), (C, J), (E, G), (F, K), and (I, L) that join the triangles (their 12 endpoints are all different), so all the degrees are 3. The Grötzsch graph is not regular: K has degree 5 (its neighbors are F, G, H, I, J), A has degree 4 (its neighbors are B, E, G, J), and F has degree 3 (its neighbors are B, E, K).

Triangle-free. The FourTriangles graph is not triangle-free: it has four triangles.

The Petersen graph is triangle-free. Among the outer vertices, the only edges are the five edges of the outer cycle, and no three of them form a triangle. The same holds for the inner vertices and the inner cycle. Finally, a triangle with both outer and inner vertices would have two vertices of one kind and a third vertex of the other kind that is adjacent to both of them. This is impossible, because every vertex has exactly one neighbor of the other kind.

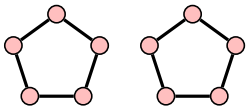
The Grötzsch graph is triangle-free. No two of F, G, H, I, J are adjacent, since each of them is adjacent only to K and to two outer vertices. Hence, a triangle contains at most one of F, G, H, I, J, and it does not contain K, because the other two vertices would both be among the neighbors F, G, H, I, J of K. As in the Petersen graph, no three of the outer vertices A, B, C, D, E form a triangle. Finally, a triangle with two outer vertices and one of F, G, H, I, J is impossible: the two outer vertices would be adjacent, that is, consecutive on the outer cycle, while the two outer neighbors of each of F, G, H, I, J are not consecutive (for example, the outer neighbors of F are B and E).

Bipartite. None of the three graphs is bipartite, because each of them contains a cycle of odd length.[13] A cycle of odd length cannot be split into two sides: going around the cycle, the sides of consecutive vertices alternate, so after an odd number of steps the starting vertex is reached on the opposite side, which is a contradiction. The Petersen graph and the Grötzsch graph contain the cycle $A - B - C - D - E - A$ of length 5, and the FourTriangles graph contains the triangle $A - B - C - A$ of length 3.

In (b)–(d), find a disconnected graph with as many vertices as the given graph and the same answers in the other four rows.

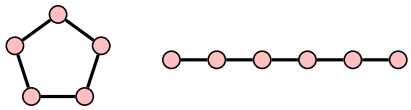
(b) The Petersen graph.

Answer: Two disjoint cycles with 5 vertices each (below). Like the Petersen graph, it has 10 vertices, it is not a tree (it has cycles), it is regular (all its degrees are 2), it is triangle-free (its only cycles have length 5), and it is not bipartite (it contains a cycle of odd length, which cannot be split into two sides, as explained in part (a)). However, it is disconnected.



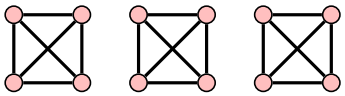
(c) The Grötzsch graph.

Answer: A cycle with 5 vertices and a path with 6 vertices (below). Like the Grötzsch graph, it has 11 vertices, it is not a tree, it is not regular (its degrees are 1 and 2), it is triangle-free, and it is not bipartite (it contains a cycle of odd length, which cannot be split into two sides, as explained in part (a)). However, it is disconnected.



(d) The FourTriangles graph.

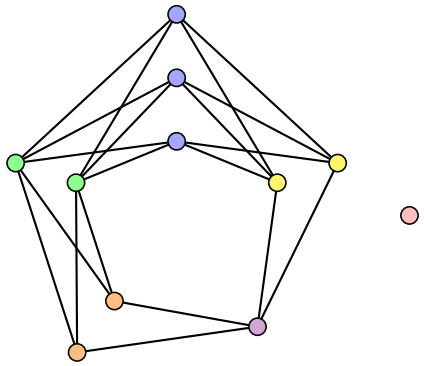
Answer: Three disjoint copies of the complete graph K_4 (below). Like the FourTriangles graph, it has 12 vertices, it is not a tree, it is regular (all its degrees are 3), it is not triangle-free, and it is not bipartite (it contains a cycle of odd length, which cannot be split into two sides, as explained in part (a)). However, it is disconnected. It even has $3 \cdot 6 = 18$ edges, as many as the FourTriangles graph.



[13] In the language of colorings, a cycle of odd length cannot be colored with two colors; see the solution of Problem 6.1(d) on page 106.

Remark: One can also require the same number of edges.

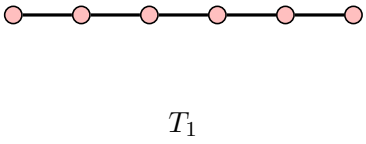
- **The Petersen graph:** this is impossible. Such a graph would have 10 vertices and 15 edges, and being regular, all its degrees would be $2 \cdot 15/10 = 3$. Each of its connected components would then have all its degrees equal to 3, and hence at least 4 vertices and an even number of vertices (the sum of the degrees in a component is twice its number of edges).[14] The only way to split 10 into at least two such numbers is $4 + 6$, and the only graph with 4 vertices in which all degrees are 3 is K_4, which has triangles.
- **The Grötzsch graph:** this is possible. Replace the vertices of a cycle with 5 vertices by groups of 3, 2, 2, 1, and 2 vertices, join every vertex of each group to every vertex of the two neighboring groups, and add a vertex with no neighbors. This graph has 11 vertices and $6 + 4 + 2 + 2 + 6 = 20$ edges, it is not regular, it is triangle-free, since no three groups are pairwise neighbors, and it is not bipartite, since it contains a cycle of length 5, which cannot be split into two sides, as explained in part (a). This graph is shown below; each group has its own color.



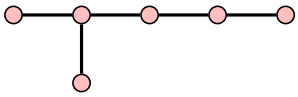
- **The FourTriangles graph:** the three copies of K_4 in part (d) already have 18 edges, as many as the FourTriangles graph.

[14] The handshake lemma is the subject of Problem 5.1 on page 18.

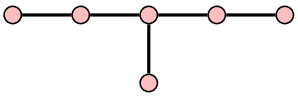
2.2. Below are all 6 trees with 6 vertices, up to isomorphism.



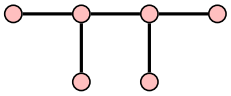
T_1



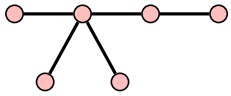
T_2



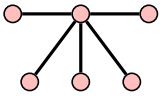
T_3



T_4



T_5



T_6

(a) For each tree, write in the table its **diameter** (in a tree, the number of edges on a longest path), its number of leaves (vertices of degree 1), its maximum degree, and its **sorted degree sequence** (its degrees in non-increasing order).

Answer:

	diameter	number of leaves	maximum degree	sorted degree sequence
T_1	**5**	**2**	**2**	$(\mathbf{2} \geq \mathbf{2} \geq \mathbf{2} \geq \mathbf{2} \geq \mathbf{1} \geq \mathbf{1})$
T_2	**4**	**3**	**3**	$(\mathbf{3} \geq \mathbf{2} \geq \mathbf{2} \geq \mathbf{1} \geq \mathbf{1} \geq \mathbf{1})$
T_3	**4**	**3**	**3**	$(\mathbf{3} \geq \mathbf{2} \geq \mathbf{2} \geq \mathbf{1} \geq \mathbf{1} \geq \mathbf{1})$
T_4	**3**	**4**	**3**	$(\mathbf{3} \geq \mathbf{3} \geq \mathbf{1} \geq \mathbf{1} \geq \mathbf{1} \geq \mathbf{1})$
T_5	**3**	**4**	**4**	$(\mathbf{4} \geq \mathbf{2} \geq \mathbf{1} \geq \mathbf{1} \geq \mathbf{1} \geq \mathbf{1})$
T_6	**2**	**5**	**5**	$(\mathbf{5} \geq \mathbf{1} \geq \mathbf{1} \geq \mathbf{1} \geq \mathbf{1} \geq \mathbf{1})$

Justifications:

- **Diameter:** In each drawing, a longest path is drawn horizontally, and the diameter is its number of edges. Indeed, in a tree there is exactly one path between any two vertices, so it suffices to check that no two vertices are farther apart than the two ends of the horizontal path. Let the horizontal path be $p_0, p_1, \ldots, p_L$. Every vertex drawn below it is a leaf attached to an inner vertex p_i with $1 \leq i \leq L-1$. Such a leaf is at distance at most $\max(i, L-i)+1 \leq L$ from every vertex of the horizontal path. Two leaves attached to p_i and p_j with $i \leq j$ are at distance $2 \leq L$ if $i = j$, and $(j-i)+2 \leq L$ otherwise, since $j - i \leq L-2$.
- **Leaves:** The leaves are the two ends of the horizontal path and all the vertices drawn below it.
- **Maximum degree:** The maximum degree is the largest number of edges touching a single vertex in the drawing. In T_1, every vertex has degree at most 2, while in T_6, the middle vertex of the horizontal path is adjacent to all the other five vertices.
- **Sorted degree sequence:** Every leaf has degree 1, and every vertex of the horizontal path that is not an end has degree 2 plus the number of leaves hanging below it. As a check, the number of ones in each sequence equals the number of leaves in the table, the first entry equals the maximum degree, and every sequence sums to 10: a tree with 6 vertices has 5 edges, and each edge contributes 1 to the degrees of both of its endpoints.

(b) Which two of the trees have the same values in all four columns of the table? Explain why they are not isomorphic.

Answer: The trees T_2 and T_3. Both have diameter 4, 3 leaves, maximum degree 3, and the sorted degree sequence $(3, 2, 2, 1, 1, 1)$. Nevertheless, they are not isomorphic.[15] An isomorphism maps the only vertex of degree 3 (red below) of one tree to the only vertex of degree 3 of the other, and it maps leaves to leaves. In T_2 (left), the vertex of degree 3 is adjacent to two leaves, while in T_3 (right), it is adjacent to only one leaf. Hence, these four properties do not determine a tree up to isomorphism.



(c) In the table, the number of leaves of each tree is at least its maximum degree. Prove that this holds for every tree with at least 2 vertices.

Proof: First, every tree with at least 2 vertices has at least 2 leaves. Indeed, consider a longest path $v_0, v_1, \ldots, v_k$ in the tree, where $k \geq 1$. Every neighbor of v_0 other than v_1 would either extend the path, if it is not on the path, or close a cycle, if it is on the path. Both are impossible, so v_0 is a leaf, and similarly v_k is a leaf.

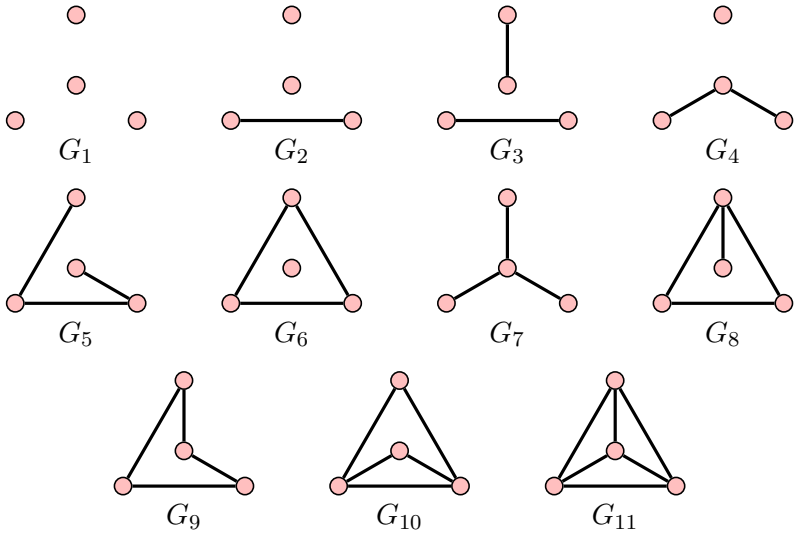
Now let T be a tree with at least 2 vertices, and let v be a vertex of maximum degree Δ, with neighbors $u_1, u_2, \ldots, u_\Delta$. Removing v and its edges splits T into Δ trees $B_1, B_2, \ldots, B_\Delta$, where B_i contains u_i. Indeed, every other vertex is joined to v by a path whose last edge is (u_i, v) for some i, so it belongs to the part of u_i, and two different vertices u_i and u_j belong to different parts, since a path between them that avoids v, together with v, would close a cycle. A leaf of T can be found in each B_i:

- If B_i has only the vertex u_i, then u_i is a leaf of T, since its only neighbor in T is v.
- Otherwise, B_i has at least 2 vertices, so it has at least 2 leaves. At least one of them, call it w, is different from u_i. This is needed, since u_i may be a leaf of B_i while it is not a leaf of T: it is also adjacent to v. The vertex w is not adjacent to v, since u_i is the only neighbor of v in B_i. Hence, the degree of w in T equals its degree in B_i, which is 1, and w is a leaf of T.

The leaves found in different trees B_i are different, so T has at least Δ leaves. The bound is tight: the star with $n \geq 3$ vertices has maximum degree $n - 1$ and exactly $n - 1$ leaves, like T_6 in the table.

[15] These two trees also appear in the solution of Problem 1.6(d) on page 34.

2.3. Below are all 11 simple graphs with 4 vertices, up to isomorphism.



(a) For each graph, mark Yes (Y) or No (N) in the table to indicate whether it is connected, a forest (no cycles), a tree (a connected forest), regular (all degrees are equal), or bipartite (the vertices can be split into two sides so that every edge joins the two sides).

Answer:

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}
connected graphs	**No**	**No**	**No**	**No**	**Yes**	**No**	**Yes**	**Yes**	**Yes**	**Yes**	**Yes**
forests	**Yes**	**Yes**	**Yes**	**Yes**	**Yes**	**No**	**Yes**	**No**	**No**	**No**	**No**
trees	**No**	**No**	**No**	**No**	**Yes**	**No**	**Yes**	**No**	**No**	**No**	**No**
regular graphs	**Yes**	**No**	**Yes**	**No**	**No**	**No**	**No**	**No**	**Yes**	**No**	**Yes**
bipartite graphs	**Yes**	**Yes**	**Yes**	**Yes**	**Yes**	**No**	**Yes**	**No**	**Yes**	**No**	**No**

Justifications:

- **Connected:** A graph is connected if every two vertices are joined by a path. In G_1, G_2, G_4, and G_6, some vertex has no neighbors, and G_3 consists of two disjoint edges; hence, these five graphs are not connected. In each of the other six graphs, the drawing shows a path between every two vertices.
- **Forests:** A forest is a graph with no cycles. The graphs $G_1, \ldots, G_5$ and G_7 have no cycles. In contrast, G_6, G_8, G_{10}, and G_{11} contain a triangle, and G_9 is a cycle of length 4.
- **Trees:** A tree is a connected forest, so the trees are exactly the graphs marked both as connected and as forests: the path G_5 and the star G_7. Both have 3 edges, as every tree with 4 vertices must.
- **Regular:** A graph is regular if all the entries of its degree sequence are equal. The sorted degree sequences of $G_1, \ldots, G_{11}$ are $(0,0,0,0)$, $(1,1,0,0)$, $(1,1,1,1)$, $(2,1,1,0)$, $(2,2,1,1)$, $(2,2,2,0)$, $(3,1,1,1)$, $(3,2,2,1)$, $(2,2,2,2)$, $(3,3,2,2)$, and $(3,3,3,3)$, respectively. Hence, exactly G_1, G_3, G_9, and G_{11} are regular.
- **Bipartite:** A graph is bipartite if its vertices can be split into two sides so that every edge joins the two sides. In all the drawings, call the vertex in the middle of the triangle the inner vertex. For G_4 and G_7, put the inner vertex on one side and the other three vertices on the other side. For G_1, G_2, G_3, G_5, and G_9, put the inner vertex and the bottom-left vertex on one side, and the top and bottom-right vertices on the other side. In contrast, G_6, G_8, G_{10}, and G_{11} contain a triangle. In any split into two sides, two of the three vertices of a triangle are on the same side, and the edge between them does not join the two sides. Hence, these graphs are not bipartite.

(b) The **complement** of a simple graph G has the same vertices as G, and two different vertices are adjacent in it if and only if they are not adjacent in G. Pair each of the 11 graphs with its complement, up to isomorphism. Which of the graphs is isomorphic to its own complement?

Answer: The pairs are G_1 and G_{11}, G_2 and G_{10}, G_3 and G_9, G_4 and G_8, and G_6 and G_7; G_5 is isomorphic to its own complement.[16] Every pair of vertices is an edge of exactly one of a graph and its complement, so if a graph has k edges, its complement has $6 - k$ edges. Hence, G_1 (no edges) and G_{11} (all 6 edges) are complements of each other. The complement of a single edge (G_2) is the graph with all the edges except one (G_{10}). The complement of two disjoint edges (G_3) is the cycle G_9. The complement of the path $a - b - c$ plus a vertex d (G_4) has the edges (a, c), (a, d), (c, d), and (b, d): the triangle a, c, d with the extra edge (b, d), which is G_8. Among the graphs with 3 edges, the complement of the triangle plus a vertex with no neighbors (G_6) is the star G_7: the vertex with no neighbors becomes adjacent to all the others, and the vertices of the triangle become pairwise non-adjacent. Finally, the complement of the path $a - b - c - d$ (G_5) has the edges (a, c), (a, d), and (b, d), which form the path $c - a - d - b$, so G_5 is isomorphic to its own complement.

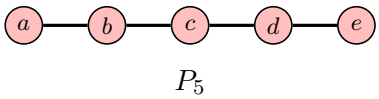
Alternatively, the pairs can be found from the degrees alone: the 11 graphs have 11 different degree sequences, and taking the complement replaces each degree d by $3 - d$, since every vertex has 3 possible neighbors. For example, G_4 has the degree sequence $(2, 1, 1, 0)$, so its complement has the degree sequence $(3, 2, 2, 1)$, which is that of G_8.

(c) Prove that the complement of a disconnected graph is connected.

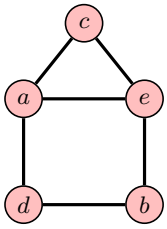
Proof: Let G be a disconnected graph, and let u and v be two different vertices. If u and v are in different connected components of G, then they are not adjacent in G, so they are adjacent in the complement. If u and v are in the same component of G, then, since G is disconnected, there is a vertex w in another component. In G, the vertex w is adjacent to neither u nor v, so in the complement, $u - w - v$ is a path. In both cases, u and v are joined by a path in the complement, so the complement is connected; in fact, every two of its vertices are at distance at most 2. In the table, the disconnected graphs G_1, G_2, G_3, G_4, and G_6 are paired with the connected graphs G_{11}, G_{10}, G_9, G_8, and G_7.

(d) Is the converse true, that is, is the complement of a connected graph always disconnected?

Answer: No. A connected graph may have a connected complement. In the table, the path G_5 is connected, and so is its complement, which is isomorphic to it. The two graphs need not be isomorphic: the path P_5, namely $a - b - c - d - e$ (below, left), is connected, and its complement (below, right) has the 6 edges (a, c), (a, d), (a, e), (b, d), (b, e), and (c, e). They form the cycle $a - c - e - b - d - a$ of length 5 and the extra edge (a, e), so the complement is connected as well. However, the two graphs are not isomorphic, since P_5 has 4 edges and its complement has 6 edges. Thus, a graph and its complement cannot both be disconnected, but they can both be connected.



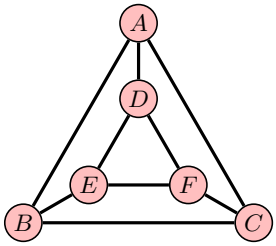
P_5



its complement

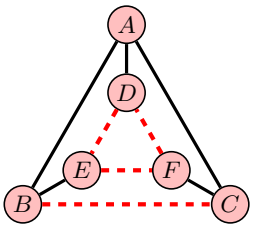
[16] Complements also appear in Problem 1.8 on page 3.

2.4. Consider the prism graph below. In each of the following parts, find the minimum number of edges whose removal leaves a graph with the required property (all the 6 vertices remain), mark such a set of edges, and explain why fewer edges do not suffice.



(a) The remaining graph is a tree.

Answer: 4 edges. A tree with 6 vertices has exactly 5 edges, so at least $9 - 5 = 4$ edges must be removed. Removing the 4 dashed edges (B, C), (D, E), (E, F), and (F, D) leaves the 5 edges (A, B), (A, C), (A, D), (B, E), and (C, F). The remaining graph is connected (every vertex is joined to A by a path of at most 2 edges) and has no cycle (the only vertex with more than two neighbors is A, and the three branches at A do not meet). Hence, it is a tree.



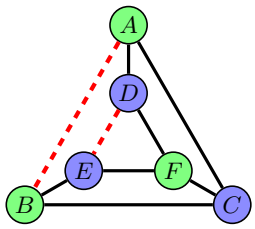
Counting the solutions. There are exactly 75 different sets of 4 edges whose removal leaves a tree. Equivalently, there are 75 ways to keep 5 of the 9 edges so that the remaining graph is a tree. A tree has no cycle, so at most 2 edges of each triangle are kept, and therefore at least one of the joining edges (A, D), (B, E), and (C, F) is kept. The count is by the number k of kept joining edges.

- $k = 1$: There are 3 choices for the kept joining edge. The other 4 kept edges are triangle edges, at most 2 in each triangle, so exactly 2 in each triangle, with 3 choices in each. Each triangle becomes a path, and the two paths are joined by the kept joining edge, which gives a tree. In total, $3 \cdot 3 \cdot 3 = 27$ trees.
- $k = 2$: There are 3 choices for the removed joining edge, say (C, F). Exactly 3 triangle edges are kept: 2 in one triangle and 1 in the other. If 2 outer edges are kept (3 choices), they form a path through A, B, C, and the joining edges (A, D) and (B, E) attach D and E to it. The single inner edge must attach F, so it is (D, F) or (E, F), and each of these 2 choices gives a tree. This gives $3 \cdot 2 = 6$ trees, and in the same way 6 more trees when 2 inner edges and 1 outer edge are kept. In total, $3 \cdot (6 + 6) = 36$ trees.
- $k = 3$: Exactly 2 triangle edges are kept. If both are in the same triangle (3 choices in each of the 2 triangles), this triangle becomes a path, and the three joining edges attach the other three vertices to it: 6 trees. If one is in each triangle ($3 \cdot 3 = 9$ choices), the remaining graph is not a tree when the two edges are "parallel", such as (A, B) and (D, E), which close the cycle $A - B - E - D - A$. Otherwise, the 5 kept edges form a path through all 6 vertices, such as $D - A - B - E - F - C$ for (A, B) and (E, F). This gives $9 - 3 = 6$ trees. In total, $6 + 6 = 12$ trees.

Altogether, there are $27+36+12 = 75$ solutions. The symmetries of the prism graph are the rotations and the reflections of the drawing, possibly combined with exchanging the outer and the inner triangle; there are 12 of them. Up to these symmetries, there are only 9 different solutions (the details are omitted).

(b) The remaining graph is bipartite.

Answer: 2 edges. A bipartite graph has no triangle: two of the three vertices of a triangle would be on the same side, joined by an edge. The prism graph has two triangles, A, B, C and D, E, F, with no common edge. Hence, at least one edge from each triangle must be removed, and at least 2 edges in total. Removing the two dashed edges (A, B) and (D, E) leaves a bipartite graph with the sides $\{A, B, F\}$ and $\{C, D, E\}$ (green and blue below): each of the remaining 7 edges (B, C), (C, A), (E, F), (F, D), (A, D), (B, E), and (C, F) joins the two sides.

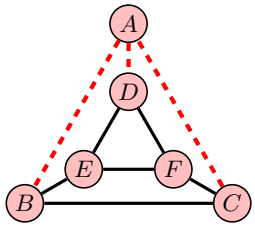


The choice matters: removing (A, B) and (E, F) breaks both triangles, but the remaining graph still contains the cycle $A - C - B - E - D - A$ of length 5, so it is not bipartite (the sides would alternate around this odd cycle), and by symmetry, the same holds for the other 5 non-"parallel" pairs. Only the three "parallel" pairs $\{(A, B), (D, E)\}$, $\{(B, C), (E, F)\}$, and $\{(C, A), (F, D)\}$ work.

Counting the solutions. There are exactly 3 solutions, the three "parallel" pairs, and up to the symmetries of the prism graph they are all the same: a rotation maps each of them to the next one.

(c) The remaining graph is disconnected.

Answer: 3 edges. Removing the 3 dashed edges of the vertex A disconnects A from the other vertices.



Removing any 2 edges leaves the graph connected. Call A, B, C the outer triangle, D, E, F the inner triangle, and (A, D), (B, E), (C, F) the joining edges. A triangle with one edge removed is still connected (it becomes a path).

- Both removed edges are in the outer triangle: the inner triangle and all the joining edges remain, and every outer vertex is joined to the connected inner triangle by its joining edge.
- Both removed edges are in the inner triangle: the same argument with the roles of the triangles exchanged.
- One removed edge is in each triangle: both triangles remain connected, and all three joining edges remain to connect them.
- At least one removed edge is a joining edge: at most one edge is removed from the two triangles, so both triangles remain connected, and at least one joining edge remains to connect them.

Counting the solutions. There are exactly 7 sets of 3 edges whose removal disconnects the graph: the 3 edges of one of the 6 vertices, and the 3 joining edges, whose removal separates the outer triangle from the inner triangle. Up to the symmetries of the prism graph, there are 2 solutions, and they leave non-isomorphic graphs: an isolated vertex and 5 connected vertices, or two triangles.

To see that there are no other solutions, note that a set of removed edges disconnects the graph only if it contains all the edges between some group S of vertices and the other vertices, where S is the smaller part, so it has 1, 2, or 3 vertices. Every vertex has degree 3. A single vertex has 3 such edges. Two vertices have at most 1 edge between them, so at least $3 + 3 - 2 \cdot 1 = 4$ such edges. Three vertices have at most 3 edges among them, so at least $3 \cdot 3 - 2 \cdot 3 = 3$ such edges, with equality only if S is a triangle, that is, the outer or the inner triangle.

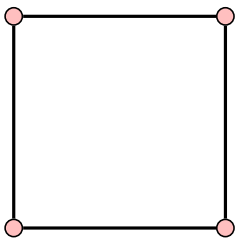
2.5. A **forest** is a graph with no cycles, and a **tree** is a connected forest. A graph is **bipartite** if its vertices can be split into two sides so that every edge joins the two sides, and it is **triangle-free** if it has no cycle of length 3, that is, no three pairwise adjacent vertices.

(a) Prove or disprove: (i) every tree is a forest; (ii) every forest is a tree.

Answer: (i) **True.** A tree is a connected forest, so in particular it is a forest. (ii) **False.** The graph with two vertices and no edges has no cycles, so it is a forest, but it is not connected, so it is not a tree. It is the smallest example, since the graph with one vertex is a tree.

(b) Prove or disprove: (i) every forest is bipartite; (ii) every bipartite graph is a forest.

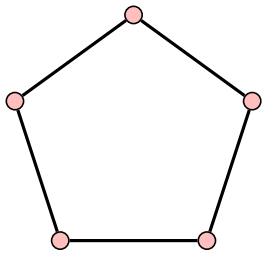
Answer: (i) **True.**[17] First, consider a tree. Pick any vertex r of the tree. Since a tree is connected and has no cycles, there is a unique simple path from r to every vertex w. Let $d(w)$ be the number of edges on this path. Let X contain all the vertices w for which $d(w)$ is even, and let Y contain all the vertices w for which $d(w)$ is odd. In particular, $r \in X$, since $d(r) = 0$. Consider an edge (u, v). If v is not on the path from r to u, then this path followed by the edge (u, v) is the path from r to v, and thus $d(v) = d(u) + 1$. Otherwise, v is the vertex just before u on this path, since otherwise the part of the path from v to u together with the edge (u, v) forms a cycle. Thus, $d(u) = d(v) + 1$. In both cases, one endpoint of the edge is in X and the other endpoint is in Y. Hence, there are no edges between two vertices from X or between two vertices from Y, and the tree is bipartite. Every connected component of a forest is a tree, so the same argument, applied to each component, shows that every forest is bipartite.
(ii) **False.** The cycle C_4 is bipartite: put two opposite vertices on one side and the other two vertices on the other side. It is connected, but it has a cycle, so it is neither a forest nor a tree. It is the smallest connected example. Indeed, a connected graph that is not a tree has a cycle (a connected graph with no cycles is a tree), and a bipartite graph has no triangle (two of the three vertices of a triangle would be on the same side, joined by an edge), so every cycle in a bipartite graph has at least 4 vertices.



C_4

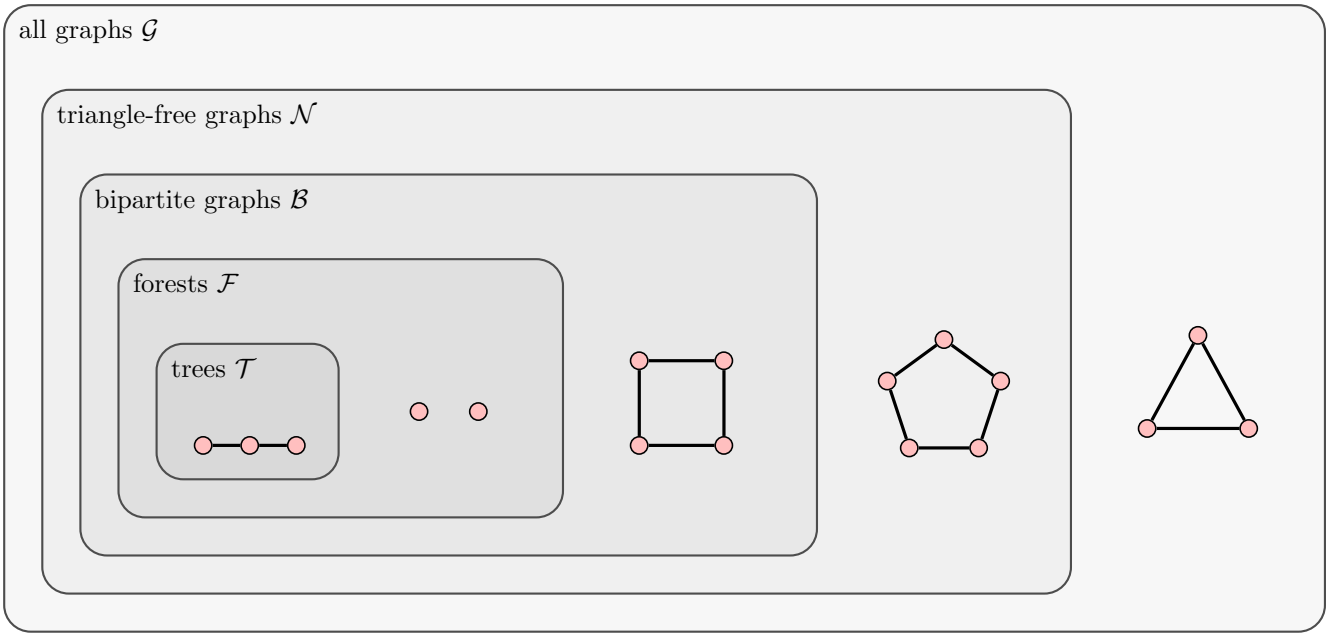
(c) Prove or disprove: (i) every bipartite graph is triangle-free; (ii) every triangle-free graph is bipartite.

Answer: (i) **True.** Assume that a bipartite graph contains a triangle. Its three vertices are split between the two sides, so two of them are on the same side. However, these two vertices are adjacent, while every edge joins the two sides, a contradiction.
(ii) **False.** The cycle C_5 with 5 vertices (below) is triangle-free, and it is not bipartite. Indeed, assume that its vertices are split into two sides so that every edge joins the two sides. Going around the cycle, the sides of consecutive vertices alternate, so after 5 steps the starting vertex is reached on the opposite side, a contradiction.



[17] Equivalently, a tree with at least 2 vertices can be colored with 2 colors; see the solution of Problem 6.1(b) on page 105.

Remark: Let $\mathcal{T}$, $\mathcal{F}$, $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{G}$ be the sets of all trees, all forests, all bipartite graphs, all triangle-free graphs, and all graphs, respectively. By the first claims of parts (a), (b), and (c), every tree is a forest, every forest is bipartite, and every bipartite graph is triangle-free. By the second claims, none of the converses holds. Finally, the triangle is not triangle-free. Hence, $\mathcal{T} \subsetneq \mathcal{F} \subsetneq \mathcal{B} \subsetneq \mathcal{N} \subsetneq \mathcal{G}$, where $\subsetneq$ means "is a subset of, but not equal to". The figure below shows the five sets, each with an example: the path with 3 vertices is a tree, the graph with two vertices and no edges is a forest but not a tree, the cycle C_4 is bipartite but not a forest, the cycle C_5 is triangle-free but not bipartite, and the triangle is not triangle-free.



2.6. The **line graph** $L(G)$ of a graph G has a vertex for every edge of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share an endpoint. Each vertex of $L(G)$ is named by the two endpoints of its edge. For example, below are a graph G (left) and its line graph $L(G)$ (right).



G $\qquad$ $L(G)$

For each of the following graphs G, find the number of vertices of $L(G)$, the degrees of its vertices, and its number of edges. If $L(G)$ is a well-known graph, name it.

(a) The matching M_k with $k \geq 1$ edges: $2k$ vertices and k edges, no two sharing an endpoint.

Answer: $L(M_k)$ is the graph with k vertices and no edges (called the **null graph** or the **empty graph**): every vertex has degree 0, since no two edges of M_k share an endpoint.

(b) The path P_n with $n \geq 2$ vertices.

Answer: $L(P_n)$ is the path P_{n-1}: it has $n-1$ vertices and $n-2$ edges, its two end vertices have degree 1, and its other vertices have degree 2 (for $n = 2$, it is a single vertex of degree 0). Indeed, let the path be $v_1, v_2, \ldots, v_n$, and let $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq n-1$. The edges e_i and e_j share an endpoint if and only if $j = i+1$ or $i = j+1$. Hence, $L(P_n)$ is the path $e_1, e_2, \ldots, e_{n-1}$.

(c) The cycle C_n with $n \geq 3$ vertices.

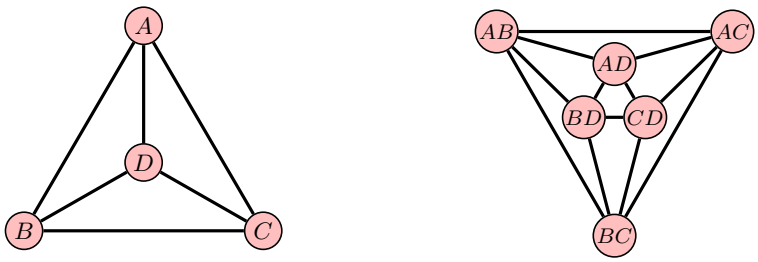
Answer: $L(C_n)$ is the cycle C_n itself: it has n vertices and n edges, and all its vertices have degree 2. Indeed, let the cycle be $v_1, v_2, \ldots, v_n, v_1$, let $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq n-1$, and let $e_n = (v_n, v_1)$. Every edge shares an endpoint exactly with the edge before it and the edge after it along the cycle, so $L(C_n)$ is the cycle $e_1, e_2, \ldots, e_n, e_1$.

(d) The star S_n with $n \geq 2$ vertices: a middle vertex adjacent to the other $n-1$ vertices, and no other edges.

Answer: $L(S_n)$ is the complete graph K_{n-1}: it has $n-1$ vertices, every vertex has degree $n-2$, and it has $(n-1)(n-2)/2$ edges. Indeed, every two of the $n-1$ edges of the star share the middle vertex.

(e) A Δ-regular graph G with n vertices, for some $\Delta \geq 1$: every vertex of G has degree Δ.

Answer: $L(G)$ has $n\Delta/2$ vertices, one for each edge of G, since the sum of the degrees of G, $n\Delta$, counts every edge twice, once for each of its endpoints (in particular, $n\Delta$ is even).[18] Every vertex of $L(G)$ has degree $2\Delta - 2$: the edge (u, v) shares the endpoint u with the other $\Delta - 1$ edges of u and the endpoint v with the other $\Delta - 1$ edges of v, and no edge is counted twice, since there is only one edge between u and v. Hence, $L(G)$ is $(2\Delta - 2)$-regular, and by the same counting it has $\frac{n\Delta}{2} \cdot \frac{2\Delta-2}{2} = n\Delta(\Delta-1)/2$ edges.
For example, the cycle C_n is 2-regular, and indeed $L(C_n)$ is 2-regular with n edges. Similarly, the matching M_k is 1-regular, and indeed $L(M_k)$ is 0-regular with no edges. The complete graph K_n is a special case of a regular graph: it is $(n-1)$-regular, so $L(K_n)$ has $n(n-1)/2$ vertices, it is $(2n-4)$-regular, and it has $n(n-1)(n-2)/2$ edges. For $n = 4$, $L(K_4)$ has 6 vertices of degree 4 and 12 edges (below). It is the **octahedron graph**: its vertices and edges are the corners and edges of an octahedron.

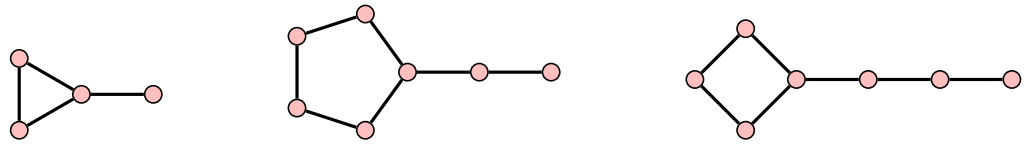


K_4 $\qquad$ $L(K_4)$

[18] The handshake lemma is the subject of Problem 5.1 on page 18.

2.7. A **balloon-graph** is a graph that is composed of a cycle and a path that share exactly 1 vertex.

- All the edges of the graph are either part of the cycle or part of the path.
- Exactly one vertex is shared by the cycle and the path, and it is an end of the path.
- The cycle has at least 3 vertices and therefore at least 2 vertices belong only to the cycle.
- The path has at least 2 vertices and therefore at least 1 vertex belongs only to the path.

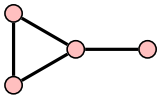


A balloon-graph must contain at least 4 vertices. Shown above are the only balloon-graph with 4 vertices and two of the four non-isomorphic balloon-graphs with 7 vertices. Of these two, one has a cycle with 5 vertices and a path with 3 vertices, and the other has a cycle with 4 vertices and a path with 4 vertices.

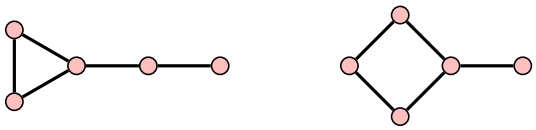
Remark: A balloon-graph must contain at least 4 vertices: the shared vertex, at least 2 additional vertices in the cycle, and at least 1 additional vertex in the path.

(a) Illustrate all non-isomorphic balloon-graphs with 4, 5, 6, and 7 vertices.

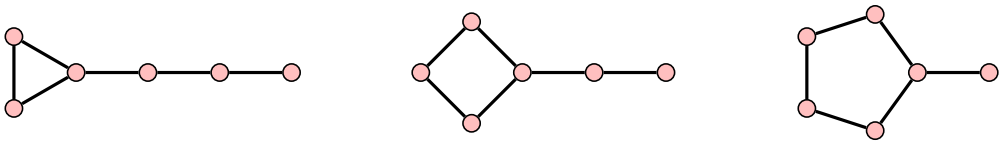
Answer: There are 1, 2, 3, and 4 such graphs, respectively (below).



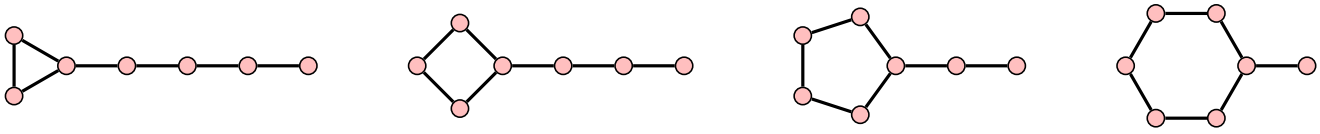
The only possible balloon-graph with 4 vertices.



The two non-isomorphic balloon-graphs with 5 vertices.



The three non-isomorphic balloon-graphs with 6 vertices.



The four non-isomorphic balloon-graphs with 7 vertices.

(b) For $n \geq 4$, how many non-isomorphic balloon-graphs with n vertices exist? Give your answer as a function of n. Justify your answer.

Answer: $n-3$. Denote by c the number of vertices in the cycle and by p the number of vertices in the path. Since the cycle and the path share exactly one vertex, $n = c+p-1$. Since n is fixed, c determines $p = n+1-c$. A cycle has at least 3 vertices, so $c \geq 3$, and the path has at least 2 vertices, so $p \geq 2$, that is, $c \leq n-1$.

Two balloon-graphs with n vertices and the same c are isomorphic. In each of them, let v be the shared vertex, number the cycle vertices $v = x_1, x_2, \ldots, x_c$ along the cycle, and number the path vertices $v = y_1, y_2, \ldots, y_p$ along the path. Mapping each vertex to the vertex with the same name in the other graph maps edges to edges.

Balloon-graphs with different values of c are not isomorphic, since the cycle is the only cycle in a balloon-graph and an isomorphism preserves the lengths of cycles. Indeed, removing any edge of the path disconnects the vertices beyond it from the rest of the graph, so no cycle uses an edge of the path.[19] Hence, every cycle uses only edges of the cycle, and the only such cycle is the cycle itself.

Alternatively, a balloon-graph has exactly one vertex of degree 1, the end of the path, and exactly one vertex of degree 3, the shared vertex. The distance between them is $p-1$, and an isomorphism maps them to the corresponding two vertices of the other graph and preserves distances. Hence, it preserves p, and therefore also $c = n+1-p$.

As a result, there are exactly $n-3$ non-isomorphic balloon-graphs with n vertices, one for each c with $3 \leq c \leq n-1$.

(c) For $n \geq 4$, all balloon-graphs with n vertices have the same number of edges. What is this number? Prove your answer.

Answer: $m = n$. Denote by c the number of vertices in the cycle and by p the number of vertices in the path. In the previous part, it was shown that $n = c+p-1$. A cycle with c vertices has c edges and a path with p vertices has $p-1$ edges. The cycle and the path have no common edge. Hence, the number of edges m of a balloon-graph with n vertices is $c+p-1$. That is, $m = n$ in a balloon-graph.

(d) For $n \geq 4$, all balloon-graphs with n vertices have the same number of vertices of each degree. Let G be a balloon-graph with n vertices. How many vertices in G have degree 0? How many vertices in G have degree 1? How many vertices in G have degree 2? How many vertices in G have degree 3? How many vertices in G have degree greater than 3? Justify your answers.

Answer: No vertex has degree 0 or more than 3, one vertex has degree 1, $n-2$ vertices have degree 2, and one vertex has degree 3. Denote by c the number of vertices in the cycle and by p the number of vertices in the path. Let v be the vertex shared by the cycle and the path.

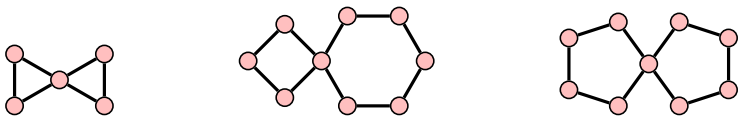
i. The degree of v is 3.
ii. Each of the other $c-1$ vertices in the cycle has degree 2.
iii. The degree of the vertex at the other end of the path is 1.
iv. Each of the $p-2$ inner vertices of the path has degree 2.

It follows that a balloon-graph with n vertices has zero vertices with degree 0, one vertex with degree 1, $n-2 = (c-1)+(p-2)$ vertices with degree 2, and one vertex with degree 3. No vertex has degree greater than 3.

[19] An edge whose removal disconnects the graph belongs to no cycle; see the solution of Problem 4.6(c) on page 85.

2.8. An **eight-graph** is a graph that is composed of two cycles that share exactly 1 vertex.

- All the edges of the graph are part of one of the two cycles.
- Exactly one vertex is shared by both cycles.
- Every other vertex belongs to exactly one of the cycles.
- Each cycle has at least 3 vertices, including the shared vertex.

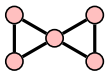


An eight-graph must contain at least 5 vertices. Shown above are the only eight-graph with 5 vertices and two of the three non-isomorphic eight-graphs with 9 vertices. Of these two, one has a cycle with 4 vertices and a cycle with 6 vertices, and the other has two cycles with 5 vertices each.

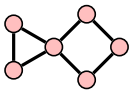
Remark: An eight-graph must contain at least 5 vertices: the shared vertex and at least 2 additional vertices in each cycle.

(a) Illustrate all non-isomorphic eight-graphs with 5, 6, 7, 8, and 9 vertices.

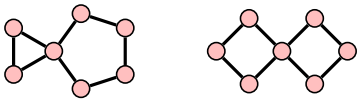
Answer: There are 1, 1, 2, 2, and 3 such graphs, respectively (below).



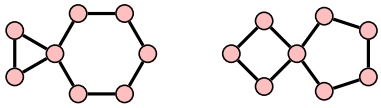
The only possible eight-graph with 5 vertices.



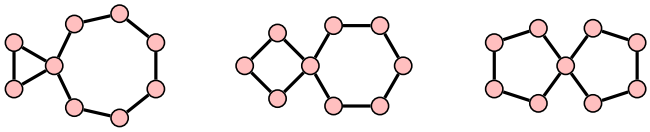
The only possible eight-graph with 6 vertices.



The two non-isomorphic eight-graphs with 7 vertices.



The two non-isomorphic eight-graphs with 8 vertices.



The three non-isomorphic eight-graphs with 9 vertices.

(b) For $n \geq 5$, how many non-isomorphic eight-graphs with n vertices exist? Give your answer as a function of n. Justify your answer.

Hint: The number is the same for the odd number $n = 2k - 1$ and the even number $n = 2k$.

Answer: $\lfloor (n-3)/2 \rfloor$. Denote by $c_1 \leq c_2$ the numbers of vertices in the two cycles. Since the two cycles share exactly one vertex, $n = c_1 + c_2 - 1$, that is, $c_1 + c_2 = n + 1$. Since n is fixed, c_1 determines $c_2 = n + 1 - c_1$. A cycle has at least 3 vertices, so $c_1 \geq 3$, and $c_1 \leq c_2$ means $2c_1 \leq n + 1$, that is, $c_1 \leq \lfloor (n+1)/2 \rfloor$ (and then $c_2 \geq c_1 \geq 3$ holds as well). Hence, the possible values are $c_1 = 3, 4, \ldots, \lfloor (n+1)/2 \rfloor$.
Two eight-graphs with the same pair of cycle lengths are isomorphic: in each of them, number the vertices of each cycle along the cycle, starting from the shared vertex, and map each vertex to the vertex with the same number in the corresponding cycle of the other graph. Eight-graphs with different pairs of cycle lengths are not isomorphic, since the two cycles are the only cycles of an eight-graph and an isomorphism preserves the lengths of cycles. Indeed, a cycle that contains vertices of both cycles other than the shared vertex v must pass through v twice, because every path between such vertices passes through v. Hence, every cycle lies within one of the two cycles, and it is that cycle itself.
It follows that the number of non-isomorphic eight-graphs with n vertices is

$$\left\lfloor \frac{n+1}{2} \right\rfloor - 2 = \left\lfloor \frac{n-3}{2} \right\rfloor .$$

For $k = \lceil n/2 \rceil$, this number is $k - 2$, which is the same for $n = 2k - 1$ and $n = 2k$.

Examples: For $n = 5$ there is $\lfloor (5-3)/2 \rfloor = 1$ non-isomorphic eight-graph, for $n = 6$ there is $\lfloor (6-3)/2 \rfloor = 1$ non-isomorphic eight-graph, for $n = 7$ there are $\lfloor (7-3)/2 \rfloor = 2$ non-isomorphic eight-graphs, for $n = 8$ there are $\lfloor (8-3)/2 \rfloor = 2$ non-isomorphic eight-graphs, and for $n = 9$ there are $\lfloor (9-3)/2 \rfloor = 3$ non-isomorphic eight-graphs.

(c) For $n \geq 5$, all eight-graphs with n vertices have the same number of edges. What is this number? Prove your answer.

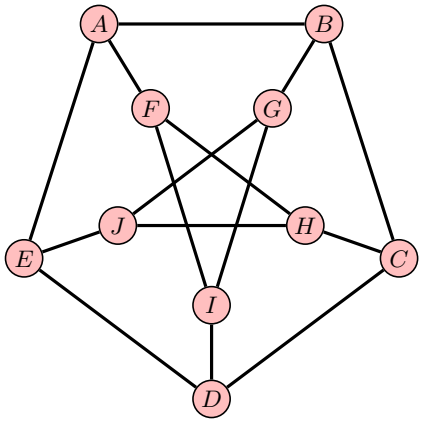
Answer: $m = n + 1$. Denote by c_1 and c_2 the numbers of vertices in the two cycles. In the previous part, it was shown that $n = c_1 + c_2 - 1$. A cycle with c vertices has c edges. The two cycles have no common edge. Hence, the number of edges m of an eight-graph with n vertices is $c_1 + c_2$. That is, $m = n + 1$ in an eight-graph.

(d) For $n \geq 5$, all eight-graphs with n vertices have the same number of vertices of each degree. Let G be an eight-graph with n vertices. How many vertices in G have degree 0? How many vertices in G have degree 1? How many vertices in G have degree 2? How many vertices in G have degree 3? How many vertices in G have degree 4? How many vertices in G have degree greater than 4? Justify your answers.

Answer: An eight-graph with n vertices has zero vertices with degree 0 or 1, $n - 1$ vertices with degree 2, zero vertices with degree 3, and one vertex with degree 4. No vertex has degree greater than 4.
Indeed, let v be the vertex shared by the two cycles. Then v has two neighbors in each cycle, so its degree is 4, and each of the other $n - 1$ vertices has two neighbors in its cycle, so its degree is 2.

3 Representing Graphs

3.1. Consider the Petersen graph below. Use the vertex order $A, B, \ldots, J$ for the rows and columns of the matrix and for the lists.



(a) Construct the adjacency matrix of this graph.

Answer:

	A	B	C	D	E	F	G	H	I	J
A	0	1	0	0	1	1	0	0	0	0
B	1	0	1	0	0	0	1	0	0	0
C	0	1	0	1	0	0	0	1	0	0
D	0	0	1	0	1	0	0	0	1	0
E	1	0	0	1	0	0	0	0	0	1
F	1	0	0	0	0	0	0	1	1	0
G	0	1	0	0	0	0	0	0	1	1
H	0	0	1	0	0	1	0	0	0	1
I	0	0	0	1	0	1	1	0	0	0
J	0	0	0	0	1	0	1	1	0	0

The matrix is symmetric with respect to its main diagonal (from top-left to bottom-right), since the graph is undirected. The main diagonal contains only zeros, since the graph has no self-loops. Each row and each column contains exactly three ones, since the Petersen graph is 3-regular.

(b) Construct the adjacency lists of this graph.

Answer:

$$
\begin{array}{lcl}
A & \to & (B, E, F) \\
B & \to & (A, C, G) \\
C & \to & (B, D, H) \\
D & \to & (C, E, I) \\
E & \to & (A, D, J) \\
F & \to & (A, H, I) \\
G & \to & (B, I, J) \\
H & \to & (C, F, J) \\
I & \to & (D, F, G) \\
J & \to & (E, G, H)
\end{array}
$$

Since the Petersen graph is 3-regular, each adjacency list has length 3.

(c) Explain how to find the common neighbors of two vertices u and v using the adjacency matrix, and using the adjacency lists. What is the complexity of each method?

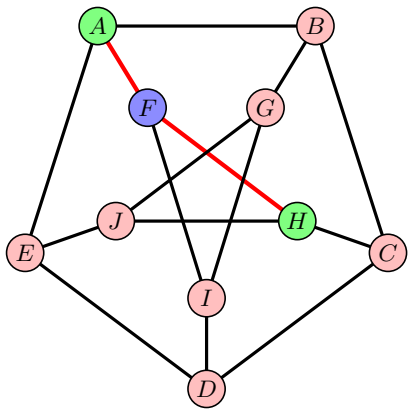
Answer: With the adjacency matrix, scan the rows of u and v together: a vertex w is a common neighbor of u and v if and only if both rows have a 1 in column w. This takes $\Theta(n)$ time for a graph with n vertices. With the adjacency lists, use an array of n marks, all of them initially off. Turn on the marks of the neighbors of u, scan the list of v and report every vertex whose mark is on, and finally turn off the marks of the neighbors of u. This takes $O(d(u) + d(v))$ time, where $d(x)$ is the degree of x, and the array is initialized only once. In the Petersen graph, every degree is 3, so the lists method reads 6 list entries, while the matrix method reads two rows of 10 entries each.

(d) Show that in the Petersen graph, every two adjacent vertices have no common neighbor, and every two non-adjacent vertices have exactly one common neighbor.

Proof: The table below gives, for every two different vertices, the number of their common neighbors, found in this way. The entry is 0 for each of the 15 adjacent pairs (the entries that are 1 in the adjacency matrix), and 1 for each of the 30 non-adjacent pairs.

	A	B	C	D	E	F	G	H	I	J
A	–	0	1	1	0	0	1	1	1	1
B	0	–	0	1	1	1	0	1	1	1
C	1	0	–	0	1	1	1	0	1	1
D	1	1	0	–	0	1	1	1	0	1
E	0	1	1	0	–	1	1	1	1	0
F	0	1	1	1	1	–	1	0	0	1
G	1	0	1	1	1	1	–	1	0	0
H	1	1	0	1	1	0	1	–	1	0
I	1	1	1	0	1	0	0	1	–	1
J	1	1	1	1	0	1	0	0	1	–

For example, in the figure below, the non-adjacent vertices A and H (green) have exactly one common neighbor, F (blue). As a check, every vertex is a common neighbor of the 3 pairs of its own neighbors, so there are $10 \cdot 3 = 30$ such pairs in total, exactly the number of non-adjacent pairs, $45 - 15 = 30$. Since two adjacent vertices have no common neighbor, the Petersen graph has no triangle. Also, every two non-adjacent vertices are at distance 2, so the Petersen graph has diameter 2 (the largest distance between two of its vertices). Off the diagonal, the table is the matrix P^2, the product $P \cdot P$ of the adjacency matrix P of part (a) with itself: the entry of P^2 in row u and column v is the sum of $P(u, w) \cdot P(w, v)$ over all vertices w, so it counts the vertices w with $P(u, w) = P(w, v) = 1$, that is, the common neighbors of u and v.[20]



[20]The matrix P^2 is also used to count triangles in the solution of Problem 3.4(c) on page 63.

(e) Use part (d) to show that the Petersen graph has no cycle of length 3 or 4, and conclude that its shortest cycle has length 5.

Proof: A cycle of length 3 is a triangle, and by part (d), two adjacent vertices have no common neighbor, so there is no triangle. Assume to the contrary that $a - b - c - d - a$ is a cycle of length 4. The vertices a and c are not adjacent, since otherwise a, b, c would form a triangle. However, both b and d are common neighbors of a and c, contradicting part (d), by which two non-adjacent vertices have exactly one common neighbor. Hence, the Petersen graph has no cycle of length 3 or 4.[21] The outer cycle $A - B - C - D - E - A$ has length 5, and every cycle of a simple graph has length at least 3, so the shortest cycle of the Petersen graph has length 5.

Remark: The Petersen graph has exactly 12 cycles of length 5. The five spokes (A, F), (B, G), (C, H), (D, I), and (E, J) are the only edges between the outer cycle and the inner cycle $F - H - J - G - I - F$, so a cycle uses an even number of spokes. A cycle of length 5 with no spoke is the outer cycle or the inner cycle. It cannot use four spokes, since it would need at least four more edges. If it uses exactly two spokes, from the outer vertices x and y, then it is completed by a path on the outer cycle between x and y and a path on the inner cycle between their inner neighbors, of total length 3. The distances between these two pairs are 1 and 2, in some order, so both paths are shortest paths, and each of them is unique. Hence, each of the 10 pairs of spokes gives exactly one cycle, and there are $2 + 10 = 12$ cycles of length 5:

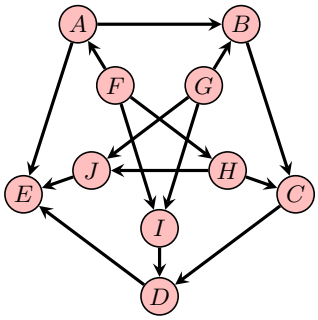
no spokes	two spokes, outer ends adjacent	two spokes, outer ends at distance 2
$A - B - C - D - E - A$	$A - B - G - I - F - A$	$A - B - C - H - F - A$
$F - H - J - G - I - F$	$B - C - H - J - G - B$	$B - C - D - I - G - B$
	$C - D - I - F - H - C$	$C - D - E - J - H - C$
	$D - E - J - G - I - D$	$D - E - A - F - I - D$
	$E - A - F - H - J - E$	$E - A - B - G - J - E$

As a check, every edge lies in exactly 4 of the cycles, and $12 \cdot 5 = 60 = 15 \cdot 4$.

[21] See also the solutions of Problems 1.1(b) and 2.1(a) on pages 28 and 38.

3.2. Consider the directed graph below, obtained by orienting the edges of the Petersen graph. Use the vertex order $A, B, \ldots, J$ for the rows and columns of the matrix and for the lists.

In the adjacency matrix, the entry in row u and column v is 1 if and only if the edge $u \to v$ exists. The outgoing list of a vertex v contains the vertices u with an edge $v \to u$, and the incoming list of v contains the vertices u with an edge $u \to v$.



(a) Construct the adjacency matrix of this directed graph.

Answer:

	A	B	C	D	E	F	G	H	I	J
A	0	1	0	0	1	0	0	0	0	0
B	0	0	1	0	0	0	0	0	0	0
C	0	0	0	1	0	0	0	0	0	0
D	0	0	0	0	1	0	0	0	0	0
E	0	0	0	0	0	0	0	0	0	0
F	1	0	0	0	0	0	0	1	1	0
G	0	1	0	0	0	0	0	0	1	1
H	0	0	1	0	0	0	0	0	0	1
I	0	0	0	1	0	0	0	0	0	0
J	0	0	0	0	1	0	0	0	0	0

Ignoring the directions of the edges gives the Petersen graph, and each of its edges appears in exactly one direction. Since the Petersen graph is 3-regular, for each vertex, the number of ones in its row plus the number of ones in its column is 3.

(b) Construct the outgoing and incoming adjacency lists of this directed graph.

Answer:

Outgoing	**Incoming**
$A \to (B, E)$	$(F) \to A$
$B \to (C)$	$(A, G) \to B$
$C \to (D)$	$(B, H) \to C$
$D \to (E)$	$(C, I) \to D$
$E \to ()$	$(A, D, J) \to E$
$F \to (A, H, I)$	$() \to F$
$G \to (B, I, J)$	$() \to G$
$H \to (C, J)$	$(F) \to H$
$I \to (D)$	$(F, G) \to I$
$J \to (E)$	$(G, H) \to J$

On the left, $v \to (\ldots)$ lists the vertices u with an edge $v \to u$. On the right, $(\ldots) \to v$ lists the vertices u with an edge $u \to v$. Similarly, for each vertex, the length of its outgoing list plus the length of its incoming list is 3.

(c) Show that the sum of each row of the adjacency matrix is the **out-degree** of its vertex (the number of edges that leave it), and the sum of each column is the **in-degree** of its vertex (the number of edges that enter it). Conclude that the sum of the out-degrees equals the sum of the in-degrees, and find both sums for this directed graph.

Answer: The entry in row u and column v is 1 if and only if the edge $u \to v$ exists. Hence, the number of ones in row u is the number of edges that leave u, its out-degree, and the number of ones in column v is the number of edges that enter v, its in-degree. The sum of all the entries of the matrix is the number of edges m. Summing it row by row gives the sum of the out-degrees, and summing it column by column gives the sum of the in-degrees. Hence, both sums are equal to m. For this directed graph, both sums are 15, the number of edges of the Petersen graph, as the table below shows. Note that E has no outgoing edges, and F and G have no incoming edges.

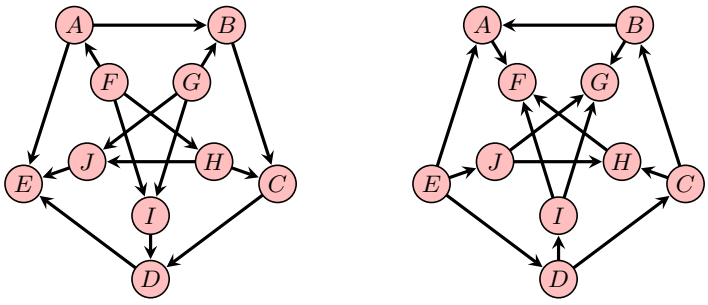
vertex	A	B	C	D	E	F	G	H	I	J	sum
out-degree	2	1	1	1	0	3	3	2	1	1	15
in-degree	1	2	2	2	3	0	0	1	2	2	15

(d) Design an algorithm that constructs the incoming lists of a directed graph with n vertices and m edges from its outgoing lists in $O(n + m)$ time.

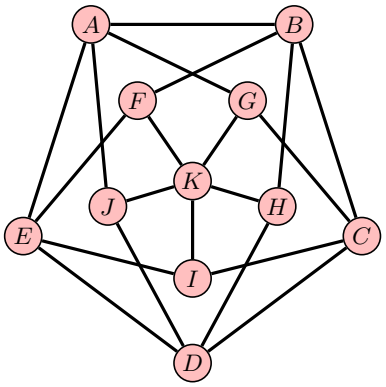
Answer: The algorithm starts with n empty incoming lists. It scans the vertices u in order, and for every vertex v in the outgoing list of u, it appends u to the incoming list of v. Every edge $u \to v$ is handled exactly once, and it contributes u to the incoming list of v, so the incoming lists are correct. The algorithm takes $O(n)$ time to create the lists and $O(1)$ time per edge, a total of $O(n + m)$. Since the vertices u are scanned in order, every incoming list is created in alphabetical order, as in part (b).

(e) The **reverse** of a directed graph is obtained by reversing the direction of every edge. Explain why the adjacency matrix of the reverse is the transpose of the adjacency matrix (rows and columns exchanged). How can the outgoing and incoming lists of the reverse be obtained from the outgoing and incoming lists of the original directed graph?

Answer: Reversing the edge $u \to v$ gives the edge $v \to u$. Hence, the entry in row u and column v of the original matrix equals the entry in row v and column u of the matrix of the reverse, so the two matrices are transposes of each other. For the lists, no computation is needed: an edge leaves v in the reverse if and only if it enters v in the original graph, so the outgoing lists of the reverse are the incoming lists of the original graph, and the incoming lists of the reverse are the outgoing lists of the original graph. For example, the outgoing list of A in the reverse is (F), and its incoming list is (B, E). Below are the directed graph (left) and its reverse (right).



3.3. Consider the Grötzsch graph below, with the 11 vertices $A, B, C, D, E, F, G, H, I, J, K$. Use the vertex order $A, B, \ldots, K$ for the rows and columns of the matrix and for the lists.



(a) Construct the adjacency matrix of the Grötzsch graph.

Answer:

	A	B	C	D	E	F	G	H	I	J	K
A	0	1	0	0	1	0	1	0	0	1	0
B	1	0	1	0	0	1	0	1	0	0	0
C	0	1	0	1	0	0	1	0	1	0	0
D	0	0	1	0	1	0	0	1	0	1	0
E	1	0	0	1	0	1	0	0	1	0	0
F	0	1	0	0	1	0	0	0	0	0	1
G	1	0	1	0	0	0	0	0	0	0	1
H	0	1	0	1	0	0	0	0	0	0	1
I	0	0	1	0	1	0	0	0	0	0	1
J	1	0	0	1	0	0	0	0	0	0	1
K	0	0	0	0	0	1	1	1	1	1	0

The matrix is symmetric with respect to its main diagonal, since the graph is undirected, and the main diagonal contains only zeros, since the graph has no self-loops. The number of ones in each row is the degree of its vertex: four in the rows of A, B, C, D, E, three in the rows of F, G, H, I, J, and five in the row of K. Altogether, there are $5 \cdot 4 + 5 \cdot 3 + 5 = 40$ ones, twice the number of edges, 20.[22]

(b) Construct the adjacency lists of the Grötzsch graph.

Answer:

$$
\begin{array}{lcl}
A & \to & (B, E, G, J) \\
B & \to & (A, C, F, H) \\
C & \to & (B, D, G, I) \\
D & \to & (C, E, H, J) \\
E & \to & (A, D, F, I) \\
F & \to & (B, E, K) \\
G & \to & (A, C, K) \\
H & \to & (B, D, K) \\
I & \to & (C, E, K) \\
J & \to & (A, D, K) \\
K & \to & (F, G, H, I, J)
\end{array}
$$

The length of each list is the degree of its vertex.

[22] The handshake lemma is the subject of Problem 5.1 on page 18.

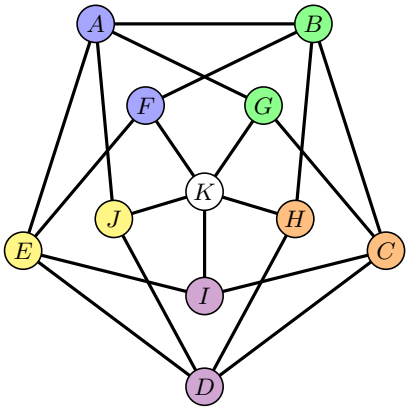
(c) Compare the two representations of the Grötzsch graph. How many entries does each of them use? How much time does each of them need to decide whether two given vertices are adjacent, and to list all the neighbors of a given vertex?

Answer: The adjacency matrix has $11 \cdot 11 = 121$ entries. The adjacency lists have 40 entries in total, the sum of the degrees: 5 vertices of degree 4, 5 vertices of degree 3, and K of degree 5, so $5 \cdot 4 + 5 \cdot 3 + 5 = 40 = 2 \cdot 20$. To decide whether u and v are adjacent, the matrix needs one entry, $O(1)$ time, while the lists need a scan of the list of u, up to 5 entries here. To list the neighbors of v, the matrix needs a scan of the whole row of v, 11 entries, while the lists give them directly, in $d(v)$ steps, where $d(v)$ is the degree of v.

In general, the matrix uses $\Theta(n^2)$ space, tests adjacency in $O(1)$ time, and lists the neighbors of a vertex in $\Theta(n)$ time. The lists use $\Theta(n+m)$ space, test adjacency in $O(d(u))$ time, and list the neighbors of v in $\Theta(d(v))$ time.

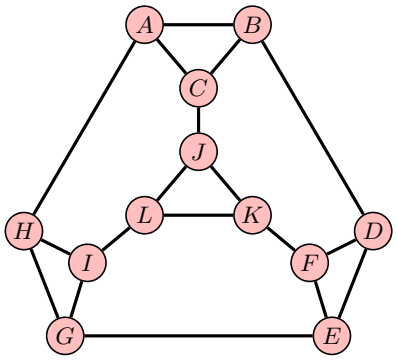
(d) In the adjacency matrix, compare the row of F with the row of A in the columns A, B, C, D, E. For each of G, H, I, J, find a vertex among A, B, C, D, E with the same property. What does this say about the structure of the Grötzsch graph?

Answer: In the columns A, B, C, D, E, the row of F and the row of A have ones exactly in the columns B and E: the vertices F and A have the same neighbors on the outer cycle. In the same way, G and B have the neighbors A and C, H and C have the neighbors B and D, I and D have the neighbors C and E, and J and E have the neighbors D and A. In the figure below, each such pair has its own color. Hence, each of F, G, H, I, J is a **shadow** of a vertex of the outer cycle: it is adjacent to the two neighbors of that vertex on the cycle, but not to the vertex itself. Finally, the row of K has ones exactly in the columns F, G, H, I, J. Thus, the Grötzsch graph is built from the cycle C_5 by adding a shadow for each of its vertices, and one more vertex K that is adjacent to all the shadows. This is known as **Mycielski's construction**.[23]



[23] See the solution of Problem 6.4 on page 109, where the shadows show that the Grötzsch graph cannot be colored with 3 colors.

3.4. Consider the FourTriangles graph below, with the 12 vertices $A, B, C, D, E, F, G, H, I, J, K, L$. Use the vertex order $A, B, \ldots, L$ for the rows and columns of the matrix and for the lists.



(a) Construct the adjacency matrix of the FourTriangles graph.

Answer:

	A	B	C	D	E	F	G	H	I	J	K	L
A	0	1	1	0	0	0	0	1	0	0	0	0
B	1	0	1	1	0	0	0	0	0	0	0	0
C	1	1	0	0	0	0	0	0	0	1	0	0
D	0	1	0	0	1	1	0	0	0	0	0	0
E	0	0	0	1	0	1	1	0	0	0	0	0
F	0	0	0	1	1	0	0	0	0	0	1	0
G	0	0	0	0	1	0	0	1	1	0	0	0
H	1	0	0	0	0	0	1	0	1	0	0	0
I	0	0	0	0	0	0	1	1	0	0	0	1
J	0	0	1	0	0	0	0	0	0	0	1	1
K	0	0	0	0	0	1	0	0	0	1	0	1
L	0	0	0	0	0	0	0	0	1	1	1	0

The matrix is symmetric with respect to its main diagonal, since the graph is undirected, and the main diagonal contains only zeros, since the graph has no self-loops. Each row contains exactly three ones, since the FourTriangles graph is 3-regular. Altogether, there are $12 \cdot 3 = 36$ ones, twice the number of edges, 18.

(b) Construct the adjacency lists of the FourTriangles graph.

Answer:

$$
\begin{array}{lcl}
A & \to & (B, C, H) \\
B & \to & (A, C, D) \\
C & \to & (A, B, J) \\
D & \to & (B, E, F) \\
E & \to & (D, F, G) \\
F & \to & (D, E, K) \\
G & \to & (E, H, I) \\
H & \to & (A, G, I) \\
I & \to & (G, H, L) \\
J & \to & (C, K, L) \\
K & \to & (F, J, L) \\
L & \to & (I, J, K)
\end{array}
$$

Since the FourTriangles graph is 3-regular, each adjacency list has length 3.

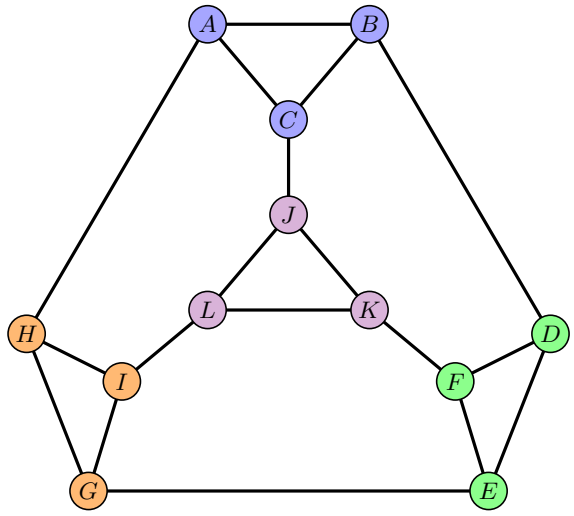
(c) Explain how the four triangles of the FourTriangles graph can be seen in its adjacency matrix. Design an algorithm that counts the triangles in a graph represented by an adjacency matrix. What is the complexity of your algorithm? How many triangles does the FourTriangles graph have?

Answer: In the order $A, B, \ldots, L$, the three vertices of each triangle are consecutive: A, B, C, then D, E, F, then G, H, I, and then J, K, L. Hence, the matrix has four 3×3 blocks along its main diagonal in which all the entries are 1, except the zeros on the diagonal itself. Outside these blocks, every row contains exactly one 1, for the edge that joins its vertex to another triangle. The matrix below shows the four blocks in four colors.

	A	B	C	D	E	F	G	H	I	J	K	L
A	0	1	1	0	0	0	0	1	0	0	0	0
B	1	0	1	1	0	0	0	0	0	0	0	0
C	1	1	0	0	0	0	0	0	0	1	0	0
D	0	1	0	0	1	1	0	0	0	0	0	0
E	0	0	0	1	0	1	1	0	0	0	0	0
F	0	0	0	1	1	0	0	0	0	0	1	0
G	0	0	0	0	1	0	0	1	1	0	0	0
H	1	0	0	0	0	0	1	0	1	0	0	0
I	0	0	0	0	0	0	1	1	0	0	0	1
J	0	0	1	0	0	0	0	0	0	0	1	1
K	0	0	0	0	0	1	0	0	0	1	0	1
L	0	0	0	0	0	0	0	0	1	1	1	0

To count the triangles, the algorithm checks every three vertices $i < j < k$, and counts the triples with $P(i, j) = P(j, k) = P(i, k) = 1$, where P is the adjacency matrix. Each triangle is counted exactly once, since its three vertices are checked in exactly one order $i < j < k$. There are $\binom{n}{3}$ triples and each of them takes $O(1)$ time, so the complexity is $\Theta(n^3)$.

For the FourTriangles graph, the algorithm checks $\binom{12}{3} = 220$ triples and finds exactly 4 triangles: $\{A, B, C\}$, $\{D, E, F\}$, $\{G, H, I\}$, and $\{J, K, L\}$, shown in four colors below. No other triangle exists: an edge between two triangles, such as (B, D), has no common neighbor, since the other neighbors of B are A and C, and the other neighbors of D are E and F.



Remark: The algorithm above checks all the $\binom{n}{3}$ triples of vertices. Three faster algorithms are described below.

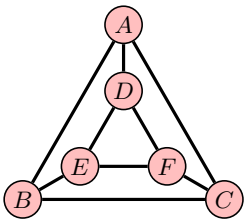
Algorithm 1: If the adjacency lists are also given, using both representations gives a faster algorithm for sparse graphs. For every vertex v, scan its adjacency list, and for every two vertices u and w in it, check in $O(1)$ time whether $P(u, w) = 1$. Every triangle is found three times, once from each of its vertices, so the final count is divided by 3. A vertex of degree d_i has $d_i(d_i - 1)/2$ pairs of neighbors, so the total time is $O(n + \sum_{i=1}^{n} d_i^2)$. This is $O(nm)$, since every degree is at most $n-1$ and the degrees sum to $2m$, so $\sum_{i=1}^{n} d_i^2 \leq \sum_{i=1}^{n} (n-1)d_i = (n-1)\sum_{i=1}^{n} d_i = 2m(n-1)$. This bound is much less than n^3 when m is much smaller than n^2. For the FourTriangles graph, every vertex has 3 pairs of neighbors, so the algorithm checks only $12 \cdot 3 = 36$ pairs instead of 220 triples. It finds 12 adjacent pairs, and reports $12/3 = 4$ triangles.

Algorithm 2: This algorithm runs in $O(n + m^{3/2})$ time. Order the vertices by their degrees, from small to large, and order vertices of equal degree alphabetically. Since the degrees are integers between 0 and $n - 1$, this order can be computed in $O(n)$ time. For every vertex v, scan its adjacency list, and check with the matrix every two neighbors u and w of v that both come after v in this order. Every triangle is found exactly once, from its vertex that comes first in the order.

The order makes the algorithm faster, and not only by a constant factor, since a vertex of large degree comes late in the order, and few of its neighbors come after it. Indeed, suppose that k neighbors of v come after v. Each of them has degree at least $d(v) \geq k$, and the sum of all the degrees is $2m$, so $k^2 \leq 2m$, that is, $k \leq \sqrt{2m}$. Hence, v checks at most $k^2/2 \leq k\sqrt{2m}/2$ pairs. Every edge is counted in k for exactly one of its endpoints, the one that comes first in the order, so the values of k sum to m, and the total number of checked pairs is at most $m\sqrt{2m}/2$. Since $\sqrt{m} < n$, this is never worse than $O(nm)$, and it can be much better. For example, in the star with n vertices, Algorithm 1 checks $(n-1)(n-2)/2$ pairs at the middle vertex, while Algorithm 2 checks none, since the middle vertex comes last in the order. For the FourTriangles graph, all the degrees are 3, so the order is alphabetical. The algorithm checks only 8 pairs: 3 at A, and 1 at each of B, D, E, G, and J, and it finds the 4 triangles from A, D, G, and J.

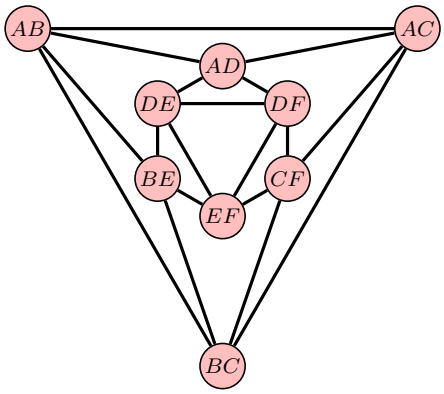
Algorithm 3: This algorithm is based on matrix multiplication, and it is only outlined here. The entry in row u and column w of the matrix $P^2 = P \cdot P$ is the number of common neighbors of u and w. A triangle $\{u, v, w\}$ is counted once for each of the 6 ordered pairs of its vertices, as a common neighbor of an adjacent pair. Hence, the number of triangles is $\frac{1}{6}\sum_{u,w} P^2(u, w)\, P(u, w)$. Multiplying two $n \times n$ matrices in the straightforward way takes $\Theta(n^3)$ time, but faster methods are known: Strassen's algorithm takes $O(n^{2.81})$ time, and the best known methods take about $O(n^{2.372})$ time. These methods are mostly of theoretical interest.

3.5. The **line graph** $L(G)$ of a graph G has a vertex for every edge of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share an endpoint. Each vertex of $L(G)$ is named by the two endpoints of its edge, for example, AB for the edge (A, B). Consider the prism graph below.

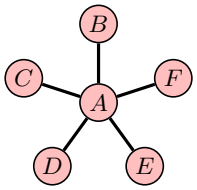


(a) Draw the line graph of the prism graph.

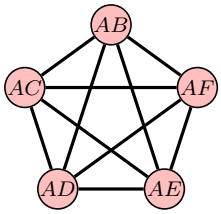
Answer: In the drawing below, the edges of the outer triangle A, B, C of the prism graph are on the outside, the edges of the inner triangle D, E, F are in the middle, and the three joining edges are between them. Each triangle of the prism graph becomes a triangle of the line graph, and so do the three edges of each vertex of the prism graph.



Planarity. A graph is **planar** if it can be drawn in the plane without crossing edges[24]. The drawing of the prism graph in the problem and the drawing of its line graph above have no crossing edges, so both graphs are planar. However, the line graph of a planar graph is not always planar. For example, the star with n vertices is planar, and its line graph is the complete graph with $n-1$ vertices, since every two edges of the star share the middle vertex. The complete graph with at least 5 vertices is not planar (this is a well-known fact, and its proof is omitted). Hence, for every $n \geq 6$, the line graph of the star with n vertices is not planar. For $n \leq 5$, the complete graph with $n - 1 \leq 4$ vertices is planar.



the star with 6 vertices



its line graph

(b) How many vertices and how many edges does the line graph of the prism graph have? What are the degrees of its vertices?

Answer: The line graph has 9 vertices, one for each edge of the prism graph. The edge (u, v) shares an endpoint with the other 2 edges of u and with the other 2 edges of v, since every vertex of the prism graph has degree 3. No edge is counted twice, since the prism graph has no parallel edges. Hence, every vertex of the line graph has degree 4. The sum of the degrees is $9 \cdot 4 = 36$, and it counts every edge of the line graph twice, once for each of its endpoints. Therefore, the line graph has $36/2 = 18$ edges.

[24] Planar graphs also appear in the solution of Problem 5.5(c) on page 97.

(c) Construct the adjacency matrix of the line graph of the prism graph. Use the vertex order $AB, AC, BC, DE, DF, EF, AD, BE, CF$ for the rows and columns.

Answer:

	AB	AC	BC	DE	DF	EF	AD	BE	CF
AB	0	1	1	0	0	0	1	1	0
AC	1	0	1	0	0	0	1	0	1
BC	1	1	0	0	0	0	0	1	1
DE	0	0	0	0	1	1	1	1	0
DF	0	0	0	1	0	1	1	0	1
EF	0	0	0	1	1	0	0	1	1
AD	1	1	0	1	1	0	0	0	0
BE	1	0	1	1	0	1	0	0	0
CF	0	1	1	0	1	1	0	0	0

(d) Construct the adjacency lists of the line graph of the prism graph, using the same vertex order.

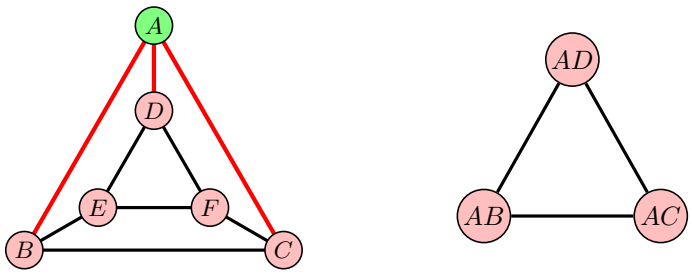
Answer:

$$\begin{array}{lcl} AB & \to & (AC, BC, AD, BE) \\ AC & \to & (AB, BC, AD, CF) \\ BC & \to & (AB, AC, BE, CF) \\ DE & \to & (DF, EF, AD, BE) \\ DF & \to & (DE, EF, AD, CF) \\ EF & \to & (DE, DF, BE, CF) \\ AD & \to & (AB, AC, DE, DF) \\ BE & \to & (AB, BC, DE, EF) \\ CF & \to & (AC, BC, DF, EF) \end{array}$$

Every list has length 4, as computed in part (b).

(e) Let G be a simple graph with n vertices of degrees $d_1, d_2, \ldots, d_n$ and with m edges. Prove that $L(G)$ has m vertices and $\sum_{i=1}^{n} d_i(d_i-1)/2$ edges. Check this formula against part (b).

Proof: $L(G)$ has a vertex for every edge of G, so it has m vertices. Two different edges of G share at most one endpoint, since G has no parallel edges. Hence, every edge of $L(G)$ corresponds to exactly one vertex v_i of G and one pair of edges that touch v_i. There are $d_i(d_i-1)/2$ such pairs, so $L(G)$ has $\sum_{i=1}^{n} d_i(d_i-1)/2$ edges.[25] In other words, the d_i edges that touch v_i are pairwise adjacent in $L(G)$. For the prism graph, $n = 6$ and every degree is 3, so $L(G)$ has $6 \cdot 3 \cdot 2/2 = 18$ edges, as in part (b): 9 vertices of degree 4 give $9 \cdot 4/2 = 18$ edges. In the figure below, the three red edges at A (left) become the triangle AB, AC, AD of the line graph (right).



[25] For a Δ-regular graph, this is $n\Delta(\Delta-1)/2$; see the solution of Problem 2.6(e) on page 49.

(f) Design an algorithm that constructs the adjacency matrix of $L(G)$ from the adjacency matrix of G. What is the complexity of your algorithm?

Answer: $\Theta(n^2 + m^2)$, where G has n vertices and m edges. First, scan the entries above the main diagonal of the adjacency matrix P of G, and number the edges $e_1, e_2, \ldots, e_m$ in the order in which they are found, recording the two endpoints of each edge. This takes $\Theta(n^2)$ time. Then, create the $m \times m$ adjacency matrix R of $L(G)$, and for every $1 \leq k, l \leq m$, set $R(k, l) = 1$ if $k \neq l$ and the edges e_k and e_l share an endpoint, and $R(k, l) = 0$ otherwise. Each entry takes $O(1)$ time, since it compares the endpoints of two edges, so this takes $\Theta(m^2)$ time. The algorithm is correct by the definition of $L(G)$. No algorithm can do better: it must read the $n(n-1)/2$ entries above the diagonal of P, and write the m^2 entries of R. For the prism graph, the algorithm scans 15 entries above the diagonal, finds 9 edges, and fills the 81 entries of the matrix of part (c), up to the vertex order. Since m can be as large as $n(n-1)/2$, the complexity can reach $\Theta(n^4)$, for example, for the complete graph.

(g) Design an algorithm that constructs the adjacency lists of $L(G)$ from the adjacency lists of G. What is the complexity of your algorithm?

Answer: $O(n + \sum_{i=1}^{n} d_i^2)$. First, create an empty list for every edge (u, v) of G, named uv, by scanning the lists of G and taking every pair u, v with v in the list of u and u before v; add a pointer to this list to the incidence lists of u and of v. Then, for every vertex v_i of G, collect the d_i edges that touch it from its incidence list, and for every two different edges e and f among them, append f to the list of e. By the proof of part (e), every edge of $L(G)$ joins two edges e and f of G with exactly one common endpoint, so it is handled exactly once, at that endpoint, where f is appended to the list of e and e is appended to the list of f. Hence, the lists are correct. Creating the lists takes $O(n + m)$ time, and the vertex v_i contributes $d_i(d_i - 1)$ list entries. Since $m = \sum_i d_i/2$, the total time is $O(n + \sum_{i=1}^{n} d_i^2)$, which is proportional to the size of G plus the size of $L(G)$. For sparse graphs, this is much less than the $\Theta(n^2 + m^2)$ time of part (f). For the prism graph, the algorithm creates 9 lists and $6 \cdot 3 \cdot 2 = 36$ list entries, the lists of part (d) up to order.

3.6. For $n \geq 1$, let X and Y be two simple undirected labeled graphs on the vertices $\{1, 2, \ldots, n\}$.

The graph Y is a **labeled subgraph** of X if every edge of Y is also an edge of X (however, X may have edges that are not edges of Y).

(a) Assume that both graphs are represented by adjacency matrices. Design an algorithm that determines whether Y is a labeled subgraph of X. What is the complexity of your algorithm as a function of n? Justify the correctness of your algorithm and your complexity claim.

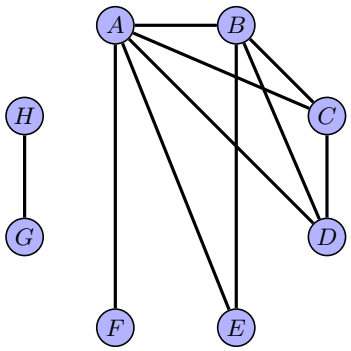
Answer: $\Theta(n^2)$ in the worst case. Let P be the adjacency matrix of X and let Q be the adjacency matrix of Y. Observe that Y is a labeled subgraph of X if and only if $Q(i,j) \leq P(i,j)$ for all $1 \leq i, j \leq n$. Indeed, $Q(i,j) \leq P(i,j)$ fails exactly when $Q(i,j) = 1$ and $P(i,j) = 0$, that is, when (i,j) is an edge of Y but not an edge of X. The algorithm compares each entry of Q with the corresponding entry of P. It answers "no" as soon as it finds an entry with $Q(i,j) > P(i,j)$, and "yes" otherwise.

The complexity of this algorithm is $\Theta(n^2)$ in the worst case. Each of the n^2 comparisons takes $\Theta(1)$ time, which gives $O(n^2)$. When Y is a labeled subgraph of X, there is no entry with $Q(i,j) > P(i,j)$, so the algorithm compares all the n^2 entries before it answers "yes", which gives $\Omega(n^2)$ in the worst case. Since both matrices are symmetric with zeros on the diagonal, it is enough to compare the $n(n-1)/2$ entries above the diagonal, which is still $\Theta(n^2)$.

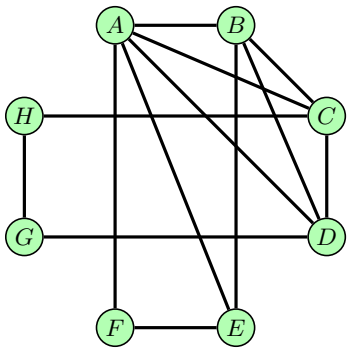
No algorithm can do better in the worst case. Let X be the complete graph and let Y have no edges, so the answer is "yes". Assume that for some pair $i < j$, an algorithm reads none of the entries $P(i,j)$, $P(j,i)$, $Q(i,j)$, and $Q(j,i)$. Then it gives the same answer when the edge (i,j) is removed from X and added to Y, although the correct answer becomes "no". Hence, every correct algorithm reads at least one entry for each of the $n(n-1)/2$ pairs, which takes $\Omega(n^2)$ time.

Example: Here $n = 8$, and the vertices are $A, B, \ldots, H$. The graph X is the middle (green) graph, Y_1 is the left (blue) graph, and Y_2 is the right (red) graph. The graph Y_1 is a labeled subgraph of X, since all its edges appear in X, while Y_2 is not, since its edges (C,G) and (D,H) are not edges of X.

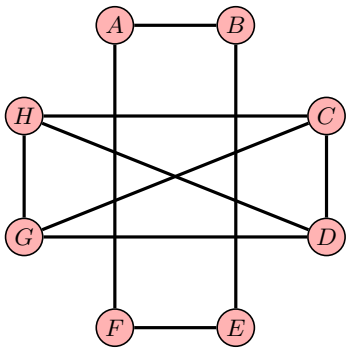
Below each graph is its adjacency matrix. The matrix of X has a 1 in every entry in which the matrix of Y_1 has a 1, while the entries (C,G) and (D,H) are 1 in the matrix of Y_2 and 0 in the matrix of X.



Y_1



X



Y_2

	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	0	0
B	1	0	1	1	1	0	0	0
C	1	1	0	1	0	0	0	0
D	1	1	1	0	0	0	0	0
E	1	1	0	0	0	0	0	0
F	1	0	0	0	0	0	0	0
G	0	0	0	0	0	0	0	1
H	0	0	0	0	0	0	1	0

	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	0	0
B	1	0	1	1	1	0	0	0
C	1	1	0	1	0	0	0	1
D	1	1	1	0	0	0	1	0
E	1	1	0	0	0	1	0	0
F	1	0	0	0	1	0	0	0
G	0	0	0	1	0	0	0	1
H	0	0	1	0	0	0	1	0

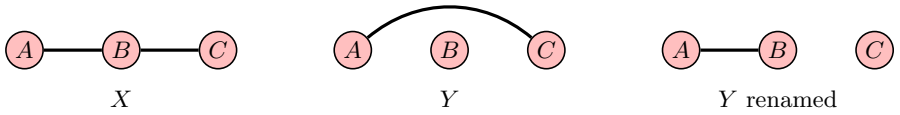
	A	B	C	D	E	F	G	H
A	0	1	0	0	0	1	0	0
B	1	0	0	0	1	0	0	0
C	0	0	0	1	0	0	1	1
D	0	0	1	0	0	0	1	1
E	0	1	0	0	0	1	0	0
F	1	0	0	0	1	0	0	0
G	0	0	1	1	0	0	0	1
H	0	0	1	1	0	0	1	0

(b) Assume that X and Y are represented by adjacency lists, where X has m_X edges and Y has m_Y edges. Design an algorithm that determines whether Y is a labeled subgraph of X in $O(n + m_X + m_Y)$ time.

Answer: The graph Y is a labeled subgraph of X if and only if, for every vertex v, every vertex in the list of v in Y also appears in the list of v in X. The algorithm uses an array of n marks, all of them initially off. For every vertex v, it turns on the marks of the vertices in the list of v in X, scans the list of v in Y, and answers "no" if it finds a vertex whose mark is off. Then it turns off the marks it turned on. If it never answers "no", it answers "yes". The vertex v takes $O(1 + d_X(v) + d_Y(v))$ time, where $d_X(v)$ and $d_Y(v)$ are its degrees in X and in Y. Since the degrees in X sum to $2m_X$ and the degrees in Y sum to $2m_Y$, the total time is $O(n + m_X + m_Y)$. For sparse graphs, this is much faster than part (a).

(c) Show an example of two labeled graphs X and Y on the same vertices, such that Y is not a labeled subgraph of X, but Y becomes a labeled subgraph of X after renaming its vertices.

Answer: Let $n = 3$, and name the vertices A, B, C. Let X be the path with the edges (A, B) and (B, C), and let Y have the single edge (A, C). The edge (A, C) is not an edge of X, so Y is not a labeled subgraph of X. However, exchanging the names B and C in Y turns its edge into (A, B), which is an edge of X. Deciding whether some renaming of the vertices of Y makes it a labeled subgraph of X is a much harder problem, and no efficient algorithm is known for it.



3.7. For $n \geq 1$, let X and Y be two simple undirected labeled graphs on the vertices $\{1, 2, \ldots, n\}$. The **union** U of X and Y is the graph on $\{1, 2, \ldots, n\}$ whose edges are exactly the edges that belong to at least one of X and Y. The **intersection** W of X and Y is the graph on $\{1, 2, \ldots, n\}$ whose edges are exactly the edges that belong to both X and Y.

(a) Assume that both graphs are represented by adjacency matrices. Design an algorithm that constructs the adjacency matrices of U and of W. What is the complexity of your algorithm as a function of n? Justify the correctness of your algorithm and your complexity claim.

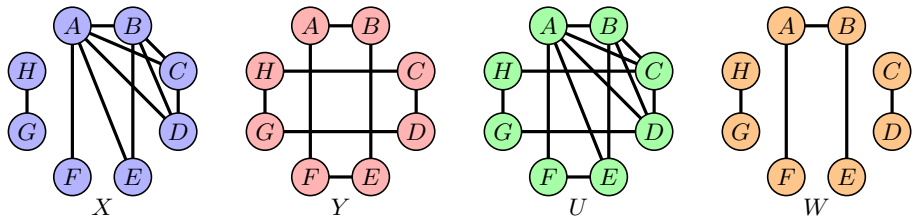
Answer: $\Theta(n^2)$. Let P be the adjacency matrix of X and let Q be the adjacency matrix of Y. Construct the adjacency matrix R_U of U and the adjacency matrix R_W of W as follows. For all $1 \leq i, j \leq n$:

$$R_U(i,j) = \max\{P(i,j), Q(i,j)\} \qquad \text{and} \qquad R_W(i,j) = \min\{P(i,j), Q(i,j)\}.$$

For all $1 \leq i, j \leq n$, it follows that $R_U(i,j) = 1$ if and only if $P(i,j) = 1$ or $Q(i,j) = 1$ (or both), that is, if and only if (i,j) is an edge of X or of Y (or both), which is exactly the definition of the union. Similarly, $R_W(i,j) = 1$ if and only if both $P(i,j) = 1$ and $Q(i,j) = 1$, that is, if and only if (i,j) is an edge of both X and Y, which is exactly the definition of the intersection. Moreover, since P and Q are symmetric with zeros on the diagonal, so are R_U and R_W; hence, they are adjacency matrices of simple undirected graphs.

The complexity of this algorithm is $\Theta(n^2)$. For each of the n^2 pairs $1 \leq i, j \leq n$, the algorithm computes $R_U(i,j)$ and $R_W(i,j)$ in $\Theta(1)$ time, which gives $O(n^2)$. Any algorithm must determine at least the $n(n-1)/2$ entries above the diagonal of each matrix, one for each pair of vertices, which gives $\Omega(n^2)$.

Example: Here $n = 8$, and the eight vertices are named A, B, C, D, E, F, G, H. Below are the graphs X (blue), Y (red), U (green), and W (orange), and their adjacency matrices. The matrix of U is the entrywise maximum of the matrices of X and Y, and the matrix of W is their entrywise minimum. Note that X has 10 edges, Y has 8 edges, U has 13 edges, and W has 5 edges.



X

	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	0	0
B	1	0	1	1	1	0	0	0
C	1	1	0	1	0	0	0	0
D	1	1	1	0	0	0	0	0
E	1	1	0	0	0	0	0	0
F	1	0	0	0	0	0	0	0
G	0	0	0	0	0	0	0	1
H	0	0	0	0	0	0	1	0

Y

	A	B	C	D	E	F	G	H
A	0	1	0	0	0	1	0	0
B	1	0	0	0	1	0	0	0
C	0	0	0	1	0	0	0	1
D	0	0	1	0	0	0	1	0
E	0	1	0	0	0	1	0	0
F	1	0	0	0	1	0	0	0
G	0	0	0	1	0	0	0	1
H	0	0	1	0	0	0	1	0

U

	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	0	0
B	1	0	1	1	1	0	0	0
C	1	1	0	1	0	0	0	1
D	1	1	1	0	0	0	1	0
E	1	1	0	0	0	1	0	0
F	1	0	0	0	1	0	0	0
G	0	0	0	1	0	0	0	1
H	0	0	1	0	0	0	1	0

W

	A	B	C	D	E	F	G	H
A	0	1	0	0	0	1	0	0
B	1	0	0	0	1	0	0	0
C	0	0	0	1	0	0	0	0
D	0	0	1	0	0	0	0	0
E	0	1	0	0	0	0	0	0
F	1	0	0	0	0	0	0	0
G	0	0	0	0	0	0	0	1
H	0	0	0	0	0	0	1	0

(b) Assume that X and Y are represented by adjacency lists, and that every list is sorted in increasing order. Design an algorithm that constructs the sorted adjacency lists of U and of W in $O(n + m_X + m_Y)$ time, where X has m_X edges and Y has m_Y edges.

Answer: For every vertex v, the algorithm merges the two sorted lists of v, as in merge sort. It keeps a pointer to the first unused vertex of each list and compares the two vertices. If they are equal, it appends this vertex to the lists of v in the union U and in the intersection W, and advances both pointers. Otherwise, it appends the smaller vertex to the list of v in U and advances its pointer. When one list is used up, it appends the rest of the other list to the list of v in U. A vertex appears in both lists of v exactly when it is a neighbor of v in both graphs, so the lists of U and W are correct, and they are sorted since they are created in increasing order. Every comparison advances at least one pointer, so the vertex v takes $O(1 + d_X(v) + d_Y(v))$ time, where $d_X(v)$ and $d_Y(v)$ are its degrees, and the total time is $O(n + m_X + m_Y)$. The table below shows the lists of two vertices of the example of part (a).

vertex	list in X	list in Y	list in U	list in W
A	B, C, D, E, F	B, F	B, C, D, E, F	B, F
C	A, B, D	D, H	A, B, D, H	D

(c) Let m_X, m_Y, m_U, and m_W be the numbers of edges of X, Y, U, and W, respectively. Prove that $m_X + m_Y = m_U + m_W$.

Proof: Consider any pair of different vertices i and j. If (i, j) is an edge of both X and Y, then it is counted twice in $m_X + m_Y$ and twice in $m_U + m_W$, once in each of the four numbers. If (i, j) is an edge of exactly one of X and Y, then it is counted once in $m_X + m_Y$ and once in $m_U + m_W$ (in m_U but not in m_W). If (i, j) is an edge of neither X nor Y, then it is not counted at all. Summing over all pairs gives $m_X + m_Y = m_U + m_W$. Equivalently, $m_U = m_X + m_Y - m_W$: the edges of the intersection are counted twice in $m_X + m_Y$, so they are subtracted once. This is the **principle of inclusion and exclusion** for two sets. For the graphs X and Y of the example in part (a), $m_X = 10$, $m_Y = 8$, $m_U = 13$, and $m_W = 5$, and indeed $10 + 8 = 13 + 5$.

3.8. For $n \geq 1$, let X be a simple undirected labeled graph on the vertices $\{1, 2, \ldots, n\}$. The **complement** $\overline{X}$ of X is the graph on $\{1, 2, \ldots, n\}$ in which two different vertices are adjacent if and only if they are not adjacent in X.

(a) If X has m edges, how many edges does $\overline{X}$ have?

Answer: $n(n-1)/2 - m$ edges. Every pair of different vertices is an edge of exactly one of X and $\overline{X}$.[26] Hence, if X has m edges, then $\overline{X}$ has $n(n-1)/2 - m$ edges.

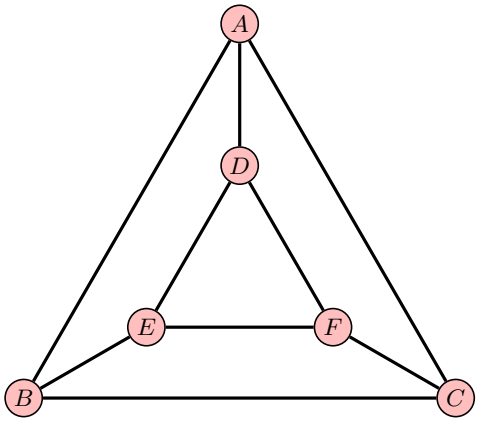
(b) Assume that X is represented by an adjacency matrix. Design an algorithm that constructs the adjacency matrix of $\overline{X}$. What is the complexity of your algorithm as a function of n? Justify the correctness of your algorithm and your complexity claim.

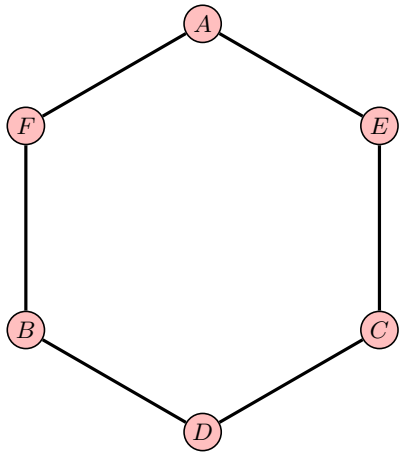
Answer: $\Theta(n^2)$. Let P be the adjacency matrix of X. Construct R, the adjacency matrix of $\overline{X}$, as follows: $R(i,i) = 0$ for all $1 \leq i \leq n$, and $R(i,j) = 1 - P(i,j)$ for all $1 \leq i, j \leq n$ with $i \neq j$.

For $i \neq j$, it follows that $R(i,j) = 1$ if and only if $P(i,j) = 0$, that is, if and only if i and j are not adjacent in X, which is exactly the definition of the complement. The diagonal of R is zero, as required, since $\overline{X}$ has no self-loops, and R is symmetric since P is symmetric. Hence, R is the adjacency matrix of $\overline{X}$.

The complexity of this algorithm is $\Theta(n^2)$. Each of the n^2 entries of R is computed in $\Theta(1)$ time, which gives $O(n^2)$. Any algorithm must compute at least the $n(n-1)/2$ entries above the diagonal, one for each pair of vertices, which is already $\Omega(n^2)$.

Example: The prism graph X (left) has $n = 6$ vertices and $m = 9$ edges, and its complement (right) has $15 - 9 = 6$ edges: it is the cycle $A - E - C - D - B - F - A$.[27] Below are the adjacency matrices of X (left) and of $\overline{X}$ (right), using the vertex order $A, B, \ldots, F$.





	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	1	0	1	0
C	1	1	0	0	0	1
D	1	0	0	0	1	1
E	0	1	0	1	0	1
F	0	0	1	1	1	0

	A	B	C	D	E	F
A	0	0	0	0	1	1
B	0	0	0	1	0	1
C	0	0	0	1	1	0
D	0	1	1	0	0	0
E	1	0	1	0	0	0
F	1	1	0	0	0	0

[26] Complements of specific graphs and self-complementary graphs appear in Problem 1.8 on page 3.
[27] See also the solution of Problem 1.8(b) on page 36.

(c) Assume that X is represented by adjacency lists. Design an algorithm that constructs the adjacency lists of $\overline{X}$. What is the complexity of your algorithm? Explain why no algorithm that reads all of its input and writes all of its output can be asymptotically faster.

Answer: $\Theta(n^2)$. The algorithm uses an array of n marks, all of them initially off. For every vertex v, it turns on the marks of v and of the vertices in its list in X. Then it scans all the vertices u in order, and appends every u whose mark is off to the list of v in $\overline{X}$. Finally, it turns off the marks it turned on. A vertex $u \neq v$ is appended if and only if it is not adjacent to v in X, which is exactly the definition of $\overline{X}$. Each vertex v takes $\Theta(n)$ time for the scan, plus $O(d(v))$ time for the marks, so the total time is $\Theta(n^2 + m) = \Theta(n^2)$.

No algorithm can be asymptotically faster: the lists of X have $2m$ entries in total, and the lists of $\overline{X}$ have $n(n-1) - 2m$ entries, so the input and the output together have $n(n-1)$ entries. Reading and writing them takes $\Omega(n^2)$ time, whatever m is. The table below gives the lists for the prism graph of the example.

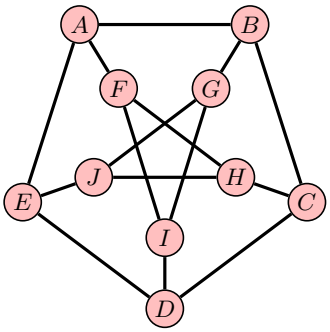
vertex	list in X	list in $\overline{X}$
A	B, C, D	E, F
B	A, C, E	D, F
C	A, B, F	D, E
D	A, E, F	B, C
E	B, D, F	A, C
F	C, D, E	A, B

4 Paths, Cycles, Distances, and Connectivity

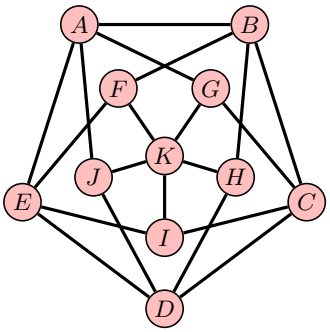
4.1. A **path** $\mathcal{P} = \langle v_0, v_1, \ldots, v_k \rangle$ of length k (number of edges) is an ordered list of $k+1$ vertices such that the edge (v_i, v_{i+1}) exists for $0 \leq i \leq k-1$ and all the edges are different. Note that a vertex may appear more than once in a path. A **cycle** $\mathcal{C} = \langle v_0, v_1, \ldots, v_{k-1}, v_0 \rangle$ of length k is a path that starts and ends at the same vertex.

An **Euler path** is a path that traverses all the edges of the graph exactly once. An **Euler cycle** is a cycle that traverses all the edges of the graph exactly once.

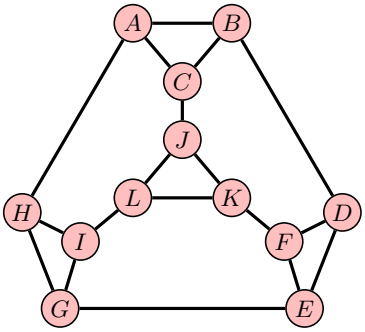
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

None of the three graphs has an Euler path or an Euler cycle.

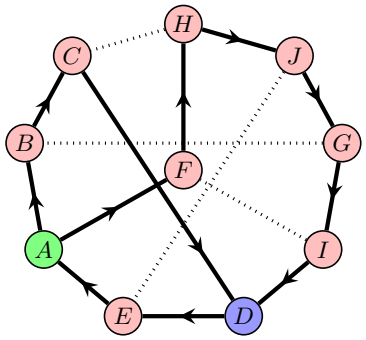
Remark: For the Petersen graph, the answers to parts (a) and (b) are shown on a different drawing of the Petersen graph.

(a) For each of the three graphs, find one of its longest paths, and explain why no longer path exists.

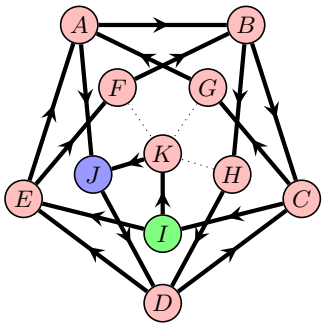
The Petersen graph: $A-F-H-J-G-I-D-E-A-B-C-D$ is a path of length 11. The four dotted edges (B,G), (C,H), (E,J), and (F,I) are not in the path. The path starts at the green vertex A and ends at the blue vertex D.

The Grötzsch graph: $I-E-A-B-C-G-A-J-D-E-F-B-H-D-C-I-K-J$ is a path of length 17. It starts at the green vertex I and ends at the blue vertex J, and the three dotted edges (K,F), (K,G), and (K,H) are not in the path.

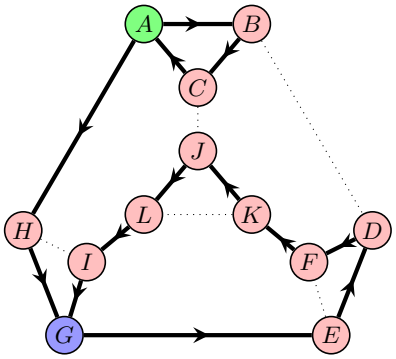
The FourTriangles graph: $A-B-C-A-H-G-E-D-F-K-J-L-I-G$ is a path of length 13. It starts at the green vertex A and ends at the blue vertex G, and the dotted edges are not in the path.



Petersen



Grötzsch



FourTriangles

Proposition (Petersen graph): The length of the longest path in the Petersen graph is 11.

Proof: The path-degree of a vertex is the number of edges of the path that touch this vertex. Each time a path passes through a vertex, it uses two of its edges. Therefore, the path-degree of each vertex is even, except possibly for the two end vertices. Since the degree of each vertex in the Petersen graph is 3, the path-degree of each of the two end vertices is at most 3, and the path-degree of each of the other eight vertices is at most 2. The number of edges in a path is the sum of the path-degrees of the vertices divided by 2, since each edge is counted twice. Therefore, a path in the Petersen graph has at most $(2 \cdot 3 + 8 \cdot 2)/2 = 11$ edges. In particular, the Petersen graph has no Euler path.

Proposition (Grötzsch graph): The length of the longest path in the Grötzsch graph is 17.

Proof: As for the Petersen graph, the path-degree of every vertex, except possibly the two end vertices, is even. The vertices F, G, H, I, J have degree 3, which is odd. Hence, each of them that is not an end vertex of the path touches at least one edge that is not in the path. At least three of these five vertices are not end vertices, and no edge joins two of them (each of them is adjacent only to K and to two vertices of the outer cycle). Therefore, at least three different edges are not in the path, and the path has at most $20 - 3 = 17$ edges. In particular, the Grötzsch graph has no Euler path.

Remark: Neither end vertex of a longest path in the Grötzsch graph can have an even degree or be K. By the proof above, a path with 17 edges misses exactly three edges. If at most one of its two end vertices is among F, G, H, I, J, then at least four of these five vertices are not end vertices, and the path misses at least four edges. Hence, both end vertices of a longest path are among F, G, H, I, J.

Proposition (FourTriangles graph): The length of the longest path in the FourTriangles graph is 13.

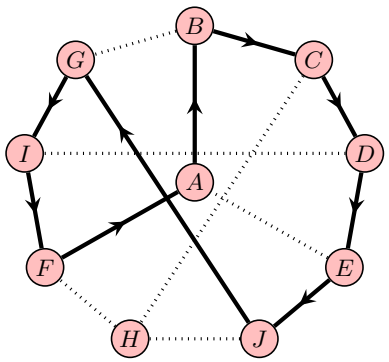
Proof: As for the Petersen graph, the path-degree of every vertex, except possibly the two end vertices, is even. Since every vertex has degree 3, the path-degree of each of the two end vertices is at most 3, and the path-degree of each of the other ten vertices is at most 2. Therefore, a path has at most $(2 \cdot 3 + 10 \cdot 2)/2 = 13$ edges. In particular, the FourTriangles graph has no Euler path.

(b) For each of the three graphs, find one of its longest cycles, and explain why no longer cycle exists.

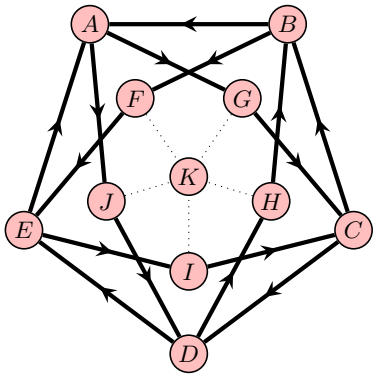
The Petersen graph: $A - B - C - D - E - J - G - I - F - A$ is a cycle of length 9. The only vertex that is not in the cycle is H.

The Grötzsch graph: $A - J - D - H - B - F - E - I - C - D - E - A - G - C - B - A$ is a cycle of length 15. It uses all the edges except the five dotted edges of the middle vertex K.

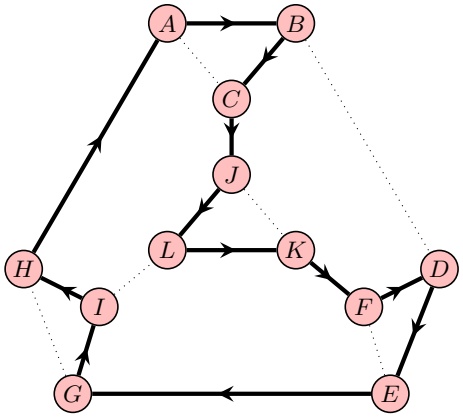
The FourTriangles graph: $A - B - C - J - L - K - F - D - E - G - I - H - A$ is a cycle of length 12, which visits every vertex exactly once.



Petersen



Grötzsch



FourTriangles

Proposition (Petersen graph): The length of the longest cycle in the Petersen graph is 9.

Proof: In any cycle, the cycle-degree of each vertex (the number of edges of the cycle that touch this vertex) is even. Since the degree of each vertex in the Petersen graph is 3, the cycle-degree of each vertex is at most 2. As a result, each vertex appears at most once in a cycle of the Petersen graph (apart from the repeated first vertex at the end). The Petersen graph does not contain such a cycle of length 10 (one that visits all its vertices),[28] and therefore the above cycle is one of the longest cycles in the Petersen graph. In particular, the Petersen graph has no Euler cycle.

Proposition (Grötzsch graph): The length of the longest cycle in the Grötzsch graph is 15.

Proof: In a cycle, the cycle-degree of every vertex is even. Hence, each of the five vertices F, G, H, I, J of degree 3 touches at least one edge that is not in the cycle, and since no edge joins two of them, these are five different edges. Therefore, the cycle has at most $20 - 5 = 15$ edges. In particular, the Grötzsch graph has no Euler cycle.

Proposition (FourTriangles graph): The length of the longest cycle in the FourTriangles graph is 12.

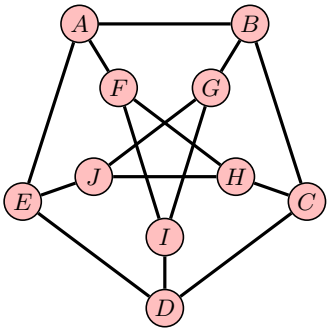
Proof: In a cycle, the cycle-degree of every vertex is even, and since every vertex has degree 3, it is at most 2. Therefore, a cycle has at most $(12 \cdot 2)/2 = 12$ edges. In particular, the FourTriangles graph has no Euler cycle.

[28] This fact is proved in the solution of Problem 4.2(b) on page 77.

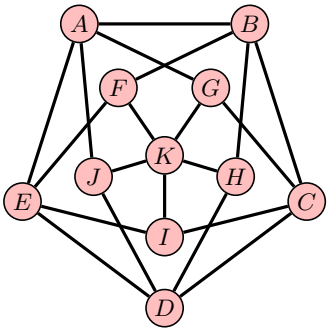
4.2. A **simple path** $\mathcal{P} = \langle v_0, v_1, \ldots, v_k \rangle$ of length k (number of edges) is an ordered list of $k+1$ distinct vertices such that the edge (v_i, v_{i+1}) exists for $0 \leq i \leq k-1$. A **simple cycle** $\mathcal{C} = \langle v_0, v_1, \ldots, v_{k-1}, v_0 \rangle$ of length $k \geq 3$ is an ordered list of k distinct vertices $v_0, v_1, \ldots, v_{k-1}$ such that the edge (v_i, v_{i+1}) exists for $0 \leq i \leq k-2$ and the edge (v_{k-1}, v_0) exists.

A **Hamiltonian path** is a simple path that visits all the vertices of the graph. A **Hamiltonian cycle** is a simple cycle that visits all the vertices of the graph.

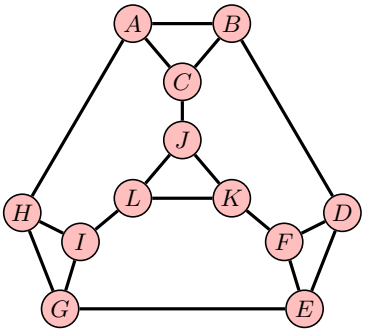
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

All three graphs have Hamiltonian paths. The Grötzsch graph and the FourTriangles graph have Hamiltonian cycles, but the Petersen graph does not.

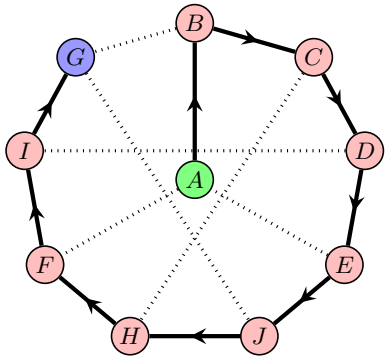
Remark: For the Petersen graph, the answers to parts (a) and (b) are shown on a different drawing of the Petersen graph.

(a) For each of the three graphs, find one of its Hamiltonian paths.

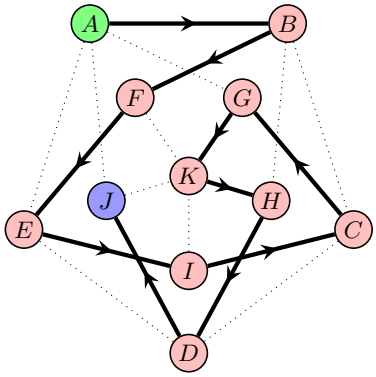
The Petersen graph: $A - B - C - D - E - J - H - F - I - G$ is a Hamiltonian path. It starts at the green vertex A and ends at the blue vertex G.

The Grötzsch graph: $A - B - F - E - I - C - G - K - H - D - J$ is a Hamiltonian path. It starts at the green vertex A and ends at the blue vertex J.

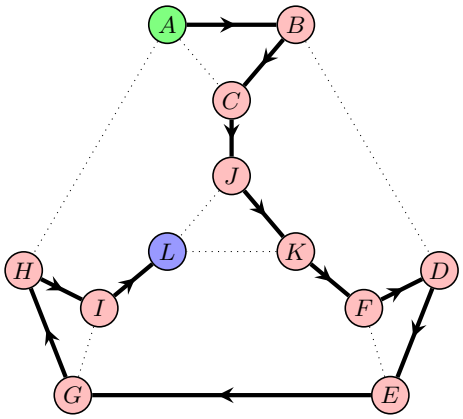
The FourTriangles graph: $A - B - C - J - K - F - D - E - G - H - I - L$ is a Hamiltonian path. It starts at the green vertex A and ends at the blue vertex L.



Petersen



Grötzsch



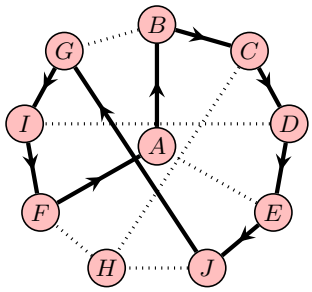
FourTriangles

(b) For each of the three graphs, find one of its longest simple cycles, and explain why no longer simple cycle exists.

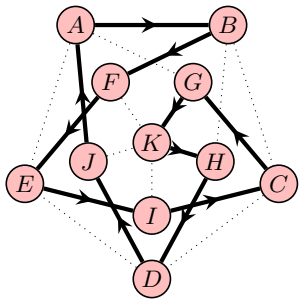
The Petersen graph: The cycle $A-B-C-D-E-J-G-I-F-A$ is a simple cycle with 9 edges and 9 vertices. No longer simple cycle exists by the proposition below.

The Grötzsch graph: $A-B-F-E-I-C-G-K-H-D-J-A$ is a Hamiltonian cycle: it visits each of the 11 vertices exactly once and returns to A. The dotted edges are not in the cycle. No longer simple cycle exists, since a simple cycle visits each vertex at most once.

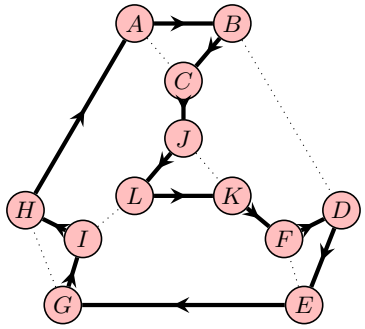
The FourTriangles graph: $A-B-C-J-L-K-F-D-E-G-I-H-A$ is a Hamiltonian cycle: it visits each of the 12 vertices exactly once and returns to A. The dotted edges are not in the cycle. No longer simple cycle exists, since a simple cycle visits each vertex at most once.



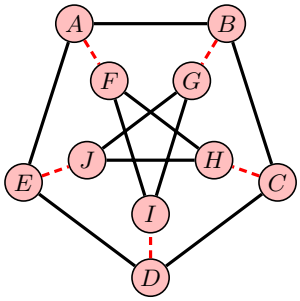
Petersen



Grötzsch



FourTriangles



Proposition (Petersen graph): The length of the longest simple cycle in the Petersen graph is 9.

Proof: It suffices to show that the Petersen graph has no Hamiltonian cycle (a simple cycle with 10 vertices).[29] Its edges are the outer cycle $A-B-C-D-E-A$, the inner cycle $F-H-J-G-I-F$, and the five spokes (A,F), (B,G), (C,H), (D,I), and (E,J). Assume to the contrary that $\mathcal{C}$ is a Hamiltonian cycle. Each time $\mathcal{C}$ uses a spoke, it moves between the outer and the inner vertices, and since it returns to its first vertex, it uses an even number of spokes. This number is not 0, since $\mathcal{C}$ visits both outer and inner vertices, so it is 2 or 4. The spokes are red and dashed in the figure above.

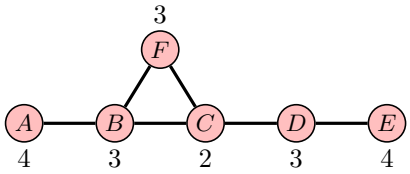
Two spokes. Then $\mathcal{C}$ visits the five outer vertices along a path with 4 edges of the outer cycle, whose two ends are adjacent on the outer cycle, say A and B by the symmetry of the graph. Their spokes go to F and G, so $\mathcal{C}$ visits the five inner vertices along a path with 4 edges of the inner cycle from F to G. The ends of such a path are adjacent on the inner cycle, but F and G are not adjacent, a contradiction.

Four spokes. By symmetry, say that the spoke (A,F) is not in $\mathcal{C}$. Since $\mathcal{C}$ uses exactly two edges at every vertex, the edges (A,B), (A,E), (F,H), and (F,I) are in $\mathcal{C}$, and each of B,C,D,E uses its spoke and exactly one outer edge. Since B uses (A,B) and E uses (A,E), the edges (B,C) and (D,E) are not in $\mathcal{C}$, and therefore (C,D) is. Similarly, (H,J) and (G,I) are not in $\mathcal{C}$, and therefore (G,J) is. Then $\mathcal{C}$ contains the cycle $A-B-G-J-E-A$ of length 5, which is impossible, since $\mathcal{C}$ is a simple cycle of length 10.

[29] Another proof appears in the solution of Problem 7.5 on page 132.

4.3. Let G be a connected graph, that is, every two of its vertices are joined by a path. The **distance** $d(u, v)$ between two vertices u and v is the number of edges of a shortest path from u to v; in particular, $d(u, u) = 0$. The **eccentricity** of a vertex v is the largest distance from v to a vertex of G. The **diameter** of G is the largest eccentricity of its vertices, the **radius** of G is the smallest eccentricity of its vertices, and the **center** of G is the set of vertices whose eccentricity equals the radius.

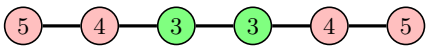
For example, in the graph below, $d(A, E) = 4$ and $d(A, F) = 2$. The number next to each vertex is its eccentricity. Hence, the diameter is 4, the radius is 2, and the center is $\{C\}$.



In the figures of the answers, every vertex is labeled with its eccentricity, and the vertices of the center are green.

(a) Find the diameter, the radius, and the center of the path P_n with $n \geq 2$ vertices $v_1, \ldots, v_n$ and the $n-1$ edges (v_i, v_{i+1}) for $1 \leq i \leq n-1$.

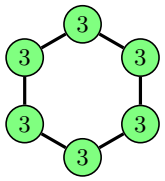
Answer: The diameter is $n-1$, the radius is $\lfloor n/2 \rfloor$, and the center is $\{v_{(n+1)/2}\}$ when n is odd and $\{v_{n/2}, v_{n/2+1}\}$ when n is even. Indeed, the distance between v_i and v_j is $|i-j|$, so the eccentricity of v_i is $\max\{i-1, n-i\}$. It is largest, $n-1$, at v_1 and v_n, and smallest, $\lceil (n-1)/2 \rceil = \lfloor n/2 \rfloor$, in the middle of the path. See below P_6 with the eccentricities of its vertices.



P_6

(b) Find the diameter, the radius, and the center of the cycle C_n with $n \geq 3$ vertices $v_1, \ldots, v_n$ and the n edges (v_i, v_{i+1}) for $1 \leq i \leq n-1$ and (v_n, v_1).

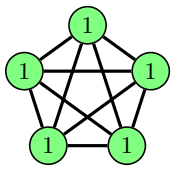
Answer: The diameter and the radius are both $\lfloor n/2 \rfloor$, and the center is the set of all the vertices. Indeed, going around the cycle in the shorter direction, every vertex is at distance at most $\lfloor n/2 \rfloor$ from any other vertex, and a vertex "opposite" to it is at distance exactly $\lfloor n/2 \rfloor$. Hence, every vertex has eccentricity $\lfloor n/2 \rfloor$. See below C_6 with the eccentricities of its vertices.



C_6

(c) Find the diameter, the radius, and the center of the complete graph K_n with $n \geq 2$ vertices, in which every two vertices are adjacent.

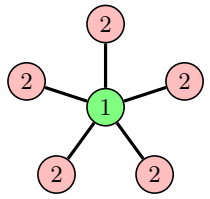
Answer: The diameter and the radius are both 1, and the center is the set of all the vertices, since every two different vertices are at distance 1. See below K_5 with the eccentricities of its vertices.



K_5

(d) Find the diameter, the radius, and the center of the star S_n with $n \geq 3$ vertices: one middle vertex that is adjacent to the other $n-1$ vertices, and no other edges.

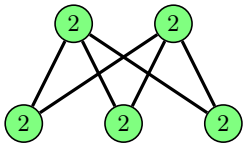
Answer: The diameter is 2, the radius is 1, and the center is the set that contains only the middle vertex. Indeed, the middle vertex is adjacent to all the other vertices, so its eccentricity is 1. Every two other vertices are at distance 2 (through the middle vertex), so each of them has eccentricity 2. See below S_6 with the eccentricities of its vertices.



S_6

(e) Find the diameter, the radius, and the center of the complete bipartite graph $K_{r,s}$ with $r, s \geq 2$: there are r vertices on one side and s vertices on the other side, and two vertices are adjacent if and only if they are on different sides.

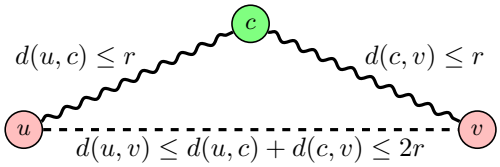
Answer: The diameter and the radius are both 2, and the center is the set of all the vertices. Indeed, two vertices on different sides are adjacent, and two vertices on the same side are not adjacent but have a common neighbor on the other side, so their distance is 2. Since every side has at least 2 vertices, every vertex has eccentricity 2. See below $K_{2,3}$ with the eccentricities of its vertices.



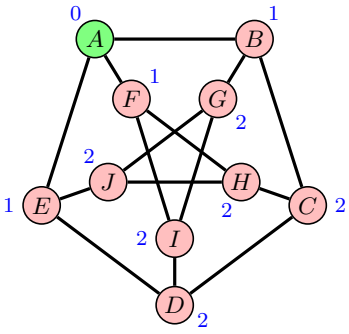
$K_{2,3}$

(f) Prove that in every connected graph, the radius is at most the diameter, and the diameter is at most twice the radius.

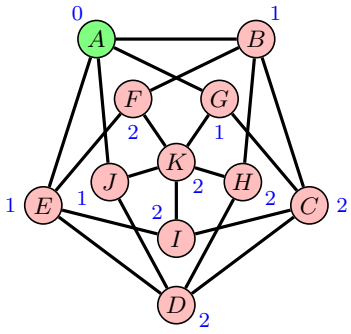
Proof: Let r be the radius and D the diameter. Since the largest eccentricity is at least the smallest one, $r \leq D$. Let c be a vertex of the center, so the eccentricity of c is r. For any two vertices u and v, go from u to c along a shortest path and then from c to v along a shortest path. This route uses $d(u,c) + d(c,v) \leq 2r$ edges, and if it visits some vertex twice, the part between the two visits can be cut out. The result is a path from u to v with at most $2r$ edges. Hence, $d(u,v) \leq 2r$, so every eccentricity is at most $2r$, and $D \leq 2r$. Both bounds are reached: in K_n of part (c), $D = r = 1$, and in P_n with an odd n of part (a), $D = n - 1 = 2r$.



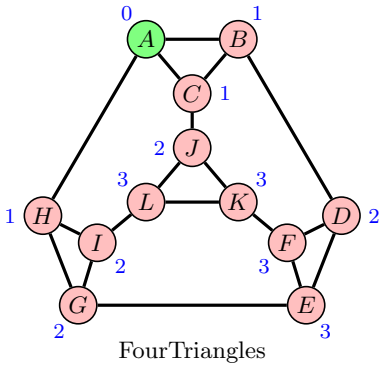
4.4. Let G be a connected graph, that is, every two of its vertices are joined by a path. The **distance** $d(u, v)$ between two vertices u and v is the number of edges of a shortest path from u to v; in particular, $d(u, u) = 0$. The **eccentricity** of a vertex v is the largest distance from v to a vertex of G. The **diameter** of G is the largest eccentricity of its vertices, the **radius** of G is the smallest eccentricity of its vertices, and the **center** of G is the set of vertices whose eccentricity equals the radius. Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



FourTriangles

In the figures above, the blue number next to each vertex is its distance from the green vertex A.

(a) Find the diameter, the radius, and the center of the Petersen graph.

Answer: The diameter and the radius are both 2, and the center is the set of all the vertices, since every vertex has eccentricity 2. No vertex is adjacent to all the others (every degree is 3), so every eccentricity is at least 2. The neighbors of A are B, E, F, and their other neighbors are C, G (of B), D, J (of E), and H, I (of F), so all the vertices are within distance 2 from A.[30] Similarly, the neighbors of F are A, H, I, and their other neighbors B, E, C, J, and D, G cover all the remaining vertices. Rotating the drawing by one fifth of a full turn maps the graph onto itself, outer vertices to outer vertices and inner vertices to inner vertices, so every vertex has eccentricity 2.

(b) Find the diameter, the radius, and the center of the Grötzsch graph.

Answer: The diameter and the radius are both 2, and the center is the set of all the vertices, since every vertex has eccentricity 2. No vertex is adjacent to all the others (the largest degree is 5), so every eccentricity is at least 2. The middle vertex K is adjacent to F, G, H, I, J, and each outer vertex is adjacent to two of them, so K has eccentricity 2. The neighbors of A are B, E, G, J, and through them A reaches C, F, H (via B), D, I (via E), and K (via G). The neighbors of F are B, E, K, and through them F reaches A, C, H (via B), D, I (via E), and G, J (via K). By the same rotation as for the Petersen graph, every vertex has eccentricity 2.

(c) Find the diameter, the radius, and the center of the FourTriangles graph.

Answer: The diameter and the radius are both 3, and the center is the set of all the vertices, since every vertex has eccentricity 3. Let v be any vertex, let x and y be the other two vertices of its triangle, and let v' be its neighbor outside the triangle. The vertices at distance 1 from v are x, y, and v'. A vertex at distance 2 is a neighbor of x, y, or v' other than these: the outside neighbor of x, the outside neighbor of y, and the two other vertices of the triangle of v'. Hence, at most $1 + 3 + 4 = 8$ of the 12 vertices are within distance 2 from v, and the eccentricity of v is at least 3. On the other hand, every two triangles are joined by an edge, so any vertex u reaches any vertex w of another triangle by at most one step inside the triangle of u, the joining edge, and at most one step inside the triangle of w. Hence, every eccentricity is exactly 3. In the figure above, the four triangles are A, B, C, D, E, F, G, H, I, and J, K, L, and for $v = A$, the vertices at distance 1 are B, C, H, the vertices at distance 2 are D, G, I, J, and the vertices at distance 3 are E, F, K, L.

[30] See the $1 + 3 + 6$ drawing in the solution of Problem 1.1(a) on page 27.

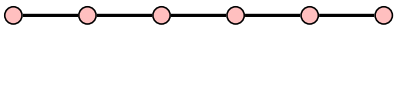
(d) Prove that a connected graph in which every vertex has degree at most 3 and whose diameter is 2 has at most 10 vertices.

Proof: Let v be any vertex. Since the diameter is 2, every other vertex is at distance 1 or 2 from v. At most 3 vertices are at distance 1: the neighbors of v. Every vertex at distance 2 is a neighbor of a neighbor of v, and each neighbor of v has at most 2 neighbors other than v. Hence, at most $3 \cdot 2 = 6$ vertices are at distance 2, and the graph has at most $1 + 3 + 6 = 10$ vertices. By part (a), the Petersen graph reaches this bound: every vertex has degree 3, the diameter is 2, and there are 10 vertices. In the $1+3+6$ drawing of the Petersen graph, the three groups are a vertex, its 3 neighbors, and the 6 vertices at distance 2 from it.[31]

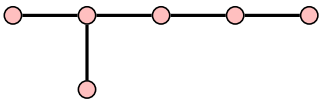
[31] See the solution of Problem 1.1(a) on page 27.

4.5. Let G be a connected graph, that is, every two of its vertices are joined by a path. The **distance** $d(u, v)$ between two vertices u and v is the number of edges of a shortest path from u to v; in particular, $d(u, u) = 0$. The **eccentricity** of a vertex v is the largest distance from v to a vertex of G. The **diameter** of G is the largest eccentricity of its vertices, the **radius** of G is the smallest eccentricity of its vertices, and the **center** of G is the set of vertices whose eccentricity equals the radius.

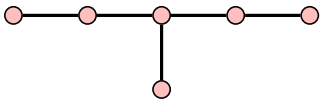
(a) Below are the 6 trees with 6 vertices, up to isomorphism. Find the diameter, the radius, and the center of each tree.



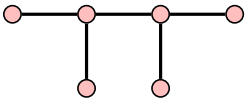
T_1



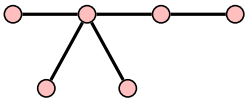
T_2



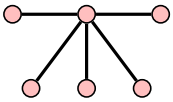
T_3



T_4

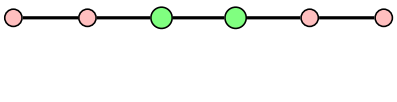


T_5

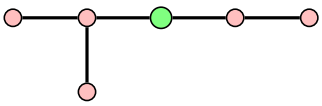


T_6

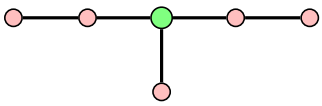
Answer: The diameter is 5 for T_1, 4 for T_2 and T_3, 3 for T_4 and T_5, and 2 for T_6. The radius is 3 for T_1, 2 for T_2, T_3, T_4, and T_5, and 1 for T_6. The centers are the green vertices below. The center is a single vertex in T_2, T_3, and T_6, whose diameters are even, and two adjacent vertices in T_1, T_4, and T_5, whose diameters are odd.[32]



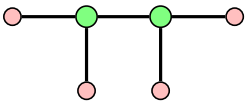
T_1



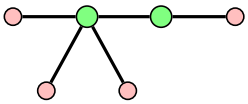
T_2



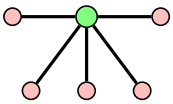
T_3



T_4



T_5



T_6

[32]These trees also appear in the solution of Problem 2.2 on page 41.

(b) Prove that the center of every tree consists of either a single vertex or two adjacent vertices.

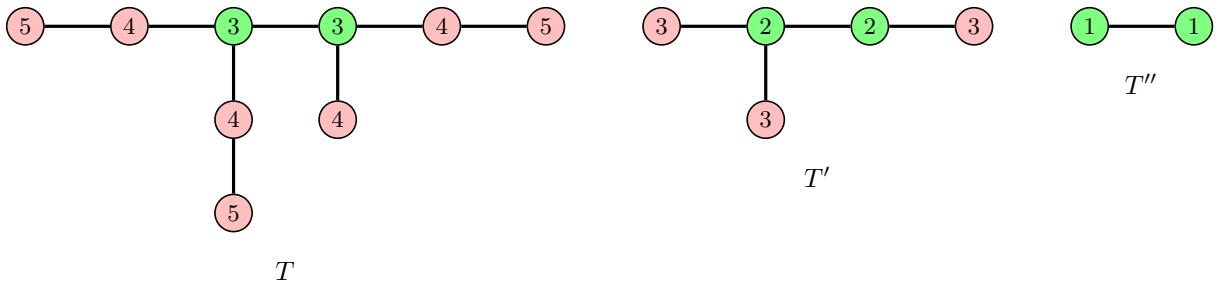
Proof: A tree with 1 or 2 vertices is its own center: a single vertex, or two adjacent vertices. Let T be a tree with $n \geq 3$ vertices, and let T' be the tree obtained from T by removing all its leaves (vertices of degree 1). T' is a tree with at least one vertex: removing leaves cannot create a cycle or disconnect the tree, since a path between two non-leaves never passes through a leaf, and not all the vertices are leaves, since $n \geq 3$. Moreover, the distance between two vertices of T' is the same in T and in T', since the unique path between them in T does not visit a leaf.

Claim: for every vertex v of T', its eccentricity in T is larger by exactly 1 than its eccentricity in T'. Indeed, a vertex w farthest from v in T is a leaf, since otherwise the path from v could be extended beyond w. Its neighbor on the path from v is in T' (or it is v), and its distance from v is $d(v, w) - 1$. Conversely, every vertex u of T' is not a leaf of T, so the path from v to u can be extended beyond u, and $d(v, u)$ is at most the eccentricity of v in T minus 1.

Since $n \geq 3$, a leaf of T is never in the center of T: its eccentricity is larger by 1 than the eccentricity of its only neighbor (every path from the leaf starts with the edge to its neighbor). By the claim, the eccentricities of the vertices of T' in T are their eccentricities in T' plus 1, so the vertices with the smallest eccentricity are the same in T and in T'. That is, T and T' have the same center.

Repeat the removal of all the leaves. Each round keeps the center and removes at least 2 vertices, since every tree with at least 2 vertices has at least 2 leaves.[33] The process ends with a tree with 1 or 2 vertices, which is the center of T: a single vertex or two adjacent vertices.

In the figure below, T' is obtained from T by removing its leaves, and T'' is obtained from T' in the same way. Each vertex keeps its position in T, which identifies it, and is labeled with its eccentricity in its own tree. The green center is the same in all three trees.



Remark: The rounds of leaf removal also show that in every tree, the radius is the diameter divided by 2, rounded up. Moreover, the center is a single vertex if the diameter is even, and two adjacent vertices if the diameter is odd, as in part (a). Indeed, by the claim, each round decreases the radius by exactly 1. Each round also decreases the diameter by exactly 2: a longest path of T ends at two leaves, and removing them leaves a path of T' that is shorter by 2, while every path of T' can be extended at both of its ends in T, since its ends are not leaves of T. The process ends with a single vertex, whose diameter and radius are 0, or with two adjacent vertices, whose diameter and radius are 1. Hence, after k rounds, either the diameter is $2k$ and the radius is k, or the diameter is $2k + 1$ and the radius is $k + 1$.

[33] This is shown in the solution of Problem 2.2(c) on page 41.

4.6. A graph is connected if every two of its vertices are joined by a path. A **bridge** in a connected graph G is an edge whose removal from G disconnects it.

In the examples below, the bridges are dashed: the left graph has one bridge and the right graph has two bridges.



(a) Let T be a tree with $n \geq 2$ vertices. Which edges of T are bridges? Justify your answer.

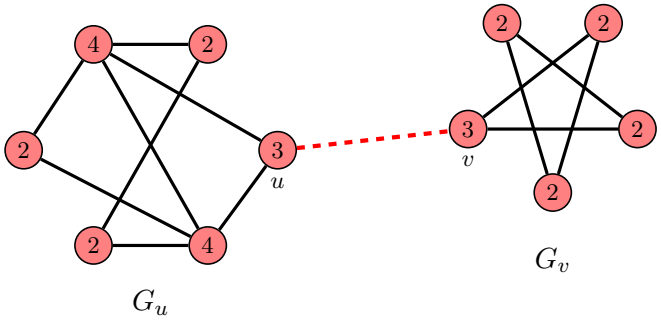
Answer: Every edge of T is a bridge. Remove an edge (u, v) of T. If u and v were still connected, there would be a path from u to v that does not use the edge (u, v). Together with the edge (u, v), this path would form a cycle in T, which is impossible since T is a tree. Hence, after the removal, u and v are disconnected.

(b) Prove that if G is a connected graph in which every vertex has an even degree, then G does not have a bridge.

Proof: Let G be a connected graph in which all the vertices have an even degree. Assume to the contrary that the edge $e = (u, v)$ is a bridge in G, and remove e from G. After the removal, since e is a bridge, by definition the graph is disconnected. Moreover, every vertex is connected to exactly one of u and v (exclusive or). It is connected to at least one of them, because every vertex had a path to u in G, and removing e can only cut such a path at e, leaving a path to u or to v. It is not connected to both of them, since otherwise u and v would be connected, and then every vertex would be connected to u, contradicting the fact that the graph is disconnected.

Let G_u be the subgraph induced by the vertices connected to u, and let G_v be the subgraph induced by the vertices connected to v. Thus, after the removal, the graph consists of two disjoint connected components G_u and G_v such that $u \in G_u$ and $v \in G_v$. The degree of any vertex in G_u except u is even, since the removal of e changes only the degrees of u and v. The degree of u in G_u is odd, because it is reduced by one after the removal of e. As a result, the sum of the degrees of all the vertices in G_u is odd, a contradiction, since the sum of the degrees counts every edge exactly twice (the handshake lemma),[34] and hence it is even.

Example: Below is a graph with 11 vertices. G_u is the subgraph composed of the 6 vertices on the left side, and G_v is the subgraph composed of the 5 vertices on the right side. The edge (u, v) (dashed and red) is a bridge because if it is removed, the graph is disconnected into G_u and G_v. Each vertex shows its degree: all the vertices except u and v have even degrees, and u and v have odd degrees.

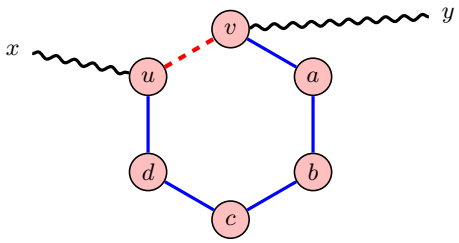


After removing the bridge, the degrees in G_u sum to $4+2+2+4+2+2 = 16$, and those in G_v sum to $2+2+2+2+2 = 10$. Both sums are even, as the handshake lemma requires. This is consistent with the proof: before the removal, u and v have the odd degree 3, so the graph is not a counterexample to the claim.

[34] The handshake lemma is the subject of Problem 5.1 on page 18.

(c) Prove that an edge of a connected graph is a bridge if and only if it does not belong to any cycle (a path that starts and ends at the same vertex and does not repeat edges).

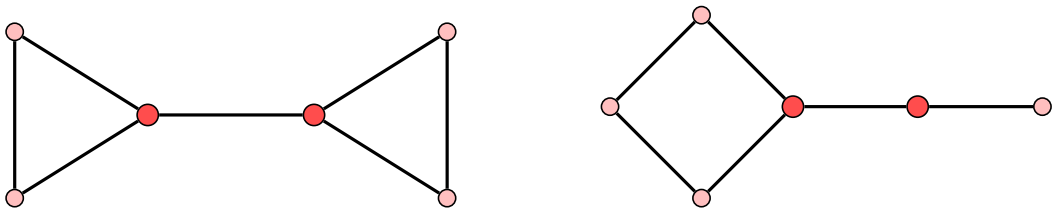
Proof: Suppose first that the edge (u, v) belongs to a cycle. After removing (u, v), the rest of the cycle is still a path from u to v. Let x and y be any two vertices, and take a path from x to y in G. If it uses the edge (u, v), replace this edge by the rest of the cycle. The result leads from x to y without the edge (u, v), and if it visits some vertex twice, the part between the two visits can be cut out, leaving a path from x to y. Hence, the graph remains connected, and (u, v) is not a bridge. Conversely, suppose that (u, v) is not a bridge. After its removal, there is still a path from u to v, and this path does not use the edge (u, v). Adding the edge (u, v) closes it into a cycle that contains (u, v).



In the figure above, a path from x to y uses the edge (u, v) (dashed), which belongs to a cycle. After the removal of (u, v), the path can go from u to v along the rest of the cycle (blue) instead.

4.7. A graph is connected if every two of its vertices are joined by a path. An **articulation point** in a connected graph G is a vertex whose removal from G, together with all the edges that touch it, disconnects it.

In the examples below, the articulation points are red: each of the two graphs has two articulation points.

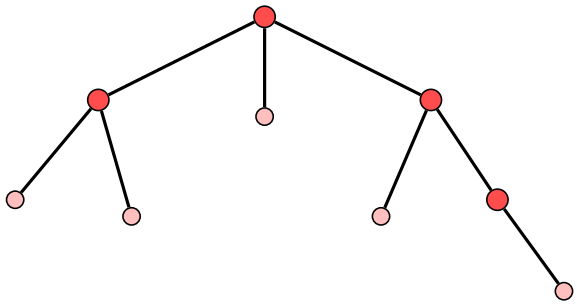


(a) Let T be a tree with $n \geq 3$ vertices. Which vertices of T are articulation points? Justify your answer.

Answer: The articulation points of T are exactly the vertices that are not leaves, that is, the vertices of degree at least 2.

A leaf is not an articulation point. Let x be a leaf, and let y and z be any two other vertices. A path cannot pass through x in its middle, since a vertex in the middle of a path uses two of its edges, while x has only one edge. Hence, the path from y to z in T does not visit x, and it survives the removal of x. Therefore, T without x is still connected.

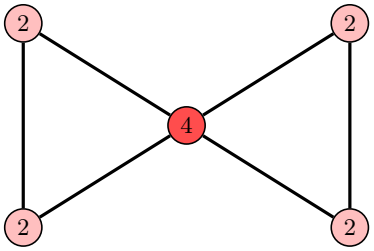
Any other vertex is an articulation point. Let v be a vertex of degree at least 2, and let a and b be two of its neighbors. In a tree, there is exactly one simple path between any two vertices, and the path between a and b is $a - v - b$. Hence, after removing v, there is no path from a to b, and the tree is disconnected. In fact, removing a vertex of degree d from a tree leaves exactly d connected components.[35]



The articulation points of a tree are the vertices that are not leaves (red).

(b) A connected graph in which every vertex has an even degree may have an articulation point. Find such a graph with the fewest vertices, and explain why no such graph with fewer vertices exists.

Answer: The example with the fewest vertices is the **bow-tie** graph below: two triangles that share one vertex. Each vertex shows its degree. The shared vertex has degree 4 and the other four vertices have degree 2, so all the degrees are even. Removing the shared vertex leaves two disjoint edges, so it is an articulation point.



[35] See also the solution of Problem 2.2 on page 41.

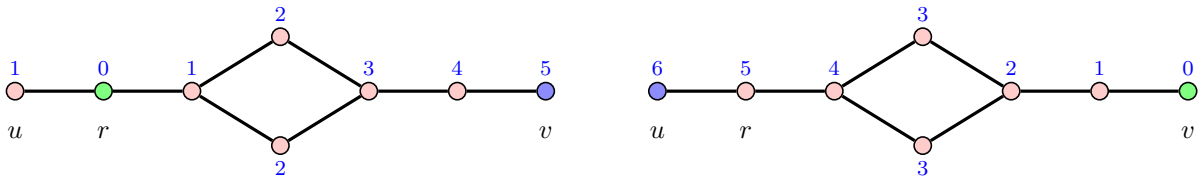
No example with fewer vertices exists. A graph with 1 vertex has no articulation point, since removing its only vertex leaves no vertices. In a graph with 2 to 4 vertices, every degree is at most 3, so the even degrees are 0 or 2. A connected graph with at least 2 vertices has no vertex of degree 0, so all its degrees are 2, and a connected graph in which all the degrees are 2 is a cycle.[36] Removing a vertex from a cycle leaves a path, which is connected. Hence, no graph with at most 4 vertices is an example.

The bow-tie is the only example with 5 vertices. An exhaustive check of all the graphs with 5 vertices confirms this, and here is a direct proof. Let v be an articulation point of such a graph. Removing v leaves at least two connected components, with 4 vertices in total. A component cannot have a single vertex x, since then the only possible neighbor of x is v, so the degree of x is at most 1, while it is even and positive (the graph is connected). Hence, there are exactly two components, each with 2 vertices. Each of these 4 vertices has at most two possible neighbors, the other vertex of its component and v, and its degree is even and positive, so it is adjacent to both. Thus, the graph consists of two triangles that share the vertex v, which is the bow-tie.

(c) Prove that every connected graph with at least 2 vertices has at least two vertices that are not articulation points.

Proof: Let the distance $d(x, y)$ be the number of edges of a shortest path from x to y. Let r be any vertex, and let v be a vertex farthest from r. Since there are at least 2 vertices, $v \neq r$. Let $w \neq v$ be any vertex, and consider a shortest path from w to r. Every vertex in the middle of this path is closer to r than w is, so its distance from r is smaller than $d(w, r) \leq d(v, r)$. Hence, v is not on this path, and the path survives the removal of v. Therefore, after removing v, every vertex is still connected to r, and v is not an articulation point. Now let u be a vertex farthest from v. By the same argument, with v in the role of r, the vertex $u \neq v$ is not an articulation point either.

In the example below, the blue numbers on the left are the distances from the green vertex r, and v (blue) is the vertex farthest from r. The blue numbers on the right are the distances from v (now green), and u (blue) is the vertex farthest from v. Indeed, neither v nor u is an articulation point.



[36] See the solution of Problem 5.4(b) on page 95.

4.8. A graph is connected if every two of its vertices are joined by a path. A **bridge** in a connected graph G is an edge whose removal from G disconnects it. An **articulation point** in a connected graph G is a vertex whose removal from G, together with all the edges that touch it, disconnects it.

(a) Let G be a connected graph with at least 3 vertices, and let (u, v) be a bridge in G. Prove that at least one of u and v is an articulation point.

Proof: If both u and v had degree 1, then the edge (u, v) alone would be the whole connected graph G, contradicting the fact that G has at least 3 vertices. Hence, without loss of generality, the degree of u is at least 2, and u has a neighbor $w \neq v$. It remains to show that removing u, together with all its edges, disconnects w from v, and hence u is an articulation point.

Assume to the contrary that after the removal of u there is still a path P from w to v. The path P does not visit u, so it does not use the edge (u, v). Now go back to G and remove only the edge (u, v). The vertices u and v are still connected: go from u to w along the edge (u, w) and then from w to v along P. Therefore, the graph remains connected. Indeed, any two vertices x and y are joined by a path in G, and if this path uses the edge (u, v), replace this edge by the route $u - w - \cdots - v$ described above. The result leads from x to y without the edge (u, v), and if it visits some vertex twice, the part between the two visits can be cut out, leaving a path. This contradicts the fact that (u, v) is a bridge.

The same argument shows that every endpoint of a bridge whose degree is at least 2 is an articulation point.

(b) Show an example of a bridge such that one of its endpoints is not an articulation point.

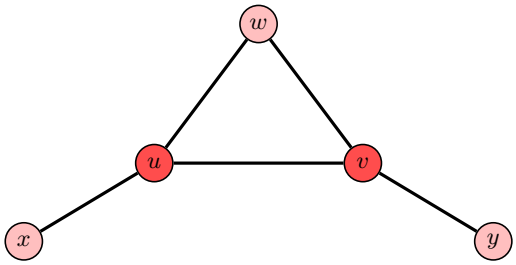
Answer: The path $a - b - c$ below is an example: both edges are bridges (dashed), the middle vertex b is an articulation point (red), but the endpoints a and c are not, since removing either of them leaves a single edge, which is connected.

More generally, any bridge with an endpoint of degree 1 is an example, since a vertex x of degree 1 is never an articulation point.[37] Indeed, let y and z be any two other vertices. Since G is connected, there is a simple path from y to z. This path does not pass through x, since a vertex in the middle of a simple path uses two of its edges, while x has only one edge. Hence, the path survives the removal of x, and the graph without x is still connected.



(c) Show that an edge whose two endpoints are articulation points is not necessarily a bridge.

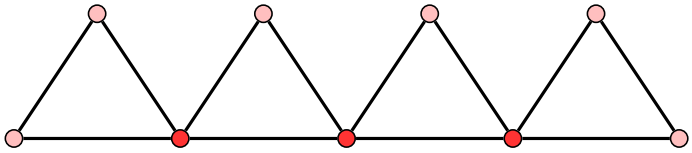
Answer: The edge (u, v) below is an example. Removing u disconnects x from the rest of the graph, and removing v disconnects y. Hence, both u and v are articulation points (red). However, the edge (u, v) is not a bridge: after its removal, u and v are still connected through w. More generally, an edge that belongs to a cycle is never a bridge.[38]



[37] See also the solution of Problem 4.7(a) on page 86.
[38] See the solution of Problem 4.6(c) on page 85.

(d) Show that a connected graph without bridges may have articulation points: for every $k \geq 2$, find a connected graph with $2k+1$ vertices and no bridges, in which $k-1$ vertices are articulation points.

Answer: Take a chain of k triangles in which every two consecutive triangles share one vertex, as shown below for $k = 4$. It has $3k - (k-1) = 2k+1$ vertices. No edge is a bridge: every edge (u, v) belongs to a triangle $u - v - w$, so after removing it, u and v are still joined through w, and the graph stays connected. Each of the $k-1$ shared vertices (red) is an articulation point: removing it separates the triangles before it from the triangles after it. No other vertex is an articulation point: removing it leaves the other two vertices of its triangle joined by an edge, so the chain stays connected. Thus, for large k, almost half of the vertices are articulation points, although there are no bridges.



The construction is the best possible. Every connected graph with $n \geq 2$ vertices and no bridges has at most $(n-3)/2$ articulation points, so for $n = 2k+1$, the chain of triangles has the largest possible number of articulation points, $k-1$. Equivalently, if such a graph has a articulation points, then $n \geq 2a+3$. The proof is by induction on a. If $a = 0$, then $n \geq 3$, since a connected graph with 2 vertices is a single edge, which is a bridge. If $a \geq 1$, let v be an articulation point, and let $C_1, \ldots, C_m$, with $m \geq 2$, be the connected components that remain after removing v. Let G_i be the graph formed by the vertices of C_i, the vertex v, and all the edges of G among them, and let $n_i \geq 2$ be its number of vertices. Each G_i is connected, and it has no bridge: an edge whose removal disconnects G_i would also disconnect G, since the rest of G is attached to G_i only at v. The vertex v is not an articulation point of G_i, since C_i is connected. Every other vertex u of G_i is an articulation point of G if and only if it is an articulation point of G_i, since after removing u, all the other graphs G_j remain attached to v. Hence, $a = 1 + a_1 + \cdots + a_m$, where $a_i < a$ is the number of articulation points of G_i, and by induction, $n_i \geq 2a_i + 3$. Since every two of the graphs G_i share only the vertex v, and $m \geq 2$,

$$n = 1 + \sum_{i=1}^{m} (n_i - 1) \;\geq\; 1 + \sum_{i=1}^{m} (2a_i + 2) \;=\; 1 + 2(a-1) + 2m \;\geq\; 2a + 3.$$

5 Degrees and Degree Sequences

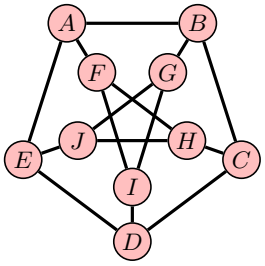
5.1. The **handshake lemma** states that in every graph, the sum of the degrees of all the vertices is twice the number of edges.

(a) Explain why the handshake lemma holds.

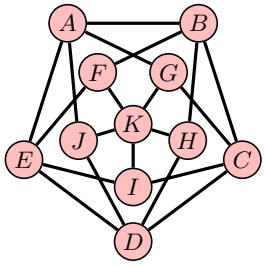
Answer: In a simple graph, each edge (u, v) contributes 1 to the degree of u and 1 to the degree of v. Hence, every edge contributes exactly 2 to the sum of the degrees, and the sum of the degrees is twice the number of edges.

The lemma also holds for graphs with parallel edges and self-loops. Each of several parallel edges between u and v still contributes 1 to the degree of u and 1 to the degree of v, and a self-loop (u, u) contributes 2 to the degree of u, one for each of its two ends. Hence, every edge still contributes exactly 2 to the sum of the degrees.

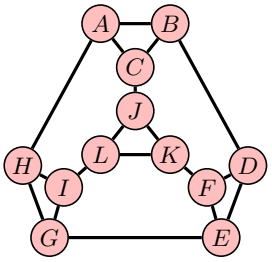
(b) Write the degree sequences (the lists of the degrees of their vertices) of the Petersen graph, the Grötzsch graph, and the FourTriangles graph drawn below. Use the handshake lemma to compute the number of edges of each graph.



Petersen



Grötzsch



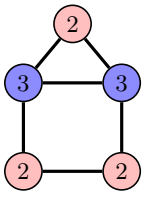
FourTriangles

Answer: In the Petersen graph, every vertex has degree 3, so its degree sequence consists of ten 3's; the sum is 30, and the graph has $30/2 = 15$ edges. In the Grötzsch graph, K has degree 5, each of A, B, C, D, E has degree 4, and each of F, G, H, I, J has degree 3. Its sorted degree sequence is $(5, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3)$; the sum is $5 + 20 + 15 = 40$, and the graph has 20 edges. In the FourTriangles graph, every vertex has degree 3, so its degree sequence consists of twelve 3's; the sum is 36, and the graph has 18 edges.

(c) Conclude that in every graph, the number of vertices with an odd degree is even.

Proof: By the handshake lemma, the sum of all the degrees is even. The sum of the even degrees is even, and therefore the sum of the odd degrees is also even. A sum of odd numbers is even only if the number of summands is even. Hence, the number of vertices with an odd degree is even.

For example, in the graph below, every vertex shows its degree. The sum of the degrees is 12, and the two vertices with an odd degree are blue.



(d) Can each of 9 people shake hands with exactly 3 of the others? Explain.

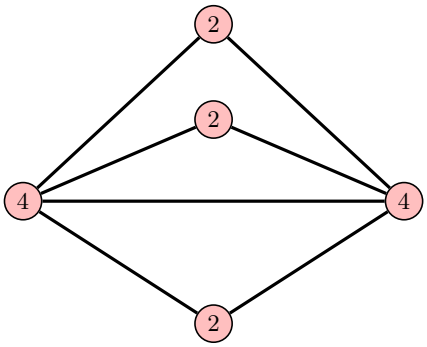
Answer: No. Consider the graph whose vertices are the 9 people and whose edges are the handshakes. Every vertex would have degree 3, so the sum of the degrees would be 27, an odd number, contradicting the handshake lemma. Equivalently, the graph would have 9 vertices with an odd degree, contradicting part (c).

5.2. Let G be a simple (no self-loops and no parallel edges) undirected graph on the $n \geq 1$ vertices $\{v_1, v_2, \ldots, v_n\}$. If vertex v_i has d_i neighbors (equivalently, the degree of v_i is d_i), then the **degree sequence** of G is $D_G = (d_1, d_2, \ldots, d_n)$. A sequence D is **graphic** if there is a simple graph G whose degree sequence is D; such a graph G is called a **realization** of D.

Among the following four sequences, one is not graphic, one is graphic with exactly one realization up to isomorphism, one is graphic with exactly two non-isomorphic realizations, and one is graphic with exactly three non-isomorphic realizations. Which is which? If a sequence is not graphic, explain why. If a sequence is graphic, illustrate all of its non-isomorphic realizations.

(a) $D = (4, 4, 2, 2, 2)$

Answer: The sequence $(4, 4, 2, 2, 2)$ is graphic and has exactly one realization up to isomorphism. Each of the two vertices with degree 4 must be adjacent to all the other four vertices. These edges already give each of the other three vertices degree 2, so no additional edges are needed. Therefore, the graph below is the only realization of $(4, 4, 2, 2, 2)$. Each vertex is labeled with its degree.

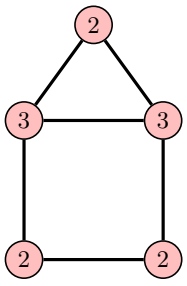


(b) $D = (3, 3, 2, 2, 2)$

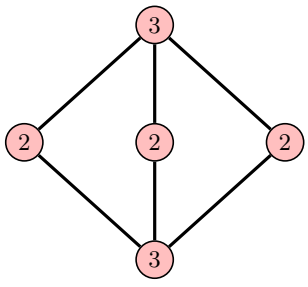
Answer: The sequence $(3, 3, 2, 2, 2)$ is graphic and has exactly two non-isomorphic realizations. Let x and y be the two vertices with degree 3, and let a, b, c be the other three vertices. Since the sum of the degrees, 12, counts every edge exactly twice (the handshake lemma), every realization has 6 edges.

- If x and y are not adjacent, then each of them is adjacent to all of a, b, c. These are 6 edges, and they give a, b, c degree 2. This is the complete bipartite graph $K_{2,3}$ (right).
- If x and y are adjacent, then each of them has two more neighbors among a, b, c, so they share at least one of them. If they share two, say a and b, then a and b already have degree 2, and c cannot get any neighbor. Hence, they share exactly one, say a, and the others are, say, x adjacent to b and y adjacent to c. Then a has degree 2, while b and c have degree 1, so the last edge is (b, c). This is the **house** graph (left): a square x, y, c, b with a roof a.

The two realizations are not isomorphic: in the house, the two vertices with degree 3 are adjacent, and in $K_{2,3}$ they are not. Each vertex is labeled with its degree.



the house



$K_{2,3}$

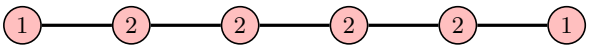
(c) $D = (4, 4, 1, 1, 1, 1)$

Answer: The sequence $(4, 4, 1, 1, 1, 1)$ is not graphic.

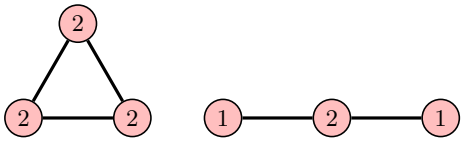
Proof: Each of the two vertices with degree 4 has at least 3 neighbors among the other four vertices, even if the two are adjacent. Hence, they have at least $3 + 3 - 4 = 2$ common neighbors among these four vertices, and each of these common neighbors has degree at least 2, a contradiction since all the other degrees are 1.

(d) $D = (2, 2, 2, 2, 1, 1)$

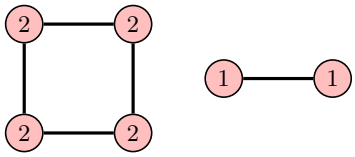
Answer: The sequence $(2, 2, 2, 2, 1, 1)$ is graphic and has exactly three non-isomorphic realizations. In a graph in which every degree is 1 or 2, every connected component is a path or a cycle:[39] moving from a vertex of degree 1 (if any) along new edges, the route can continue through every vertex of degree 2, so it either ends at a vertex of degree 1 (a path) or returns to its start (a cycle). A path with at least 2 vertices has exactly two vertices with degree 1 (its ends), and a cycle has none. Hence, a realization consists of one path with $p \geq 2$ vertices, and of cycles, each with at least 3 vertices, on the other $6 - p$ vertices. Since $6 - p \leq 4$, there is at most one cycle, so $6 - p \in \{0, 3, 4\}$, that is, $p = 6$, $p = 3$, or $p = 2$. This gives the three realizations below: the path P_6, a triangle and a path with 3 vertices, and a cycle with 4 vertices and a single edge. They are not isomorphic: the first is connected, the second contains a triangle, and the third contains a cycle with 4 vertices. Each vertex is labeled with its degree.



P_6



C_3 and P_3



C_4 and P_2

[39] See also the solution of Problem 5.4(b) on page 95.

5.3. A sequence is **graphic** if it is the degree sequence of a simple graph, called a **realization** of the sequence. The **Havel–Hakimi algorithm** constructs a realization in rounds. Every vertex has a **residual degree**, which is initially its degree in the sequence. In each round, a **pivot** vertex with a positive residual degree d becomes adjacent to d other vertices with the highest residual degrees. Then the residual degree of the pivot becomes 0, and the residual degree of each of its d new neighbors decreases by 1.

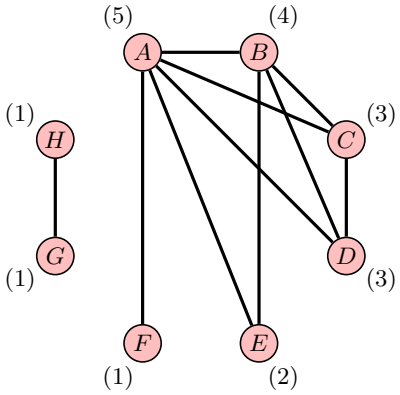
The algorithm stops either when all the residual degrees are 0, and then the new edges of all the rounds form a realization, or when a pivot with residual degree d cannot find d other vertices with positive residual degrees, and then the sequence is not graphic.

The sequence $D = (5, 4, 3, 3, 2, 1, 1, 1)$ is graphic. Let $A, B, \ldots, H$ be its eight vertices, with initial residual degrees $5, 4, 3, 3, 2, 1, 1, 1$, respectively. In parts (a), (b), and (d), for each round, show the sequence before the round, the new edges, and the sequence after the round, until the sequence becomes $(0, 0, 0, 0, 0, 0, 0, 0)$, and draw only the final realization.

(a) Construct a realization of the sequence D by running the Havel–Hakimi algorithm, in which the pivot in each round is a vertex with the highest residual degree. Tie-breaker rule: select the pivot and its neighbors in alphabetical order from A to H.

Answer: The table lists the rounds, in which the pivots and their new neighbors are selected by the rules above. The graph on the right is the resulting realization.

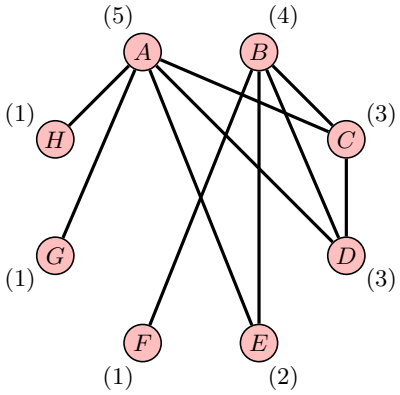
Round	Before	Pivot	New neighbors	After
1	$(5, 4, 3, 3, 2, 1, 1, 1)$	A	B, C, D, E, F	$(0, 3, 2, 2, 1, 0, 1, 1)$
2	$(0, 3, 2, 2, 1, 0, 1, 1)$	B	C, D, E	$(0, 0, 1, 1, 0, 0, 1, 1)$
3	$(0, 0, 1, 1, 0, 0, 1, 1)$	C	D	$(0, 0, 0, 0, 0, 0, 1, 1)$
4	$(0, 0, 0, 0, 0, 0, 1, 1)$	G	H	$(0, 0, 0, 0, 0, 0, 0, 0)$



(b) Construct a realization of the sequence D by running the Havel–Hakimi algorithm, in which the pivot in each round is a vertex with the smallest positive residual degree. Tie-breaker rules: select the pivot in alphabetical order from H to A, and select its neighbors in alphabetical order from A to H.

Answer: The table lists the rounds, in which the pivots and their new neighbors are selected by the rules above. The graph on the right is the resulting realization.

Round	Before	Pivot	New neighbors	After
1	$(5, 4, 3, 3, 2, 1, 1, 1)$	H	A	$(4, 4, 3, 3, 2, 1, 1, 0)$
2	$(4, 4, 3, 3, 2, 1, 1, 0)$	G	A	$(3, 4, 3, 3, 2, 1, 0, 0)$
3	$(3, 4, 3, 3, 2, 1, 0, 0)$	F	B	$(3, 3, 3, 3, 2, 0, 0, 0)$
4	$(3, 3, 3, 3, 2, 0, 0, 0)$	E	A, B	$(2, 2, 3, 3, 0, 0, 0, 0)$
5	$(2, 2, 3, 3, 0, 0, 0, 0)$	B	C, D	$(2, 0, 2, 2, 0, 0, 0, 0)$
6	$(2, 0, 2, 2, 0, 0, 0, 0)$	D	A, C	$(1, 0, 1, 0, 0, 0, 0, 0)$
7	$(1, 0, 1, 0, 0, 0, 0, 0)$	C	A	$(0, 0, 0, 0, 0, 0, 0, 0)$



(c) Prove that the realizations from parts (a) and (b) are not isomorphic.

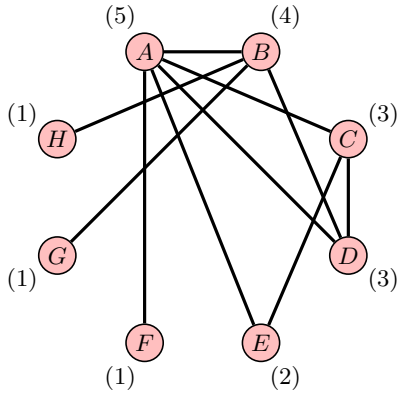
Proof: The two realizations G_a in part (a) and G_b in part (b) are not isomorphic. The following are four differences.

- In G_a the vertex with degree 5 is adjacent to the vertex with degree 4. This is not the case in G_b.
- In G_a two of the vertices with degree 1 are adjacent to each other. This is not the case in G_b.
- G_a has two connected components while G_b is a connected graph.
- In G_a there is a complete subgraph with four vertices. This is not the case in G_b.

(d) Find another sequence of pivots (in each round, any vertex with a positive residual degree may be the pivot) that generates a realization of D that is not isomorphic to the realizations from parts (a) and (b).

Answer: The table lists the rounds of one such sequence of pivots. The graph on the right is the resulting realization.

Round	Before	Pivot	New neighbors	After
1	$(5,4,3,3,2,1,1,1)$	A	B,C,D,E,F	$(0,3,2,2,1,0,1,1)$
2	$(0,3,2,2,1,0,1,1)$	H	B	$(0,2,2,2,1,0,1,0)$
3	$(0,2,2,2,1,0,1,0)$	G	B	$(0,1,2,2,1,0,0,0)$
4	$(0,1,2,2,1,0,0,0)$	C	D,E	$(0,1,0,1,0,0,0,0)$
5	$(0,1,0,1,0,0,0,0)$	D	B	$(0,0,0,0,0,0,0,0)$



(e) Prove that the third realization is not isomorphic to the realizations from parts (a) and (b).

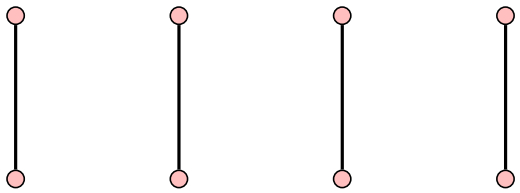
Proof: The realization G_d in part (d) is isomorphic neither to the realization G_a in part (a) nor to the realization G_b in part (b). This is because in G_d the vertex with degree 4 is adjacent to two vertices with degree 1, while in G_a none of the neighbors of the vertex with degree 4 has degree 1, and in G_b the vertex with degree 4 is adjacent to only one vertex with degree 1.

Remark: The sequence D has exactly 14 realizations, up to isomorphism, and 13 of them are connected. The only realization that is not connected is the realization G_a of part (a). This can be checked by an exhaustive search.

5.4. A simple graph is 1**-regular** if every vertex has degree 1, and it is 2**-regular** if every vertex has degree 2. For example, the cycle C_n with $n \geq 3$ vertices is 2-regular. For $n \geq 3$, let $p(n)$ be the number of ways to write n as a sum of integers that are at least 3, where the order of the summands does not matter. For example, $p(7) = 2$, since $7 = 7$ and $7 = 4 + 3$.

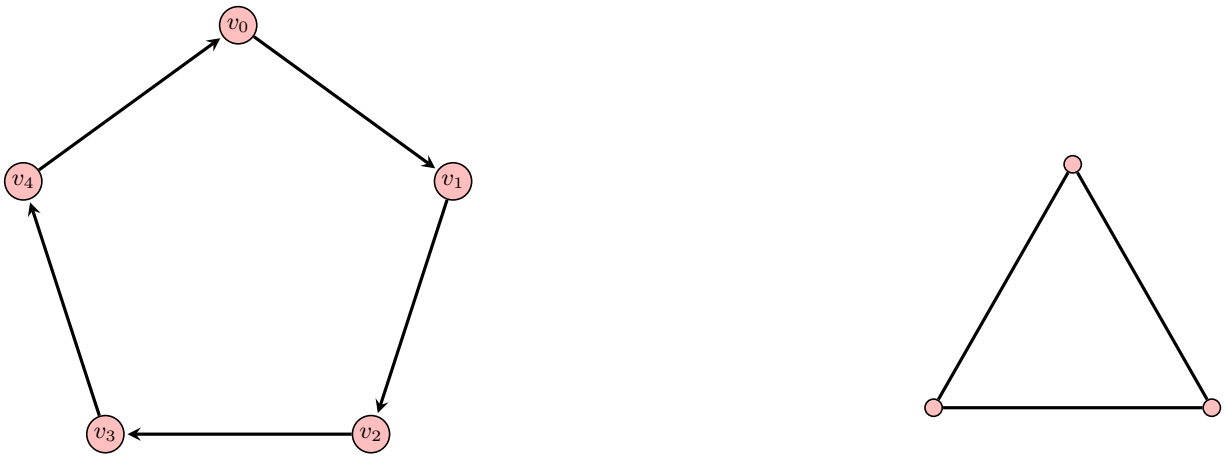
(a) Find all n for which there exists a 1-regular graph with n vertices. For each such n, find a 1-regular graph with n vertices, and prove that it is the only one up to isomorphism.

Answer: A 1-regular graph with n vertices exists if and only if n is even, and then the only one, up to isomorphism, is the **matching** with $n/2$ edges, no two of which share an endpoint. For every even $n \geq 2$, this matching is 1-regular. Conversely, in a 1-regular graph, every vertex v has exactly one neighbor, its **partner** v', and the partner of v' is v itself, since v is a neighbor of v'. Hence, the vertices are split into pairs $\{v, v'\}$, every pair is joined by an edge, and these are all the edges. Consequently, n is even, and the graph is a matching with $n/2$ edges. Finally, any two matchings with $n/2$ edges are isomorphic: map the pairs of one of them to the pairs of the other, in any order. The figure below shows the 1-regular graph with 8 vertices.



(b) Prove that every 2-regular graph is a disjoint union of cycles, each with at least 3 vertices.

Proof: Consider a connected component of a 2-regular graph, and move from one of its vertices v_0 to a neighbor v_1, then from v_1 to its other neighbor v_2, and so on: from each vertex, continue along the edge that was not used to enter it. Since the graph is finite, some vertex is eventually reached for the second time. The first such vertex must be v_0: every other vertex v_i of the route already has both of its edges, to v_{i-1} and to v_{i+1}, so the route cannot enter it again. Hence, the route closes a cycle $v_0, v_1, \ldots, v_{k-1}, v_0$. Every vertex of this cycle has both of its edges on the cycle, so no edge leaves the cycle, and the cycle is the whole component. Finally, every cycle has at least 3 vertices, since the graph has no self-loops and no parallel edges. In the figure below, a 2-regular graph has two components: the route that starts at v_0 closes the cycle $v_0, v_1, v_2, v_3, v_4, v_0$, and the other component is a cycle with 3 vertices.



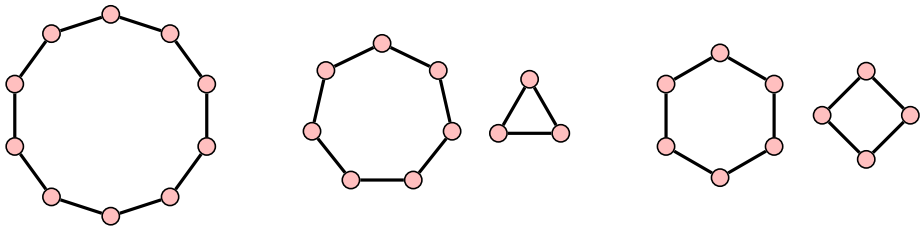
(c) Prove that for every $n \geq 3$, the number of non-isomorphic 2-regular graphs with n vertices is $p(n)$.

Proof: By part (b), a 2-regular graph with n vertices is a disjoint union of cycles whose numbers of vertices are integers that are at least 3 and sum to n. Two such graphs are isomorphic if and only if they have the same numbers of vertices in their cycles, up to order. Indeed, an isomorphism maps every cycle (connected component) to a cycle with the same number of vertices. Conversely, if the numbers are the same, then pairing the cycles of equal length and mapping each cycle onto its partner along the cycle is an isomorphism. Moreover, every way to write n as such a sum is obtained by taking disjoint cycles with the given numbers of vertices. Hence, the non-isomorphic 2-regular graphs with n vertices correspond exactly to the ways to write n as a sum of integers that are at least 3, and there are $p(n)$ of them.

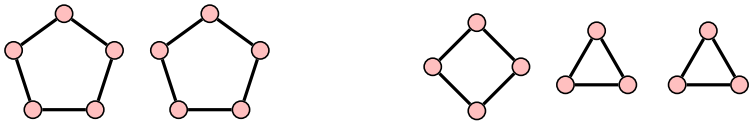
(d) Find $p(n)$ for $3 \leq n \leq 10$, and draw all the non-isomorphic 2-regular graphs with 10 vertices.

Answer: The values of $p(n)$ for $3 \leq n \leq 10$ are given in the table below. For example, $p(10) = 5$, since $10 = 10 = 7 + 3 = 6 + 4 = 5 + 5 = 4 + 3 + 3$, and the five 2-regular graphs with 10 vertices are drawn below.

n	3	4	5	6	7	8	9	10
$p(n)$	1	1	1	2	2	3	4	5



C_{10} $\quad$ C_7 and C_3 $\quad$ C_6 and C_4



C_5 and C_5 $\quad$ C_4, C_3, and C_3

Remark 1: The ways to write n as a sum of positive integers, where the order of the summands does not matter, are called the **partitions** of n. Thus, $p(n)$ is the number of partitions of n into parts that are at least 3. There is no simple closed formula for $p(n)$, and it grows quickly: for example, $p(20) = 49$ and $p(30) = 331$.

Remark 2: In most texts, $p(n)$ denotes the number of all the partitions of n, with no restriction on the parts. For example, 20 has 627 partitions, and only 49 of them have all their parts at least 3.

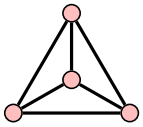
5.5. A simple graph is 3-**regular**, also called **cubic**, if every vertex has degree 3.

(a) Explain why there is no 3-regular graph with an odd number of vertices.

Answer: In a 3-regular graph with n vertices, the sum of the degrees is $3n$. This sum counts every edge exactly twice (the handshake lemma),[40] so it is even, while $3n$ is odd when n is odd.

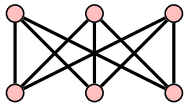
(b) Draw the only 3-regular graph with 4 vertices.

Answer: The complete graph K_4. In a 3-regular graph with 4 vertices, every vertex is adjacent to all the other 3 vertices, so the graph must be K_4.

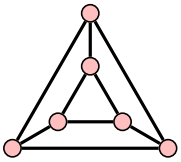


(c) Draw two non-isomorphic 3-regular graphs with 6 vertices, and explain why they are not isomorphic. Prove that there are no others, up to isomorphism.

Answer: The complete bipartite graph $K_{3,3}$ (left) and the prism graph (right). They are not isomorphic because the prism contains a triangle while $K_{3,3}$ does not: in $K_{3,3}$, every edge joins the top side to the bottom side, so any cycle alternates between the two sides and has an even length.



$K_{3,3}$



prism

Claim: Every 3-regular graph with 6 vertices is isomorphic to $K_{3,3}$ or to the prism.

Proof: Let G be a 3-regular graph with 6 vertices. *Case 1:* G has a triangle a, b, c. Let x, y, z be the other three vertices. Each of a, b, c has exactly one neighbor outside the triangle, so exactly 3 edges join the triangle to x, y, z. The degrees of x, y, z sum to 9, and only 3 of them come from edges to the triangle, so the edges among x, y, z contribute 6, that is, there are 3 such edges. These are all the possible edges among x, y, z, so x, y, z form a triangle, and each of them has exactly one neighbor among a, b, c. Hence, the 3 edges between the two triangles join their vertices in pairs, and G is the prism.
Case 2: G has no triangle. Let a be any vertex, let b, c, d be its neighbors, and let x and y be the other two vertices. No two of b, c, d are adjacent, since otherwise they would form a triangle with a. Hence, each of b, c, d has its two other neighbors among x and y, so it is adjacent to both of them. Then x and y already have their 3 neighbors b, c, d, so they are adjacent neither to a nor to each other. Thus, every edge joins the side $\{a, x, y\}$ to the side $\{b, c, d\}$, all 9 such edges exist, and G is $K_{3,3}$.

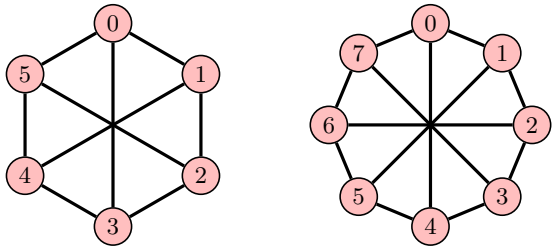
Another proof of the claim: Every vertex of a 3-regular graph with 6 vertices is non-adjacent to exactly $5 - 3 = 2$ other vertices. Hence, the complement graph (the graph whose edges are exactly the non-edges) is a graph in which every vertex has degree 2. Such a graph on 6 vertices is a collection of disjoint cycles, each with at least 3 vertices: either one cycle with 6 vertices or two triangles. The complement of two triangles is $K_{3,3}$, and the complement of a cycle with 6 vertices is the prism. Since two graphs are isomorphic if and only if their complements are isomorphic, there are exactly two 3-regular graphs with 6 vertices.[41]

Remark: The two graphs differ in two more properties. *Bipartite:* $K_{3,3}$ is bipartite: its two sides are the top and the bottom vertices. The prism is not bipartite since it contains a triangle, and in any split into two sides, two of the three vertices of a triangle are on the same side. *Planar:* A graph is **planar** if it can be drawn in the plane without crossing edges. The drawing of the prism above has no crossings, so the prism is planar. In contrast, it is known that $K_{3,3}$ is not planar: every drawing of $K_{3,3}$ has crossing edges (the details are omitted). Thus, each of the two graphs has exactly one of the two properties.

[40] The handshake lemma is the subject of Problem 5.1 on page 18.
[41] See the solutions of Problems 5.4(b) and 1.8(b),(c) on pages 95 and 36.

(d) Prove that for every even $n \geq 4$ there exists a 3-regular graph with n vertices.

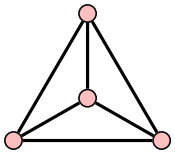
Proof by an explicit construction: For an even $n \geq 4$, place the vertices $0, 1, \ldots, n-1$ on a cycle, and add a chord between every vertex and the vertex opposite to it on the cycle. That is, the three neighbors of vertex $0 \leq i \leq n-1$ are $i-1$, $i+1$, and $i+n/2$, where the additions and subtractions are done modulo n. The definition is consistent: $j = i + n/2$ if and only if $i = j + n/2$, since $n/2 + n/2 = n$. Moreover, since $n \geq 4$, the three neighbors of i are different from each other and from i: $i - 1 \neq i + 1$ since $n \geq 3$, and $i + n/2$ differs from $i - 1$ and $i + 1$ since $n/2$ is neither 1 nor $n - 1$. Therefore, the graph is simple, and the degree of every vertex is exactly 3. For $n = 4$, this is the complete graph K_4. For $n = 6$, it is $K_{3,3}$, with the even and the odd vertices as its two sides, and for $n = 8$, it is known as the **Wagner graph**. See the figure below for the case $n = 6$ on the left side and the case $n = 8$ on the right side.



Proof by building blocks: Every even $n \geq 4$ can be written as $n = 4h$ or as $n = 4h + 2$ for some integer $h \geq 1$. If $n = 4h$, take h disjoint copies of K_4. If $n = 4h + 2$, take $h - 1$ disjoint copies of K_4 and one copy of $K_{3,3}$ (or of the prism) from part (c). In both cases, every vertex has degree 3, since it has degree 3 in its own copy and no edge joins two different copies.

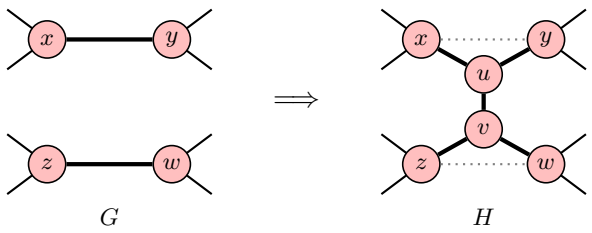
Remark: Except for $n = 4$ and $n = 6$, the graphs of this proof are not connected, while the other two proofs construct connected 3-regular graphs.

Proof by induction: The base case is $n = 4$: the degree of every vertex in the complete graph K_4 is 3. See the figure below.

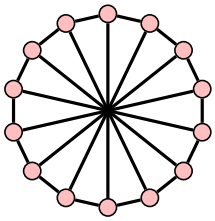


Assume that for some even $n \geq 6$ there exists a simple 3-regular graph G with $n - 2$ vertices. Since the sum of the degrees, $3(n-2)$, counts every edge exactly twice (the handshake lemma), G has $3(n-2)/2 \geq 6$ edges. Let (x, y) be an edge of G. Since the degrees of x and y are 3, at most $3 + 3 - 1 = 5$ edges of G share a vertex with (x, y), including (x, y) itself. Hence, G has another edge (z, w) that shares no vertex with (x, y), and the four vertices x, y, z, w are different.

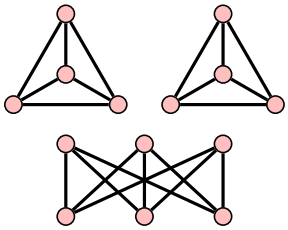
Let H be the graph G without the edges (x, y) and (z, w), with two new vertices u and v, and with the five new edges (u, x), (u, y), (v, z), (v, w), and (u, v). The degree of u in H is 3 (its neighbors are x, y, v), and so is the degree of v (its neighbors are z, w, u). Each of the four vertices x, y, z, w lost one edge and gained one edge, so its degree remains 3, and the degrees of all the other vertices did not change. The graph H is simple, since u and v are new vertices, $x \neq y$, and $z \neq w$. Hence, H is a 3-regular graph with n vertices, and by induction the claim holds for every even $n \geq 4$. The figure below shows the change from G (left) to H (right).



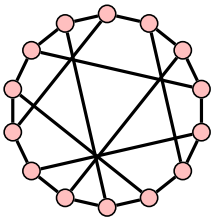
Three 3-regular graphs with 14 vertices: The figure below shows one 3-regular graph with 14 vertices from each of the three proofs: the cycle with chords between opposite vertices (left), two copies of K_4 and one copy of $K_{3,3}$ (middle), and a graph obtained from K_4 by five induction steps, drawn as a cycle with chords (right; the steps are omitted). No two of them are isomorphic: the middle graph is not connected, the left graph has a cycle of length 4, and the right graph has no cycle of length 3 or 4.



explicit construction

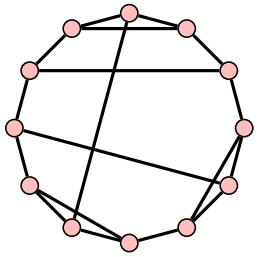


building blocks

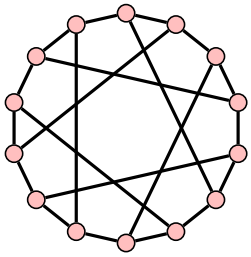


induction from K_4

Famous 3-regular graphs: Many famous graphs are 3-regular. The complete graph K_4 (4 vertices) is the smallest one. The complete bipartite graph $K_{3,3}$ and the prism (6 vertices) appear in part (c). The Petersen graph (10 vertices) is the smallest 3-regular graph with no cycle of length 3 or 4.[42] The FourTriangles graph (12 vertices), also known as the truncated tetrahedron graph, consists of four triangles, every two of them joined by one edge. The Frucht graph (12 vertices) has no symmetry at all: the only isomorphism from it to itself maps every vertex to itself. The Heawood graph (14 vertices) is the smallest 3-regular graph with no cycle of length 3, 4, or 5. The Frucht graph and the Heawood graph are drawn below, each as a cycle with chords.

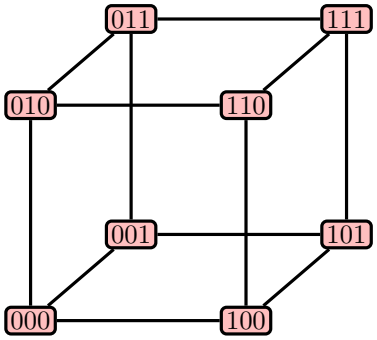


the Frucht graph



the Heawood graph

The **3-dimensional hypercube** (the cube) has 8 vertices, the binary strings of length 3, and two strings are adjacent if and only if they differ in exactly one bit. Every string differs from exactly 3 other strings in exactly one bit (flip one of its 3 bits), so the hypercube is 3-regular. The drawing below shows it as a cube: the front face contains the strings that end with 0, the back face contains the strings that end with 1, and every edge changes exactly one bit. The hypercube is bipartite: put the strings with an even number of ones on one side and the strings with an odd number of ones on the other side; flipping one bit always changes the parity of the number of ones, so every edge joins the two sides.



[42] See the solution of Problem 3.1(e) on page 56.

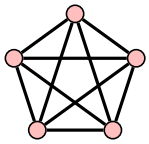
5.6. A simple graph is 4-**regular** if every vertex has degree 4.

(a) Explain why there is no 4-regular graph with fewer than 5 vertices.

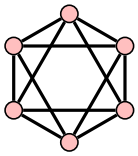
Answer: In a simple graph with n vertices, every vertex has at most $n-1$ neighbors. A vertex with degree 4 needs 4 different neighbors, so $n-1 \geq 4$, that is, $n \geq 5$.

(b) Draw a 4-regular graph with n vertices for each $n = 5, 6, 7, 8, 9$.

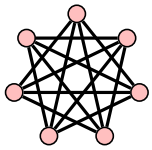
Answer: Here are five 4-regular graphs with 5, 6, 7, 8, and 9 vertices, drawn below: the complete graph K_5; the complete graph with 6 vertices without three pairwise disjoint edges (every vertex loses exactly one edge); the complement of the cycle C_7 (every vertex is adjacent to all the vertices except itself and its two neighbors on the cycle);[43] the complete bipartite graph $K_{4,4}$; and the graph whose 9 vertices are the cells of a 3×3 table, in which two cells are adjacent if and only if they are in the same row or in the same column (every cell has 2 neighbors in its row and 2 in its column).[44]



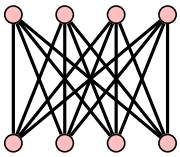
K_5
$n = 5$



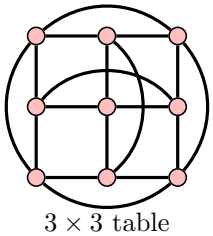
K_6 minus 3 edges
$n = 6$



complement of C_7
$n = 7$



$K_{4,4}$
$n = 8$



3×3 table
$n = 9$

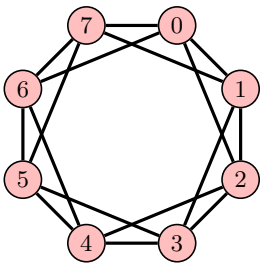
(c) Prove that for every $n \geq 5$ there exists a 4-regular graph with n vertices.

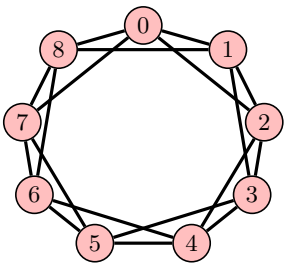
Proof by an explicit construction: For $n \geq 5$, construct the following graph with the vertices $0, 1, \ldots, n-1$. The four neighbors of vertex $0 \leq i \leq n-1$ are $i-2, i-1, i+1, i+2$, where the additions and subtractions are done modulo n (for example, $(n-1)+1 = 0$ and $1-2 = n-1$).

For example, for $n \geq 11$, the four neighbors of vertex 8 are vertices $\{6, 7, 9, 10\}$ and the four neighbors of vertex 1 are $\{n-1, 0, 2, 3\}$.

Observe that the edge definitions are consistent: if $j = i+1$ then $i = j-1$, and if $j = i+2$ then $i = j-2$. That is, j is a neighbor of i if and only if i is a neighbor of j. Moreover, since $n \geq 5$, the four neighbors $i-2, i-1, i+1, i+2$ of i are different from each other and from i (for $n = 4$, the vertices $i-2$ and $i+2$ are the same). Therefore, the graph is simple, and the degree of each vertex is exactly 4 as required.

See the figure below for the case $n = 8$ on the left side and the case $n = 9$ on the right side.

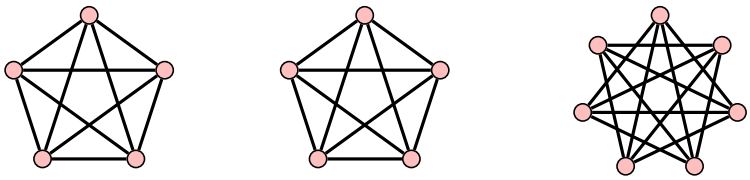




[43] By the solution of Problem 1.8(c) on page 36, there are exactly two 4-regular graphs with 7 vertices.
[44] This graph is the line graph $L(K_{3,3})$ in the solution of Problem 6.6(b) on page 112.

Proof by building blocks: Use the five graphs of part (b). Every $n \geq 5$ can be written as $n = 5k + r$ with $k \geq 1$ and $0 \leq r \leq 4$. Take $k - 1$ disjoint copies of K_5 and one copy of the building block with $5 + r$ vertices. Every vertex has degree 4, since it has degree 4 in its own copy and no edge joins two different copies.

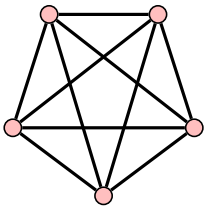
For example, $17 = 5 \cdot 3 + 2$, so $k = 3$ and $r = 2$. Hence, the resulting 4-regular graph with 17 vertices consists of two disjoint copies of K_5 and one copy of the complement of C_7, as shown below.



K_5 K_5 complement of C_7

Remark: Except for $5 \leq n \leq 9$, the graphs of this proof are not connected, while the other two proofs construct connected 4-regular graphs.

Proof by induction: The base case is $n = 5$. The degree of each vertex in the complete graph K_5 is 4. See the figure below.



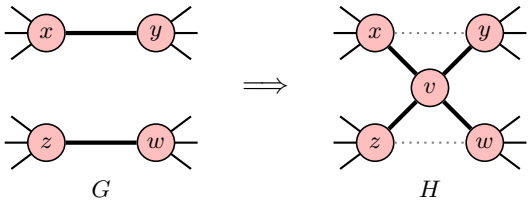
Assume that for some $n \geq 6$ there exists a simple graph G with $n-1$ vertices $0, 1, \ldots, n-2$, in which the degree of every vertex is 4. Therefore, the sum of the degrees of all the vertices of G is $4(n-1)$. Since this sum counts every edge exactly twice (the handshake lemma),[45] G has exactly $2n - 2$ edges.

Let (x, y) be any edge of G. Since the degrees of x and y are 4, at most $4 + 4 - 1 = 7$ edges of G share a vertex with (x, y), including (x, y) itself. Since $n \geq 6$, the graph G has $2n - 2 \geq 10$ edges. As a result, G has another edge (z, w) that shares no vertex with (x, y), and the four vertices x, y, z, w are different.

Let H be the graph G without the edges (x, y) and (z, w) and with a new vertex $v = n-1$ that is adjacent to the vertices x, y, z, w. By definition the degree of v in H is 4. Also, the degree in H of any vertex that is not x, y, z, w remains 4 as it was in G. Finally, the degrees in H of the four vertices x, y, z, w remain 4 because each one of them lost an edge and gained a new edge with v. This relies on the fact that the edges (x, y) and (z, w) share no vertex: each of the four vertices x, y, z, w loses exactly one edge, and v has four different neighbors. Moreover, H is simple, since v is a new vertex and its four neighbors are different.

It follows that H has n vertices and that the degrees of all of them are 4 as required. Hence, by induction, the claim holds for every $n \geq 5$.

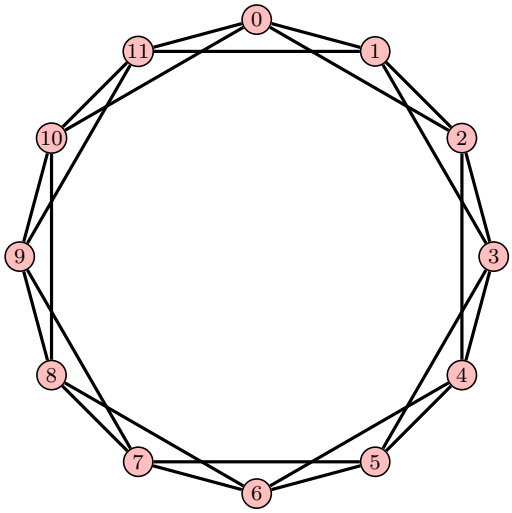
The figure below shows the changes from G (left) to H (right) for the four vertices x, y, z, w and the new vertex v. Note that there could be edges in G, and therefore also in H, between other pairs of vertices among the four vertices x, y, z, w.



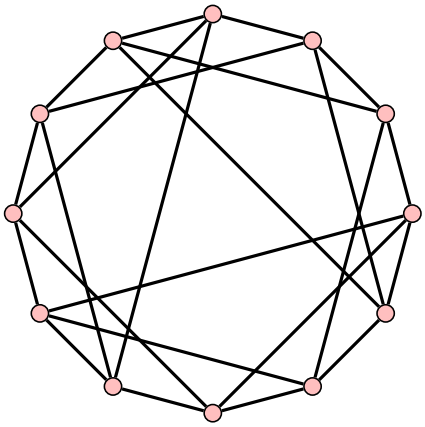
G H

[45] The handshake lemma is the subject of Problem 5.1 on page 18.

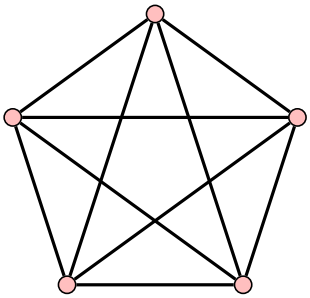
Three 4-regular graphs with 12 vertices: The figure below shows one 4-regular graph with 12 vertices from each of the three proofs: the explicit construction, in which every vertex is adjacent to the two nearest vertices on each side of the cycle (top left); K_5 and the complement of C_7 (bottom); and a graph obtained from K_5 by seven induction steps, drawn as a cycle with chords (top right; the steps are omitted). No two of them are isomorphic: the bottom graph is not connected, the top left graph has triangles (for example, three consecutive vertices of the cycle), and the top right graph has no triangle.

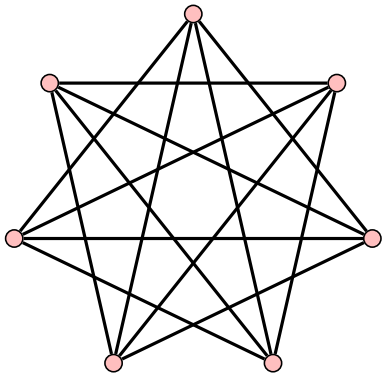


explicit construction



induction from K_5





building blocks

5.7. Prove that in every simple graph with at least 2 vertices, two vertices have the same degree. That is, prove that the degree sequence of such a graph contains at least one repeated value.

Proof: Consider a simple graph with $n \geq 2$ vertices. There are only n possible values for a degree of a vertex in a simple graph: $0, 1, \ldots, n-1$. However, if the degree of a vertex is 0 (no neighbors), then no other vertex may have degree $n-1$ (being a neighbor of all the other vertices), and if the degree of a vertex is $n-1$ (being a neighbor of all the other vertices), then no other vertex may have degree 0 (no neighbors). As a result, the n degrees take at most $n-1$ different values: either $0, 1, \ldots, n-2$ or $1, \ldots, n-1$. By the pigeonhole principle, two of the n degrees must be equal.

Generalization: For every $n \geq 2$, the only graphic sequence of length n without zeros (that is, of a graph without isolated vertices) in which all the degrees are different except one degree that appears twice is:

$$(n-1, n-2, \ldots, \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor, \lfloor n/2 \rfloor, \lfloor n/2 \rfloor - 1, \ldots, 2, 1),$$

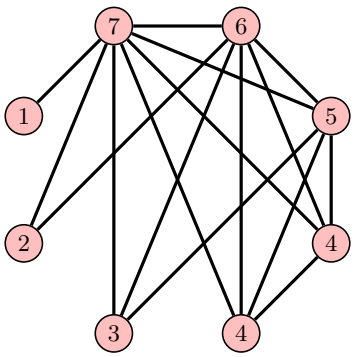
where $\lfloor n/2 \rfloor$ is $n/2$ rounded down, that is, $n/2$ for an even n and $(n-1)/2$ for an odd n. Moreover, this sequence has exactly one realization, up to isomorphism. These realizable sequences for $n = 2, 3, 4, 5, 6, 7$ are $(1, 1)$, $(2, 1, 1)$, $(3, 2, 2, 1)$, $(4, 3, 2, 2, 1)$, $(5, 4, 3, 3, 2, 1)$, and $(6, 5, 4, 3, 3, 2, 1)$, respectively. For $n = 8$ and $n = 9$, the unique realizations of the sequences $(7, 6, 5, 4, 4, 3, 2, 1)$ and $(8, 7, 6, 5, 4, 4, 3, 2, 1)$ are drawn at the end of this solution.

Proof sketch: Every graph with this degree sequence has a vertex u with degree $n-1$, which is adjacent to all the other vertices, and a vertex w with degree 1, which is therefore adjacent only to u. Removing u and w decreases the degree of each of the other $n-2$ vertices by exactly 1, and the result is the sequence of the same form for $n-2$, since $\lfloor n/2 \rfloor - 1 = \lfloor (n-2)/2 \rfloor$. Repeating this process ends with the sequence $(1, 1)$ for an even n, whose only realization is a single edge, or with the sequence $(2, 1, 1)$ for an odd n, whose only realization is a path with two edges. Since every step is forced, the realization is unique. The same reduction also shows that no other sequence of this kind is graphic (the details are omitted).

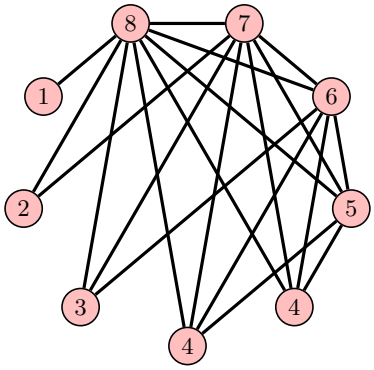
Remark 1: If zeros are allowed (that is, graphs with isolated vertices), then for every $n \geq 3$ there is exactly one more such sequence: the sequence of length $n-1$ described above followed by a single 0. For example, $(1, 1, 0)$ for $n = 3$ and $(3, 2, 2, 1, 0)$ for $n = 5$. Its unique realization is the realization for $n-1$ together with an isolated vertex. Examples of realizations with and without a zero are drawn at the end of this solution.

Remark 2: A sequence of this kind with two zeros is not graphic for $n \geq 3$: removing one of the two isolated vertices would leave a simple graph with $n - 1 \geq 2$ vertices in which all the degrees are different, which is impossible by the first part of this solution. (In a sequence with three or more zeros, the value 0 appears more than twice.) For $n = 2$, however, the sequence $(0, 0)$ with two zeros is graphic: its realization consists of two isolated vertices.

Realizations: The figure below shows the realizations of the sequences without zeros: on the left, the sequence $(7, 6, 5, 4, 4, 3, 2, 1)$ for the even $n = 8$, and on the right, the sequence $(8, 7, 6, 5, 4, 4, 3, 2, 1)$ for the odd $n = 9$. Each vertex is labeled with its degree, and the vertices are placed clockwise in non-increasing order of their degrees. In both graphs, the vertex with the highest degree is adjacent to all the other vertices, and the vertex with degree 1 is adjacent only to it. Removing these two vertices leaves the realization for $n - 2$.

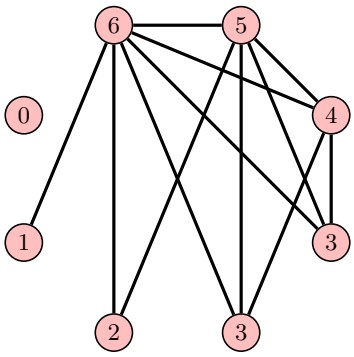


$n = 8$
$(7, 6, 5, 4, 4, 3, 2, 1)$

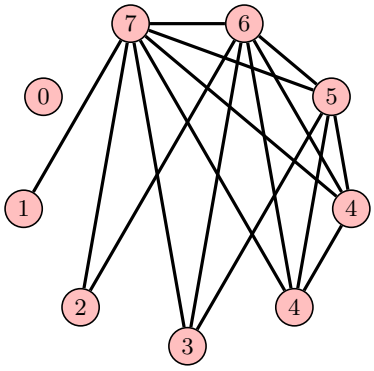


$n = 9$
$(8, 7, 6, 5, 4, 4, 3, 2, 1)$

The figure below shows the realizations of the sequences with one zero: on the left, the sequence $(6, 5, 4, 3, 3, 2, 1, 0)$ for $n = 8$, which is the realization for $n = 7$ together with an isolated vertex, and on the right, the sequence $(7, 6, 5, 4, 4, 3, 2, 1, 0)$ for $n = 9$, which is the realization for $n = 8$ together with an isolated vertex.



$n = 8$
$(6, 5, 4, 3, 3, 2, 1, 0)$



$n = 9$
$(7, 6, 5, 4, 4, 3, 2, 1, 0)$

6 Graph Coloring

6.1. A **vertex coloring** (or simply a **coloring**) of a graph assigns colors to its vertices so that every two adjacent vertices get different colors.

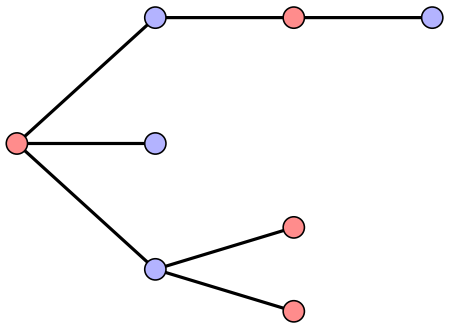
Notation: In the answers below, the colors are the positive integers $1, 2, \ldots$, and in the figures, color 1 is red, color 2 is blue, color 3 is green, and color 4 is orange. For a graph G, let $\chi(G)$ be the minimum number of colors needed to color G.

(a) What is the minimum number of colors needed to color a null graph (a graph with no edges) with $n \geq 1$ vertices? Explain.

Answer: 1 color. Since there are no edges, assigning the color 1 to all n vertices is a coloring, and at least one color is needed, since $n \geq 1$.

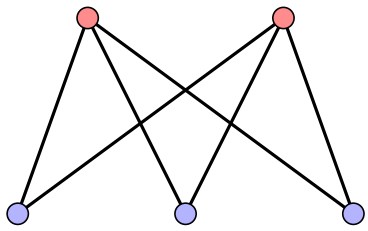
(b) What is the minimum number of colors needed to color a tree with $n \geq 2$ vertices? Explain.

Answer: 2 colors. A graph with at least one edge needs at least 2 colors, since the two endpoints of an edge must get different colors, and a tree with $n \geq 2$ vertices has an edge, since it is connected. To color a tree with 2 colors, fix a vertex r, and for every vertex w, let $d(w)$ be the number of edges on the unique path from r to w. Color w with 1 if $d(w)$ is even and with 2 if $d(w)$ is odd. For every edge (u, v), one of u and v is the vertex just before the other on the path from r, so $d(u)$ and $d(v)$ differ by exactly 1, and u and v get different colors.[46] Below is such a coloring of a tree, where r is the leftmost vertex: the vertices at even distance from r are red, and the vertices at odd distance from r are blue.



(c) What is the minimum number of colors needed to color a bipartite graph (a graph whose vertices can be split into two sides so that every edge joins the two sides) with at least one edge? Explain.

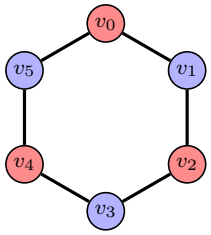
Answer: 2 colors. By definition, the vertices of a bipartite graph can be split into two sides X and Y so that every edge joins X and Y. Hence, coloring all the vertices of X with 1 and all the vertices of Y with 2 is a coloring. As in part (b), at least 2 colors are needed, since the graph has an edge. Below is such a coloring of the complete bipartite graph $K_{2,3}$, in which each of the 2 vertices on one side is adjacent to each of the 3 vertices on the other side. This graph is not a tree, since it contains cycles of length 4, so part (b) does not apply to it.



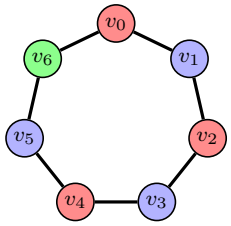
[46] This is shown in the solution of Problem 2.5(b) on page 47. Equivalently, every tree is bipartite.

(d) What is the minimum number of colors needed to color the cycle C_n with $n \geq 3$ vertices? Explain. Note that the answer is different when n is even and when n is odd.

Answer: $\chi(C_n) = 2$ for an even $n \geq 4$ and $\chi(C_n) = 3$ for an odd $n \geq 3$. Let the edges of the cycle be (v_i, v_{i+1}) for $0 \leq i \leq n-1$, where the indices are taken modulo n. Color v_i with $c_i = 1$ if i is even and with $c_i = 2$ if i is odd. If n is even, then $c_{n-1} = 2 \neq 1 = c_0$, and this is a coloring with two colors; one color is not enough, since C_n has edges. If n is odd, recolor v_{n-1} with 3, which differs from $c_{n-2} = 2$ and from $c_0 = 1$. Finally, assume to the contrary that C_n can be colored with two colors for an odd n. Since the colors alternate along the cycle, all the even-indexed vertices $v_0, v_2, \ldots, v_{n-1}$ have the same color. This is a contradiction since v_{n-1} and v_0 are adjacent. See below the colorings of C_6 with 2 colors and of C_7 with 3 colors (color 1 is red, color 2 is blue, and color 3 is green).



C_6: 2 colors

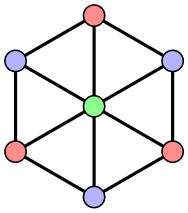


C_7: 3 colors

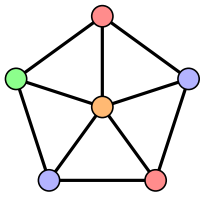
In particular, the cycle C_5 is triangle-free, but it cannot be colored with 2 colors, so it is not bipartite.[47]

(e) The **wheel** W_n with $n \geq 4$ vertices consists of a cycle with $n-1$ vertices and one more vertex, the **hub**, that is adjacent to all the vertices of the cycle. What is the minimum number of colors needed to color W_n? Explain.

Answer: $\chi(W_n) = 3$ for an odd $n \geq 5$ and $\chi(W_n) = 4$ for an even $n \geq 4$. The hub is adjacent to all the vertices of the cycle, so its color differs from all the colors used on the cycle. Hence, $\chi(W_n) \geq \chi(C_{n-1}) + 1$. Conversely, coloring the cycle with $\chi(C_{n-1})$ colors and the hub with one more color gives a coloring, so $\chi(W_n) = \chi(C_{n-1}) + 1$. By part (d), if n is odd, then the cycle has an even number $n-1$ of vertices and needs 2 colors, so $\chi(W_n) = 3$; if n is even, then the cycle has an odd number of vertices and needs 3 colors, so $\chi(W_n) = 4$. For example, W_4 is the complete graph K_4, which needs 4 colors. Below are colorings of W_7 with 3 colors and of W_6 with 4 colors.



W_7



W_6

Remark: A graph can be colored with 2 colors if and only if it has no cycle of odd length. If it has such a cycle, then by part (d), this cycle alone needs 3 colors. Conversely, assume that the graph has no cycle of odd length, and color each connected component separately, as in part (b): fix a vertex r of the component, and color every vertex w with 1 if its distance $d(r, w)$ is even and with 2 if it is odd. The distances of two adjacent vertices u and v from r differ by at most 1. If they were equal, then shortest paths from r to u and to v, together with the edge (u, v), would form a closed route of odd length, and such a route always contains a cycle of odd length, a contradiction. Hence, u and v get different colors. Thus, the bipartite graphs of part (c) are exactly the graphs with no cycle of odd length, and they include the trees of part (b) and the cycles with an even number of vertices of part (d).

[47] See also the solution of Problem 2.5(c) on page 47.

6.2. An **edge coloring** of a graph assigns colors to its edges so that every two edges that share an endpoint get different colors.

(a) Explain why an edge coloring of any graph uses at least as many colors as the maximum degree of the graph.

Answer: Let v be a vertex whose degree Δ is the maximum degree. The Δ edges of v share the endpoint v, so every two of them must have different colors. Hence, at least Δ colors are needed.

(b) What is the minimum number of colors needed to color the edges of the star S_n with $n \geq 2$ vertices (one middle vertex adjacent to the other $n-1$ vertices, and no other edges)? Explain.

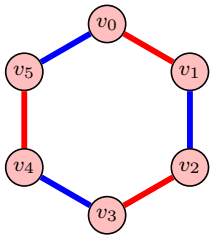
Answer: $n-1$ colors. All the $n-1$ edges of the star share the middle vertex, so by part (a) at least $n-1$ colors are needed, and assigning a different color to each edge uses exactly $n-1$ colors.

(c) What is the minimum number of colors needed to color the edges of the path P_n with $n \geq 3$ vertices? Explain.

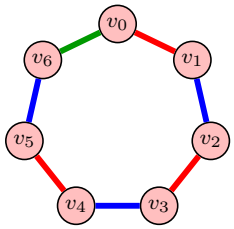
Answer: 2 colors. The maximum degree is 2, so by part (a) at least 2 colors are needed. Coloring the edges alternately $1, 2, 1, 2, \ldots$ along the path uses 2 colors, and every two consecutive edges have different colors.

(d) What is the minimum number of colors needed to color the edges of the cycle C_n with $n \geq 3$ vertices? Explain. Note that the answer is different when n is even and when n is odd.

Answer: 2 colors if n is even and 3 colors if n is odd. By part (a), at least 2 colors are needed. Let $e_1, e_2, \ldots, e_n$ be the edges in the order of the cycle. If n is even, color them alternately with 1 and 2. If n is odd, two colors must alternate, so e_1 and e_n would get the same color although they share a vertex; instead, color $e_1, \ldots, e_{n-1}$ alternately and e_n with color 3. See below the edge colorings of C_6 with 2 colors and of C_7 with 3 colors, where color 1 is red, color 2 is blue, and color 3 is green.



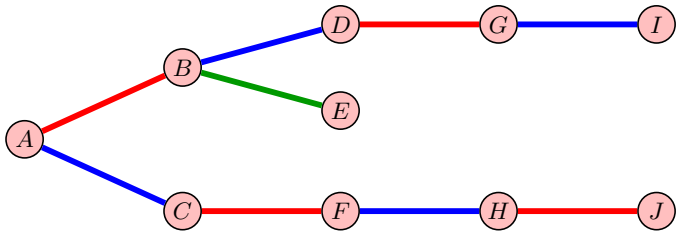
C_6: 2 colors



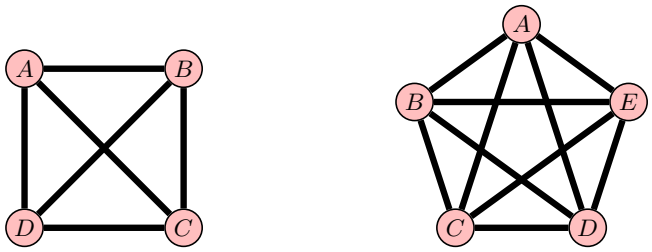
C_7: 3 colors

(e) Prove that the edges of any tree can be colored with as many colors as its maximum degree.

Proof: Let Δ be the maximum degree of the tree, and choose any vertex r as the root. Give the edges of r different colors from $\{1, \ldots, \Delta\}$. Then consider the other vertices in order of their distance from r. When a vertex $v \neq r$ is considered, the edge to its parent is already colored, and v has at most $\Delta - 1$ edges to its children; give these edges different colors from the $\Delta - 1$ colors that differ from the color of the parent edge. Two edges that share a vertex v are either two edges to children of v, which got different colors, or the parent edge of v and an edge to a child of v, which also got different colors. Hence, Δ colors suffice. For example, see below a tree with the root A, depth 4, and maximum degree $\Delta = 3$. Its edges are colored by this process, starting with the colors 1 and 2 for the edges of A.



6.3. A **vertex coloring** of a graph assigns colors to its vertices so that every two adjacent vertices get different colors, and an **edge coloring** of a graph assigns colors to its edges so that every two edges that share an endpoint get different colors. In the complete graph K_n with n vertices, every two vertices are adjacent. Below are K_4 (left) and K_5 (right).

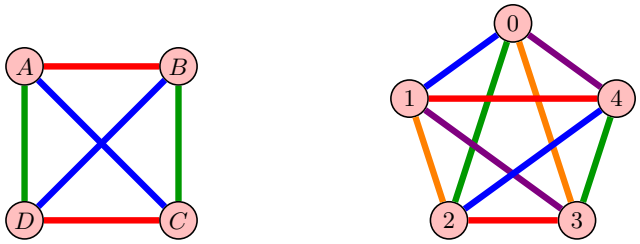


(a) What is the minimum number of colors needed to color the **vertices** of the complete graph K_n with $n \geq 1$ vertices? Explain.

Answer: n colors. Each vertex must have a different color since all possible edges exist. Therefore, coloring the n vertices with the n different colors $1, 2, \ldots, n$ is an optimal coloring of K_n.

(b) What is the minimum number of colors needed to color the **edges** of the complete graph K_4? And of K_5? Explain. **Hint:** How many edges can have the same color?

Answer: For K_4: 3 colors. The maximum degree is 3, so at least 3 colors are needed (the three edges of a vertex of degree 3 need three different colors), and the three pairs of disjoint edges $\{(A,B),(C,D)\}$, $\{(A,C),(B,D)\}$, $\{(A,D),(B,C)\}$ get the three colors.
For K_5: 5 colors. Edges of the same color do not share endpoints, so each color is used for at most $\lfloor 5/2 \rfloor = 2$ of the 10 edges, and at least $10/2 = 5$ colors are needed. Five colors suffice: name the vertices $0, 1, 2, 3, 4$ and color the edge (i, j) with the color $1 + ((i + j) \bmod 5)$; two edges (i, j) and (i, k) with the same color satisfy $j = k$, so edges with a common endpoint get different colors.



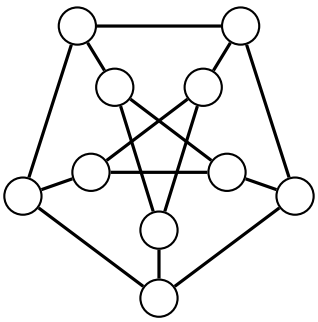
Optimal edge colorings of K_4 with 3 colors (left) and of K_5 with 5 colors (right), where color 1 is red, color 2 is blue, color 3 is green, color 4 is orange, and color 5 is violet.

Remark: For vertex coloring, the answer for K_n follows directly from the definition, while for edge coloring, the answer is $n - 1$ for K_4 but n for K_5, and both require an argument. In general, the edges of K_n need $n - 1$ colors when n is even and n colors when n is odd.

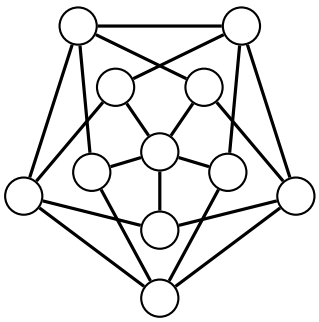
Odd n. The edges of one color share no endpoints, so when n is odd, a color has at most $(n-1)/2$ edges, and the $n(n-1)/2$ edges of K_n need at least n colors. To see that n colors suffice, name the vertices $0, 1, \ldots, n-1$ and color the edge (i, j) with the color $1 + ((i + j) \bmod n)$, as for K_5 above. Two edges (i, j) and (i, k) with the same color satisfy $j \equiv k \pmod{n}$, that is, $j = k$. Geometrically, if the vertices are the corners of a regular polygon in this order, each color is a side with its parallel diagonals.

Even n. Here $n - 1$ colors are needed, since every vertex has degree $n - 1$. To see that they suffice, put one vertex v aside and color the edges among the other vertices $0, 1, \ldots, n-2$ (an odd number of vertices) with the $n - 1$ colors $1 + ((i + j) \bmod (n - 1))$ as above. The vertex i has $n - 2$ edges with different colors, so it misses exactly one color, namely $1 + (2i \bmod (n - 1))$, the color of the missing "edge" (i, i). Different vertices miss different colors, since $2i \equiv 2i' \pmod{n-1}$ implies $i = i'$ when $n - 1$ is odd. Now bring v back and connect it to each vertex i with the color that i misses. The new edges at v get different colors, and no conflict is created at i. In the edge coloring of K_4 above, C plays the role of v. This is also how a round-robin tournament with n teams is scheduled in $n - 1$ rounds.

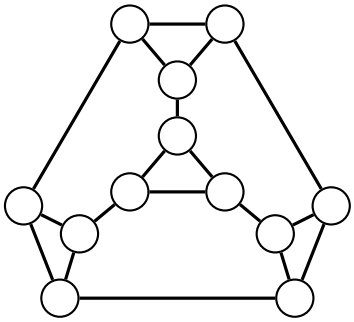
6.4. A **vertex coloring** (or simply a **coloring**) of a graph assigns colors to its vertices so that every two adjacent vertices get different colors. Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



Grötzsch



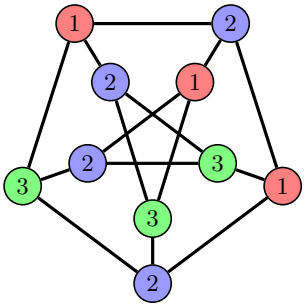
FourTriangles

For each of the three graphs, color its vertices with as few colors as possible, and explain why it is impossible to color its vertices with fewer colors.

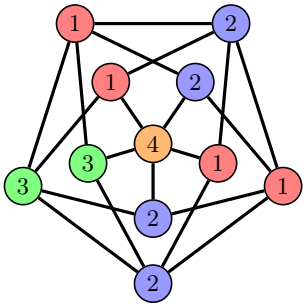
The Petersen graph: See below a coloring of the Petersen graph with three colors. A cycle of odd length cannot be colored with two colors: the two colors must alternate along the cycle, which is impossible when its length is odd.[48] The outer cycle of the Petersen graph has odd length 5, so it alone requires three colors.

The Grötzsch graph: See below a coloring of the Grötzsch graph with four colors. The Grötzsch graph consists of a middle vertex, five inner vertices that are the neighbors of the middle vertex, and five vertices on the outer cycle. Each inner vertex is the **shadow** of one outer-cycle vertex: it is adjacent to the two neighbors of this vertex on the outer cycle.[49] Assume to the contrary that the Grötzsch graph can be colored with three colors, and without loss of generality let the middle vertex have color 3. Then the five inner vertices have the colors 1 and 2. Recolor every outer-cycle vertex whose color is 3 with the color of its shadow. The outer cycle remains properly colored: two neighbors on the outer cycle are never both recolored, and a recolored vertex gets the color of an inner vertex that is adjacent to both of its neighbors on the outer cycle. Hence, the outer cycle, whose length is 5, is colored with the two colors 1 and 2, which is impossible, since the colors must alternate along a cycle of odd length. The coloring with four colors below is built from the shadows: the outer cycle is colored with the colors 1, 2, and 3, every inner vertex gets the color of the outer-cycle vertex whose shadow it is, and the middle vertex gets color 4. This is a coloring, since the two neighbors of a shadow on the outer cycle are the two neighbors of its vertex, whose colors differ from the color of this vertex, and no two inner vertices are adjacent.

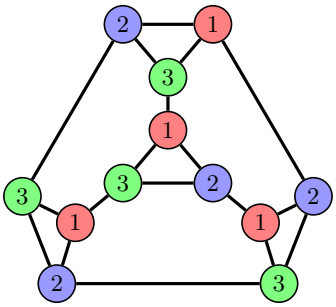
The FourTriangles graph: See below a coloring of the FourTriangles graph with three colors. The FourTriangles graph contains four cycles of length 3 (triangles). Each triangle by itself requires three colors, since its three vertices are pairwise adjacent. Therefore, it is impossible to color the whole graph with only two colors.



Petersen



Grötzsch



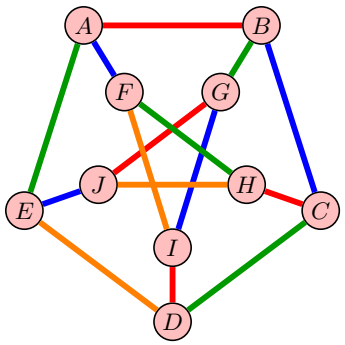
FourTriangles

[48] See also the solution of Problem 6.1(d) on page 106.
[49] See also the solution of Problem 3.3(d) on page 60.

6.5. An **edge coloring** of a graph assigns colors to its edges so that every two edges that share an endpoint get different colors. Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.

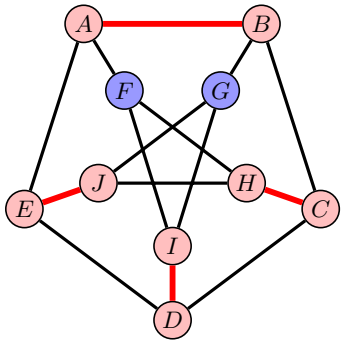
For each of the three graphs, color its edges with as few colors as possible, and explain why it is impossible to color its edges with fewer colors.

The Petersen graph: See below an edge coloring of the Petersen graph with four colors: color 1: (A, B), (C, H), (D, I), (G, J); color 2: (A, F), (B, C), (E, J), (G, I); color 3: (A, E), (B, G), (C, D), (F, H); color 4: (D, E), (F, I), (H, J). Every vertex has degree 3, so at least 3 colors are needed.



Petersen

Assume to the contrary that 3 colors suffice. Then every vertex has exactly one edge of each color, and two edges with the same color share no vertex. Hence, each color appears on at most two of the five edges of the outer cycle, so the three colors appear on 2, 2, and 1 of them, and some color c appears on exactly one of them, say on the edge (u, v). The five edges (A, F), (B, G), (C, H), (D, I), and (E, J) join the outer cycle to the inner cycle; call them **spokes**. Each of the three outer vertices other than u and v has no outer edge of color c, so its spoke has color c, while the spokes of u and v do not. Therefore, the inner endpoints of the spokes of u and v are the only two inner vertices without a spoke of color c, and each of them needs an edge of color c, so a single inner edge of color c must join the two of them. However, the inner endpoints of the spokes of two neighbors on the outer cycle are never adjacent. For example, A and B are adjacent, but F and G are not. Indeed, the inner cycle $F - H - J - G - I - F$ joins F to H and I, whose outer partners C and D are at distance 2 from A, and in the same way, every inner edge joins the inner endpoints of the spokes of two outer vertices at distance 2. This is a contradiction. The figure below shows this for $u = A$ and $v = B$: the edges of color c are red, and the two inner vertices F and G (blue) have no edge of color c.



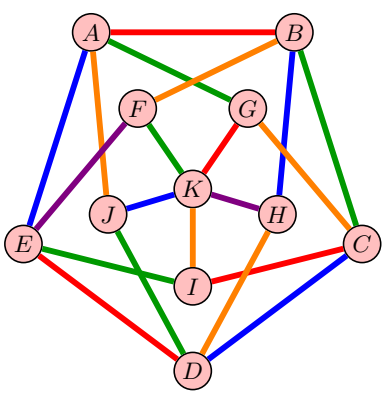
the proof

Another proof for the Petersen graph: Assume to the contrary that 3 colors suffice. Consider the edges of colors 1 and 2: every vertex touches exactly two of them, so they form disjoint cycles that together contain all 10 vertices. Along each such cycle the two colors alternate, so every cycle has an even length. The Petersen graph has no cycles of length 3 or 4,[50] so each of these cycles has an even length of at least 6, and their lengths sum to 10. The only possibility is a single cycle of length 10 that visits each vertex exactly once, that is, a **Hamiltonian cycle**. However, the Petersen graph has no Hamiltonian cycle.[51] This is a contradiction.

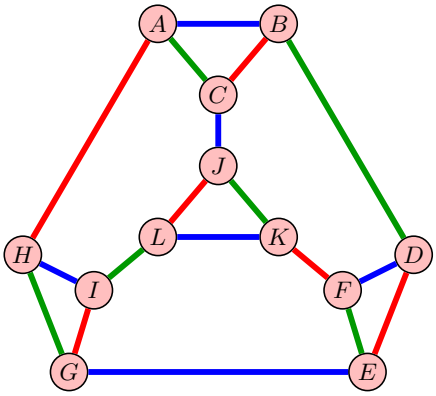
[50] See the solution of Problem 3.1(e) on page 56.
[51] This fact is proved in the solution of Problem 4.2(b) on page 77.

The Grötzsch graph: See below an edge coloring of the Grötzsch graph with five colors: color 1: (A, B), (C, I), (D, E), (G, K); color 2: (A, E), (B, H), (C, D), (J, K); color 3: (A, G), (B, C), (D, J), (E, I), (F, K); color 4: (A, J), (B, F), (C, G), (D, H), (I, K); color 5: (E, F), (H, K). The vertex K has degree 5, so at least 5 colors are needed.

The FourTriangles graph: See below an edge coloring of the FourTriangles graph with three colors: color 1: (A, H), (B, C), (D, E), (F, K), (G, I), (J, L); color 2: (A, B), (C, J), (D, F), (E, G), (H, I), (K, L); color 3: (A, C), (B, D), (E, F), (G, H), (I, L), (J, K). In this coloring, every vertex has exactly one edge of each color. Every vertex has degree 3, so at least 3 colors are needed.



Grötzsch



FourTriangles

In the edge colorings, color 1 is red, color 2 is blue, color 3 is green, color 4 is orange, and color 5 is violet.

Remark: Let Δ be the maximum degree of a graph. The edges of a vertex of degree Δ need Δ different colors, so at least Δ colors are always needed. By a theorem of Vizing, the edges of every simple graph can be colored with at most $\Delta + 1$ colors, so only two answers are possible: Δ or $\Delta + 1$. The Petersen graph needs $\Delta + 1 = 4$ colors, while the Grötzsch graph and the FourTriangles graph need only Δ colors, 5 and 3, respectively. By a theorem of Kőnig, the edges of every bipartite graph can be colored with Δ colors.[52]

[52] See the solution of Problem 6.6(c) on page 112, where $K_{m,m}$ needs $m = \Delta$ colors.

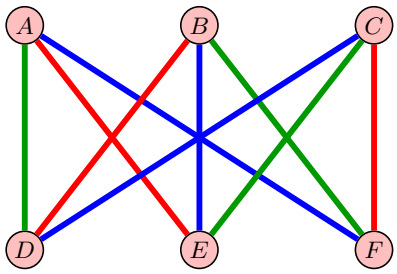
6.6. The **line graph** $L(G)$ of a graph G has a vertex for every edge of G, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G share an endpoint. Each vertex of $L(G)$ is named by the two endpoints of its edge. A **vertex coloring** of a graph assigns colors to its vertices so that every two adjacent vertices get different colors, and an **edge coloring** of a graph assigns colors to its edges so that every two edges that share an endpoint get different colors.

(a) Prove that for every graph G, the minimum number of colors needed to color the edges of G equals the minimum number of colors needed to color the vertices of $L(G)$.

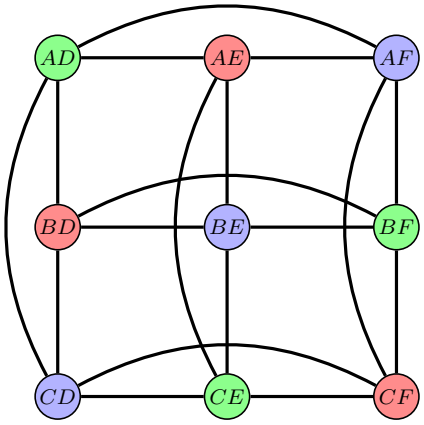
Proof: The vertices of $L(G)$ are the edges of G, and two of them are adjacent in $L(G)$ if and only if they share an endpoint in G.[53] Hence, an assignment of colors to the edges of G is an edge coloring of G if and only if it is a vertex coloring of $L(G)$, and the two minimum numbers of colors are equal.

(b) Let $K_{3,3}$ be the complete bipartite graph with 3 vertices on each side, in which two vertices are adjacent if and only if they are on different sides. Draw $K_{3,3}$ and its line graph $L(K_{3,3})$. Color the edges of $K_{3,3}$ with 3 colors, and color the vertices of $L(K_{3,3})$ accordingly.

Answer: See below $K_{3,3}$ (left) and $L(K_{3,3})$ (right). Label the top vertices A, B, C and the bottom vertices D, E, F. Color the edge between the i-th top vertex and the j-th bottom vertex with the color $1 + ((i + j) \bmod 3)$, where color 1 is red, color 2 is blue, and color 3 is green. The vertex of this edge in $L(K_{3,3})$ is placed in row i and column j of a 3×3 table. Two vertices are adjacent if and only if they are in the same row or in the same column, and each vertex gets the color of its edge.[54]



$K_{3,3}$



$L(K_{3,3})$

(c) For $m \geq 2$, let $K_{m,m}$ be the complete bipartite graph with m vertices on each side, in which two vertices are adjacent if and only if they are on different sides. Find the minimum number of colors needed to color its vertices and the minimum number of colors needed to color its edges. Conclude that the vertices of a graph and the vertices of its line graph may require very different numbers of colors.

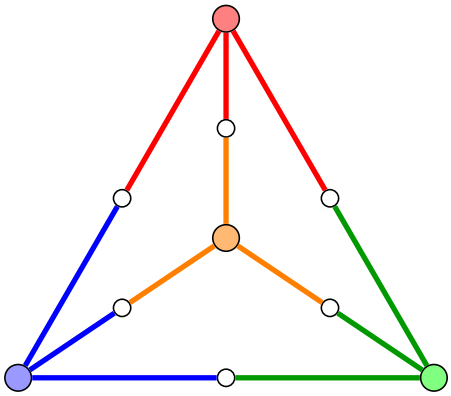
Answer: **Vertices:** 2 colors. Let $u_1, \ldots, u_m$ and $w_1, \ldots, w_m$ be the two sides, and color $u_1, \ldots, u_m$ with color 1 and $w_1, \ldots, w_m$ with color 2; every edge joins the two sides, so its endpoints get different colors. One color is not enough, since the graph has edges. **Edges:** m colors. Every vertex has degree m, and its m edges share it, so at least m colors are needed. To see that m colors suffice, color the edge (u_i, w_j) with the color $1 + ((i + j) \bmod m)$. Two different edges (u_i, w_j) and $(u_i, w_{j'})$ of u_i have $j \neq j'$, and therefore $i + j \not\equiv i + j' \pmod{m}$, so they get different colors. The same argument applies to two edges of w_j. By part (a), the vertices of $L(K_{m,m})$ require m colors, while the vertices of $K_{m,m}$ require only 2 colors. The gap grows with m.

[53] See also the solutions of Problems 2.6 and 3.5 on pages 49 and 64.
[54] The same graph appears in the solution of Problem 5.6(b) on page 100.

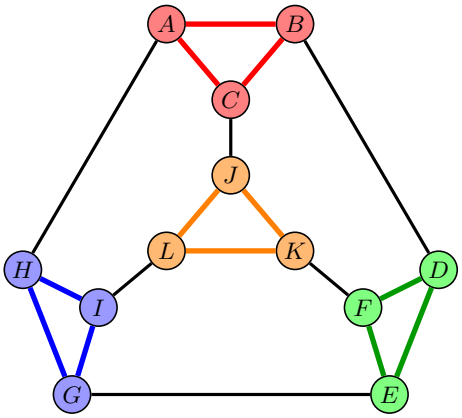
(d) Prove that the star $K_{1,3}$ (the star S_4), with one middle vertex adjacent to the other 3 vertices and no other edges, is not the line graph of any graph. Conclude that every edge coloring problem is a vertex coloring problem, but not every vertex coloring problem is an edge coloring problem.

Proof: Assume to the contrary that $K_{1,3} = L(H)$ for some graph H. The middle vertex of $K_{1,3}$ is an edge (u, v) of H, and its three neighbors are three edges of H that share an endpoint with (u, v). Each of these three edges contains u or v, so two of them contain the same endpoint, say u. These two edges share u, so their vertices in $L(H)$ are adjacent. However, no two of the three leaves of $K_{1,3}$ are adjacent. This is a contradiction. By part (a), coloring the edges of any graph H is the same as coloring the vertices of $L(H)$. Since $K_{1,3}$ is not $L(H)$ for any H, coloring the vertices of $K_{1,3}$ is not the same as coloring the edges of any graph. Thus, edge coloring is a special case of vertex coloring, but not the other way around.

Remark: By the same argument, the Petersen graph and the Grötzsch graph are not line graphs: they have no triangles, so the three neighbors of any vertex are pairwise non-adjacent. In contrast, the FourTriangles graph is a line graph: it is the line graph of K_4 with a new vertex in the middle of each of its 6 edges. In the figure below, the 3 edges at each vertex of K_4 have the color of this vertex, and they become a triangle of the same color in the line graph. The 2 edges at each middle vertex become the edge that joins two triangles.



K_4 with middle vertices



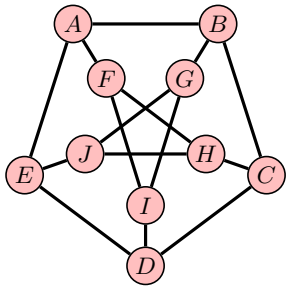
its line graph: FourTriangles

7 Special Sets of Vertices and Edges

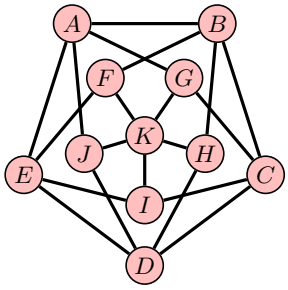
7.1. Let G be a simple undirected graph with $n \geq 1$ vertices. A **clique** in G is a set CL of vertices, every two of which are adjacent. That is, for any two different vertices $u, v \in CL$, the graph G includes the edge (u, v).

For example, the set of all the vertices of the complete graph K_n (every two vertices are adjacent) is a clique of size n. In contrast, in the null graph (no edges) a clique contains at most one vertex.

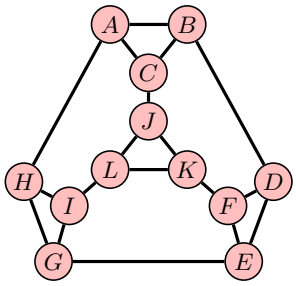
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



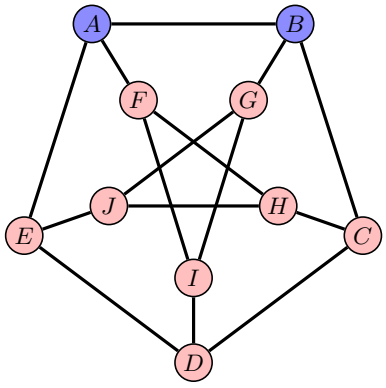
Grötzsch



FourTriangles

(a) What is the size of the **largest** clique in the Petersen graph? Give an example of such a clique and prove that no larger clique exists.

Answer: 2. See below an example. The blue vertices form a clique.

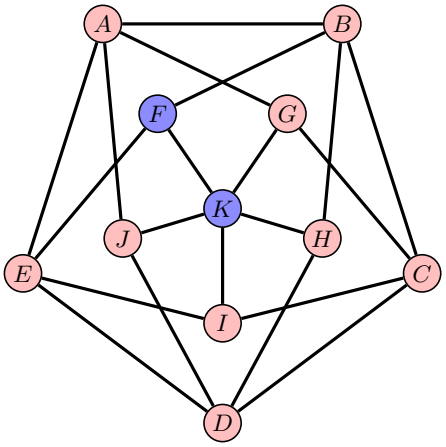


Proposition (Petersen graph): No clique in the Petersen graph has more than 2 vertices.

Proof: The Petersen graph does not contain a triangle. The outer cycle and the inner cycle are cycles of length 5, and thus do not contain a triangle. A triangle with vertices in both cycles would contain a vertex with two neighbors in the other cycle. However, each vertex has only one neighbor in the other cycle. Therefore, any edge is a largest clique in the Petersen graph.

(b) What is the size of the **largest** clique in the Grötzsch graph? Give an example of such a clique and prove that no larger clique exists.

Answer: 2. See below an example. The blue vertices form a clique.

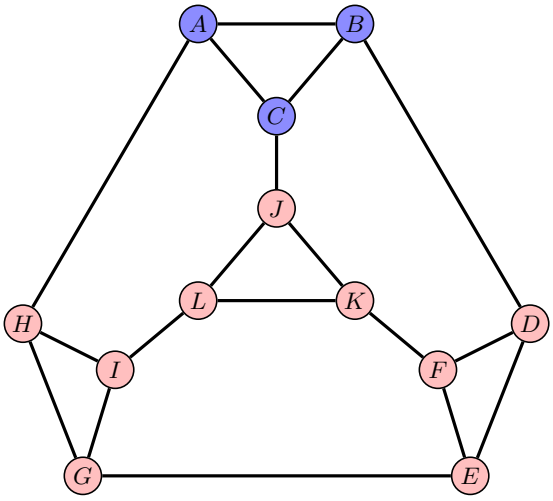


Proposition (Grötzsch graph): No clique in the Grötzsch graph has more than 2 vertices.

Proof: It suffices to show that the Grötzsch graph has no triangle. The five neighbors F, G, H, I, J of the middle vertex K are not adjacent to each other, so no triangle contains K. The outer cycle $A - B - C - D - E - A$ has length 5, so a triangle cannot have all its vertices on it. Hence, a triangle without K contains at least one of F, G, H, I, J, and since these vertices are not adjacent to each other, it contains exactly one of them together with two of its neighbors on the outer cycle. However, each of F, G, H, I, J is adjacent to two outer-cycle vertices that are not adjacent: F to B and E, G to A and C, H to B and D, I to C and E, and J to A and D. Hence, there is no triangle.

(c) What is the size of the **largest** clique in the FourTriangles graph? Give an example of such a clique and prove that no larger clique exists.

Answer: 3. See below an example. The blue vertices form a clique: the triangle $\{A, B, C\}$.



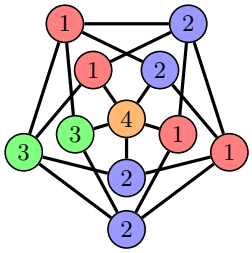
Proposition (FourTriangles graph): No clique in the FourTriangles graph has more than 3 vertices.

Proof: Assume to the contrary that there is a clique of size 4, and let v be one of its vertices. Since v has degree 3, the clique consists of v and its three neighbors: its two neighbors x and y in its own triangle, and its neighbor w in another triangle. However, w has only one neighbor outside its own triangle, namely v, so w is not adjacent to x, a contradiction.

(d) A **vertex coloring** (or simply a **coloring**) of a graph assigns colors to its vertices so that every two adjacent vertices get different colors. Prove that every coloring of a graph uses at least as many colors as the size of its largest clique. What lower bound does this give for each of the three graphs above?

Answer: The vertices of a clique are pairwise adjacent, so a coloring assigns them pairwise different colors. Hence, every coloring uses at least as many colors as the size of the largest clique. By parts (a), (b), and (c), this lower bound is 2 for the Petersen graph, 2 for the Grötzsch graph, and 3 for the FourTriangles graph.

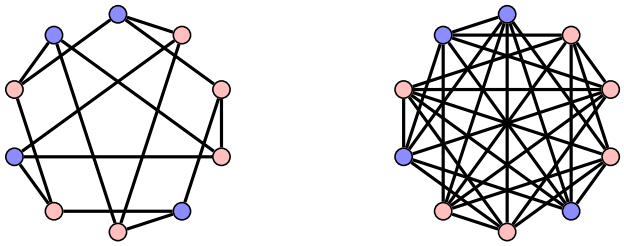
Remark: The minimum numbers of colors needed to color the Petersen graph, the Grötzsch graph, and the FourTriangles graph are 3, 4, and 3, respectively.[55] Hence, the bound is tight only for the FourTriangles graph. The Grötzsch graph shows that the bound can be far from the truth: its largest clique has size 2, but it needs 4 colors. See below a coloring of the Grötzsch graph with 4 colors.



(e) The **complement** $\overline{G}$ of a graph G has the same vertices as G, and two different vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. An **independent set** is a set of vertices, no two of which are adjacent. Prove that a set of vertices is a clique in G if and only if it is an independent set in $\overline{G}$. What is the size of the largest independent set in the complement of the Petersen graph?

Answer: Assume that CL is a clique in G. By definition, every pair of vertices in CL is an edge in G. Therefore, no pair of vertices in CL is an edge in $\overline{G}$. Hence, CL is an independent set in $\overline{G}$. Conversely, assume that CL is an independent set in $\overline{G}$. By definition, no pair of vertices in CL is an edge in $\overline{G}$. Therefore, every pair of vertices in CL is an edge in G. Hence, CL is a clique in G.

See below the Petersen graph (left) and its complement (right).[56] The left drawing is the Petersen graph: every other vertex on the circle, starting from the top one, belongs to the outer cycle. Each of the remaining five vertices is adjacent to one of its two neighbors on the circle, and these five vertices form the inner cycle. Moving them inside the outer cycle gives the traditional drawing. Every pair of vertices is an edge in exactly one of the two drawings. Hence, the right drawing is the complement of the Petersen graph.



It follows that the size of the largest independent set in a graph is the same as the size of the largest clique in its complement graph, and vice versa. By part (a), the largest clique in the Petersen graph has size 2, so the largest independent set in its complement has size 2.

Remark: Similarly, the largest independent set in the Petersen graph has size 4,[57] so the largest clique in its complement has size 4. For example, the four blue vertices in the drawings above (A, C, I, and J in the traditional drawing) form an independent set in the Petersen graph (left) and a clique in its complement (right).

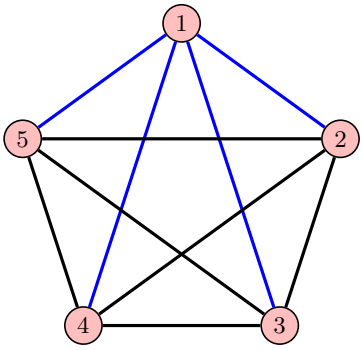
[55] See the solution of Problem 6.4 on page 109.
[56] Complements are the subject of Problem 1.8 on page 3.
[57] See the solution of Problem 7.2 on page 118.

Symmetry and counting: All three graphs are very symmetric, but the Petersen graph is the most symmetric. It can be mapped onto itself in 120 ways, so that any vertex can go to any other vertex and any edge to any other edge. The FourTriangles graph has 24 such mappings: any vertex can go to any other vertex, but an edge of a triangle can never go to an edge between triangles. The Grötzsch graph has only 10: the rotations and reflections of its drawing, all of which keep K in place. The details of these mappings are omitted.[58] Below, the optimal solutions are counted in the labeled graphs, in which every vertex has a distinct name, and up to "symmetry".

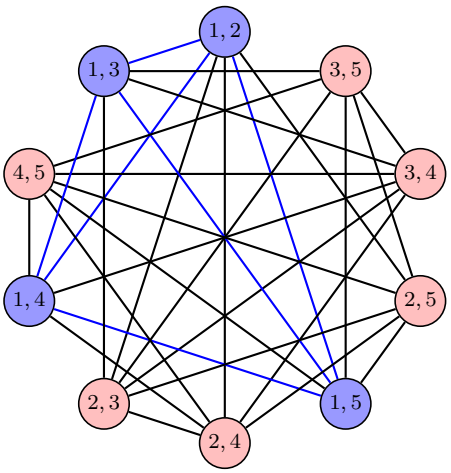
The Petersen graph: A clique of size 2 is an edge. Since the Petersen graph has fifteen edges, the labeled Petersen graph has exactly fifteen different cliques of size 2. Nevertheless, up to "symmetry", there is only one clique of size 2 in the Petersen graph, since the Petersen graph can be mapped onto itself so that any edge is mapped to any other edge.

The other two graphs: The labeled Grötzsch graph has 20 cliques of size 2, namely its edges, and up to "symmetry" there are 3 of them: an edge at K, an edge of the outer cycle, and an edge between the outer cycle and one of F, G, H, I, J. The labeled FourTriangles graph has 4 cliques of size 3, namely its triangles, and up to "symmetry" there is only one. The details are omitted.

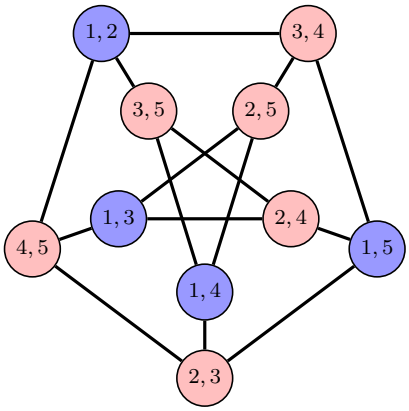
Remark: The complement of the Petersen graph, from part (e), is a line graph.[59] Name the vertices of the Petersen graph by the 10 pairs of numbers from $\{1, 2, 3, 4, 5\}$, as in the figure below. Then two vertices are adjacent if and only if their pairs have no common number. Hence, two vertices are adjacent in the complement if and only if their pairs share a number. Thinking of the pair $\{i, j\}$ as the edge (i, j) of the complete graph K_5, the complement of the Petersen graph is exactly the line graph $L(K_5)$. For example, the four blue vertices are the four pairs that contain 1, that is, the four edges of K_5 at the vertex 1. They form a clique of size 4 in $L(K_5)$, and an independent set of size 4 in the Petersen graph. The figure below also shows K_5, with the four edges at the vertex 1 in blue, and $L(K_5)$, drawn as in part (e), with the four pairs that contain 1 in blue.



K_5



$L(K_5)$



Petersen

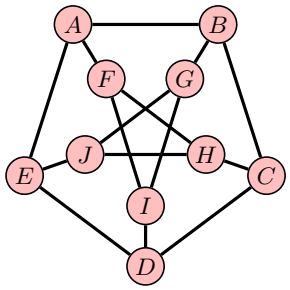
[58] The 10 mappings of the Grötzsch graph and the 24 mappings of the FourTriangles graph are found in the solutions of Problems 1.2(b) and 1.3(b) on pages 29 and 30.

[59] Line graphs are the subject of Problems 2.6 and 6.6 on pages 6 and 21.

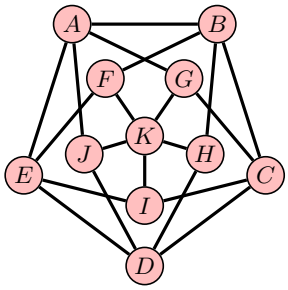
7.2. Let G be a simple undirected graph with $n \geq 1$ vertices. An **independent set** in G is a set IS of vertices, no two of which are adjacent. That is, for any two different vertices $u, v \in IS$, the graph G does not include the edge (u, v).

For example, the set of all the vertices of the null graph (no edges) is an independent set of size n. In contrast, in the complete graph K_n (every two vertices are adjacent) an independent set contains at most one vertex.

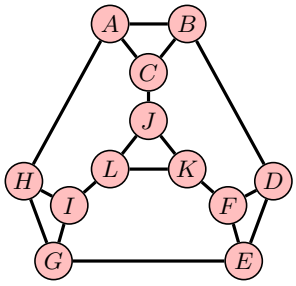
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



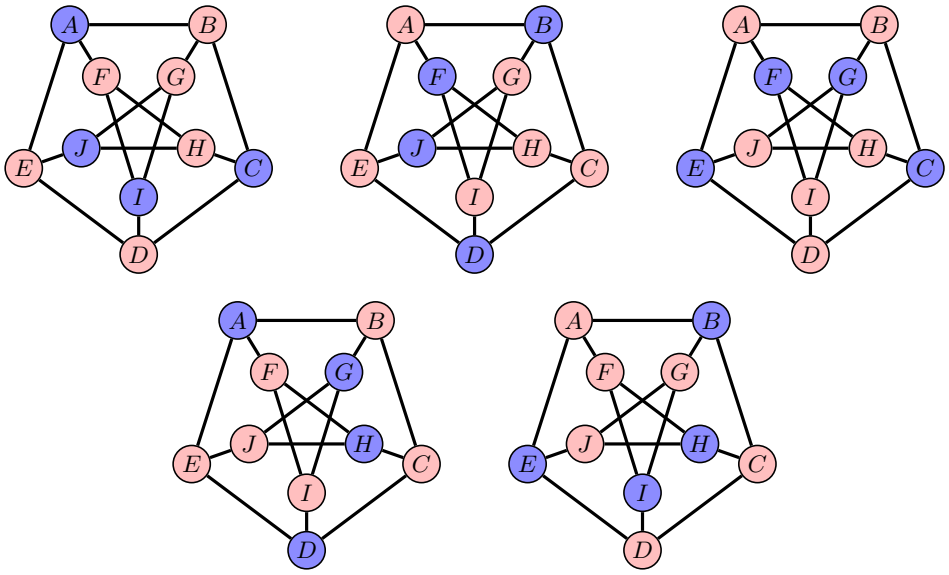
Grötzsch



FourTriangles

(a) What is the size of the **largest** independent set in the Petersen graph? Give an example of such a set and prove that no larger independent set exists.

Answer: 4. See below five examples.[60] In each example, the blue vertices form an independent set.



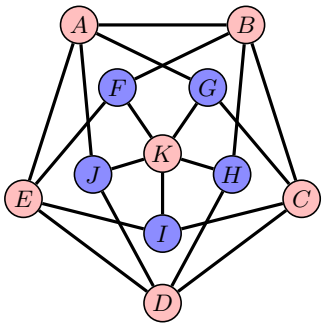
[60] A coloring of the Petersen graph with three colors (see the solution of Problem 6.4 on page 109) has a color class with at least $\lceil 10/3 \rceil = 4$ vertices, which is an independent set.

Proposition (Petersen graph): No independent set in the Petersen graph has more than 4 vertices.

Proof: The largest independent set in the cycle of length 5, denoted by C_5, has size 2. To see this, let v be one of the vertices in an independent set of C_5. Then v's two neighbors cannot be in this independent set, and since the remaining two vertices are adjacent, only one of them can be in this independent set. The vertices of the Petersen graph are split between two cycles of length 5: the outer cycle and the inner cycle. Each cycle can contribute at most two vertices to an independent set, even without the restrictions imposed by the five edges that connect the two cycles. Therefore, the size of the largest independent set in the Petersen graph is at most 4.

(b) What is the size of the **largest** independent set in the Grötzsch graph? Give an example of such a set and prove that no larger independent set exists.

Answer: 5. See below an example. The blue vertices form an independent set.

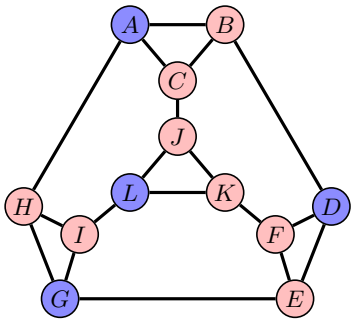


Proposition (Grötzsch graph): No independent set in the Grötzsch graph has more than 5 vertices.

Proof: Let IS be an independent set. The outer cycle $A - B - C - D - E - A$ has length 5, so IS contains at most two of its vertices. If $K \in IS$, then none of its neighbors F, G, H, I, J is in IS, and therefore $|IS| \leq 1 + 2 = 3$. Otherwise, $K \notin IS$. Each outer-cycle vertex is adjacent to exactly two of F, G, H, I, J. If IS contains exactly one outer-cycle vertex, then two of F, G, H, I, J are excluded, and $|IS| \leq 1 + 3 = 4$. If IS contains two outer-cycle vertices, then they are not adjacent, and together they are adjacent to three of F, G, H, I, J (for example, A is adjacent to G and J, and C is adjacent to G and I). Hence, $|IS| \leq 2 + 2 = 4$. Finally, if IS contains no outer-cycle vertex, then $IS \subseteq \{F, G, H, I, J\}$, and $|IS| \leq 5$.

(c) What is the size of the **largest** independent set in the FourTriangles graph? Give an example of such a set and prove that no larger independent set exists.

Answer: 4. See below an example. The blue vertices form an independent set with one vertex from each triangle.



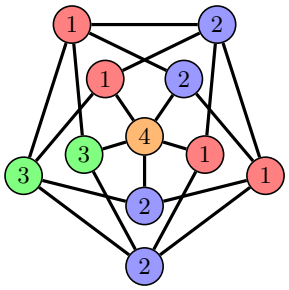
Proposition (FourTriangles graph): No independent set in the FourTriangles graph has more than 4 vertices.

Proof: The vertices of the FourTriangles graph are split into the four triangles $\{A, B, C\}$, $\{D, E, F\}$, $\{G, H, I\}$, and $\{J, K, L\}$, and an independent set contains at most one vertex of each triangle, since every two vertices of a triangle are adjacent.

(d) A **vertex coloring** (or simply a **coloring**) of a graph assigns colors to its vertices so that every two adjacent vertices get different colors. Prove that if a graph has n vertices and its largest independent set has size α, then every coloring of the graph uses at least n/α colors. What lower bound does this give for each of the three graphs above?

Answer: The vertices of each color form an independent set, since adjacent vertices have different colors. Hence, each color is used for at most α vertices. If a coloring uses k colors, then the n vertices are split into k independent sets, so $n \le k\alpha$, and $k \ge n/\alpha$. By parts (a), (b), and (c), the lower bound is $\lceil 10/4 \rceil = 3$ for the Petersen graph, $\lceil 11/5 \rceil = 3$ for the Grötzsch graph, and $12/4 = 3$ for the FourTriangles graph.

Remark: The minimum numbers of colors needed to color the Petersen graph, the Grötzsch graph, and the FourTriangles graph are 3, 4, and 3, respectively.[61] Hence, the bound is tight for the Petersen graph and the FourTriangles graph, but not for the Grötzsch graph, which needs 4 colors. See below a coloring of the Grötzsch graph with 4 colors.



Symmetry and counting: All three graphs are very symmetric, but the Petersen graph is the most symmetric. It can be mapped onto itself in 120 ways, so that any vertex can go to any other vertex and any edge to any other edge. The FourTriangles graph has 24 such mappings: any vertex can go to any other vertex, but an edge of a triangle can never go to an edge between triangles. The Grötzsch graph has only 10: the rotations and reflections of its drawing, all of which keep K in place. The details of these mappings are omitted. Below, the optimal solutions are counted in the labeled graphs, in which every vertex has a distinct name, and up to "symmetry".

The Petersen graph: By the proof in part (a), an independent set of size 4 must contain two non-adjacent vertices of the outer cycle and two non-adjacent vertices of the inner cycle. Moreover, each pair of non-adjacent vertices of the outer cycle can be combined with exactly one pair of non-adjacent vertices of the inner cycle: the inner neighbors of the outer pair are excluded and adjacent to each other, so the other three inner vertices form a path, and only its two ends are non-adjacent. Since the outer cycle has five pairs of non-adjacent vertices, the labeled Petersen graph has exactly five different independent sets of size 4. These are the five examples in the answer to part (a).

Rotating the drawing of the Petersen graph by one fifth of a full turn (72°) maps outer-cycle vertices to outer-cycle vertices, inner-cycle vertices to inner-cycle vertices, and edges to edges. Thus, this rotation maps the Petersen graph onto itself. Each of these examples is obtained from the previous one by this rotation, and rotating the last example gives the first one. Hence, if the vertices are not labeled, all five independent sets are essentially the same: up to "symmetry", there is only one independent set of size 4 in the Petersen graph.

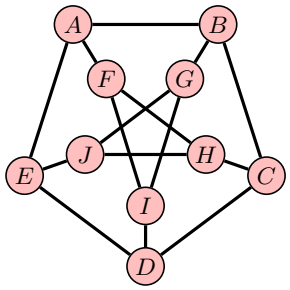
The other two graphs: The labeled Grötzsch graph has only one independent set of size 5, namely $\{F, G, H, I, J\}$, as the proof in part (b) also shows. The labeled FourTriangles graph has 30 independent sets of size 4, and up to "symmetry" there are 2 of them. By part (c), such a set has one vertex in each triangle. Each vertex of a triangle is the endpoint of the edge that joins this triangle to one of the other three triangles, its *target*, and the set is independent if and only if no two triangles are each other's targets. Of the $3^4 = 81$ choices of the targets, two given triangles are each other's targets in 9, and the 3 splits of the four triangles into two such pairs are subtracted twice, so there are $81 - 6 \cdot 9 + 3 = 30$ independent sets: in 6 of them the targets form a cycle through all four triangles, and in 24 they form a cycle through three triangles.

[61] See the solution of Problem 6.4 on page 109.

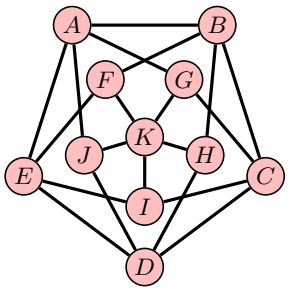
7.3. Let G be a simple undirected graph with $n \geq 1$ vertices. A **vertex cover** in G is a set VC of vertices that contains at least one endpoint of every edge. That is, for any edge (u, v) of G, $u \in VC$ or $v \in VC$ (or both).

For example, the empty set is a vertex cover of the null graph (no edges), because there are no edges to cover. In contrast, in the complete graph K_n (every two vertices are adjacent) any vertex cover must contain at least $n - 1$ vertices, because if a set misses two vertices, then no vertex in the set covers the edge between them.

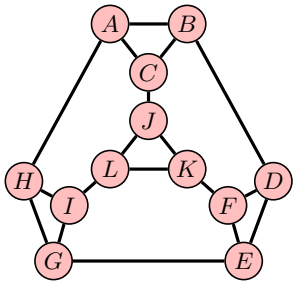
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



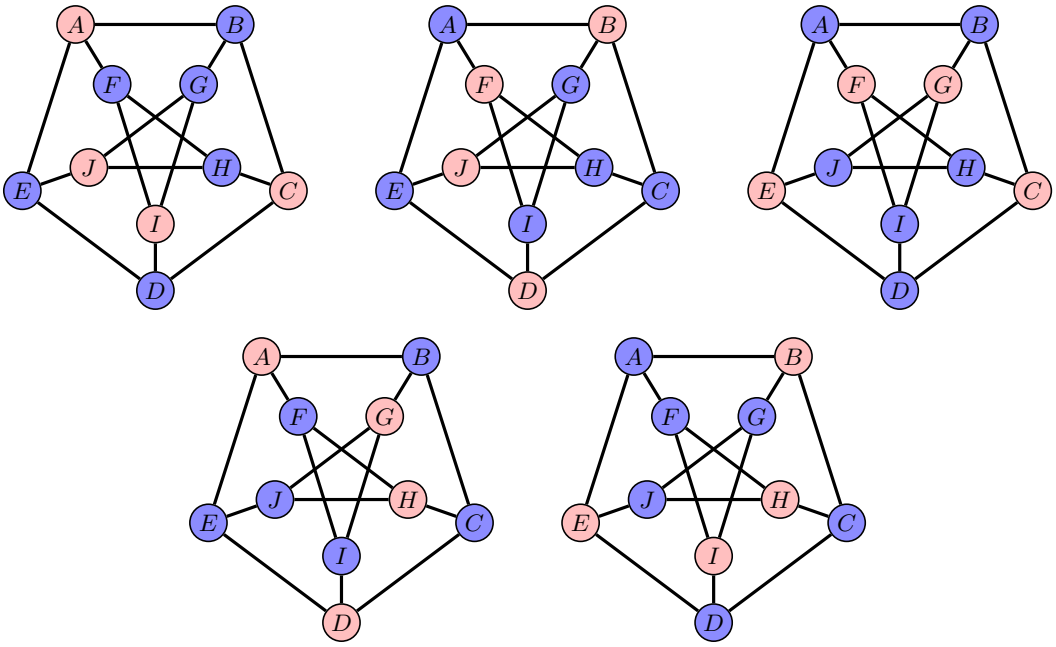
Grötzsch



FourTriangles

(a) What is the size of the **smallest** vertex cover in the Petersen graph? Give an example of such a vertex cover and prove that no smaller vertex cover exists.

Answer: 6. See below five examples. In each example, the blue vertices form a vertex cover.

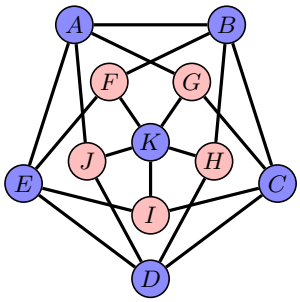


Proposition (Petersen graph): No vertex cover in the Petersen graph has fewer than 6 vertices.

Proof: The smallest vertex cover in the cycle of length 5, denoted by C_5, has size 3, since each vertex covers at most two edges and the cycle has five edges. The vertices of the Petersen graph are split between two cycles of length 5: the outer cycle and the inner cycle. Each cycle must contribute at least three vertices to a vertex cover, even without the restrictions imposed by the five edges that connect the two cycles. Therefore, the size of the smallest vertex cover in the Petersen graph is at least 6.

(b) What is the size of the **smallest** vertex cover in the Grötzsch graph? Give an example of such a vertex cover and prove that no smaller vertex cover exists.

Answer: 6. See below an example. The blue vertices form a vertex cover: the middle vertex K and the five outer-cycle vertices.

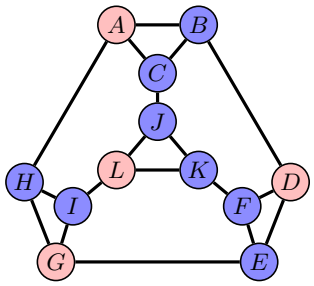


Proposition (Grötzsch graph): No vertex cover in the Grötzsch graph has fewer than 6 vertices.

Proof: Let VC be a vertex cover. The outer cycle $A-B-C-D-E-A$ has five edges, and each vertex covers at most two of them, so VC contains at least three outer-cycle vertices. If $K \notin VC$, then the five edges of K force $F, G, H, I, J \in VC$, and $|VC| \geq 3+5 = 8$. Otherwise, $K \in VC$. If VC contains at least two of F, G, H, I, J, then $|VC| \geq 3+1+2 = 6$. Otherwise, at least four of F, G, H, I, J are not in VC, so all their outer-cycle neighbors are in VC. Each outer-cycle vertex is adjacent to two of F, G, H, I, J, and hence to at least one of these four. Therefore, all five outer-cycle vertices are in VC, and $|VC| \geq 5+1 = 6$. Moreover, this vertex cover of size 6 is unique. In the case with two of F, G, H, I, J, the size 6 requires exactly three outer-cycle vertices, so three of F, G, H, I, J are not in VC, and their outer-cycle neighbors are in VC. Each of F, G, H, I, J is adjacent to the two outer-cycle neighbors of one outer-cycle vertex (F of A, G of B, and so on), and among three outer-cycle vertices two are adjacent, say B and C. Then G and H have the four outer-cycle neighbors A, C, B, D, which is impossible. In the last case, the size 6 requires that none of F, G, H, I, J is in VC, so $VC = \{A, B, C, D, E, K\}$.

(c) What is the size of the **smallest** vertex cover in the FourTriangles graph? Give an example of such a vertex cover and prove that no smaller vertex cover exists.

Answer: 8. See below an example. The blue vertices form a vertex cover: all the vertices except A, D, G, and L.



Proposition (FourTriangles graph): No vertex cover in the FourTriangles graph has fewer than 8 vertices.

Proof: A vertex covers at most two of the three edges of a triangle, so a vertex cover contains at least two vertices of each of the four triangles, and these $4 \cdot 2 = 8$ vertices are all different.

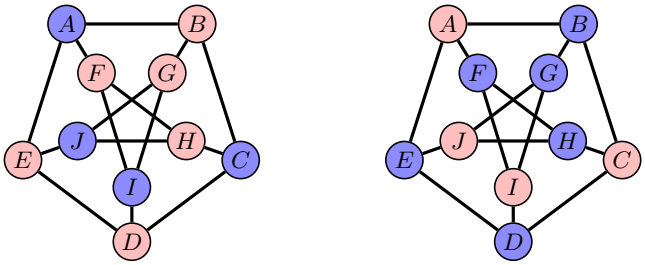
(d) An **independent set** is a set of vertices, no two of which are adjacent. Let G be a graph with n vertices, and let V be the set of its vertices. Prove that a set IS of vertices is an independent set in G if and only if $V \setminus IS$ is a vertex cover in G. Conclude that the size of the largest independent set plus the size of the smallest vertex cover is n.

Proof: Let $VC = V \setminus IS$.

- Assume that IS is an independent set in G. By definition, IS does not contain both endpoints of any edge. Hence, each edge has at least one endpoint that does not belong to IS, and therefore belongs to VC. Thus, VC is a vertex cover in G.
- Conversely, assume that VC is a vertex cover in G. By definition, VC contains at least one endpoint of each edge. Hence, IS does not contain both endpoints of any edge. Thus, IS is an independent set in G.

It follows that the complement of an independent set with the largest size in the graph is a vertex cover with the smallest size in the graph. As a result, for any graph G, the size of G's largest independent set plus the size of G's smallest vertex cover is the number of vertices in G.

For example, in the Petersen graph below, the four blue vertices on the left form an independent set, and the other six vertices, which are blue on the right, form a vertex cover.



Remark: By parts (a), (b), and (c), the smallest vertex covers of the Petersen graph, the Grötzsch graph, and the FourTriangles graph have sizes 6, 6, and 8. Hence, their largest independent sets have sizes $10 - 6 = 4$, $11 - 6 = 5$, and $12 - 8 = 4$.[62]

Symmetry and counting: All three graphs are very symmetric, but the Petersen graph is the most symmetric. It can be mapped onto itself in 120 ways, so that any vertex can go to any other vertex and any edge to any other edge. The FourTriangles graph has 24 such mappings: any vertex can go to any other vertex, but an edge of a triangle can never go to an edge between triangles. The Grötzsch graph has only 10: the rotations and reflections of its drawing, all of which keep K in place. The details of these mappings are omitted. Below, the optimal solutions are counted in the labeled graphs, in which every vertex has a distinct name, and up to "symmetry".

The Petersen graph: By the proof in part (a), a vertex cover of size 6 must contain exactly three vertices of the outer cycle and exactly three vertices of the inner cycle. In each cycle, the two vertices that are not in the vertex cover must be non-adjacent, since otherwise the edge between them is not covered. Moreover, each such choice of the three outer-cycle vertices can be combined with exactly one choice of the three inner-cycle vertices. Since the outer cycle has five pairs of non-adjacent vertices, the labeled Petersen graph has exactly five different vertex covers of size 6. These are the five examples in the answer to part (a).

Rotating the drawing of the Petersen graph by one fifth of a full turn (72°) maps the Petersen graph onto itself.[63] Each of these examples is obtained from the previous one by this rotation, and rotating the last example gives the first one. Hence, if the vertices are not labeled, all five vertex covers are essentially the same: up to "symmetry", there is only one vertex cover of size 6 in the Petersen graph.

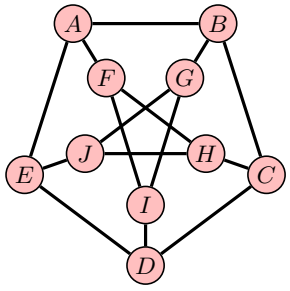
The other two graphs: The labeled Grötzsch graph has only one vertex cover of size 6, namely K and the five outer-cycle vertices. The labeled FourTriangles graph has 30 vertex covers of size 8, and up to "symmetry" there are 2 of them. By part (d), these are the complements of the 30 independent sets of size 4.[64]

[62] These sizes are found directly in the solution of Problem 7.2 on page 118.
[63] The same rotation is discussed for independent sets in the solution of Problem 7.2 on page 120.
[64] See the solution of Problem 7.2 on page 120.

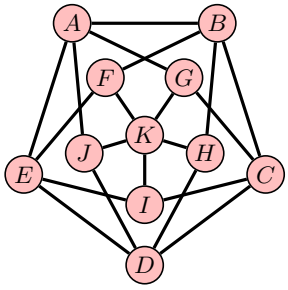
7.4. Let G be a simple undirected graph with $n \geq 1$ vertices. A **dominating set** in G is a set DS of vertices for which every vertex not in DS has at least one neighbor in DS. That is, for any vertex v of G, $v \in DS$ or v has a neighbor in DS.

For example, in the null graph (no edges) any dominating set must contain all the vertices, and therefore this is the only possible dominating set. In contrast, in the complete graph K_n (every two vertices are adjacent) any single vertex forms a dominating set, because it is adjacent to all the other vertices.

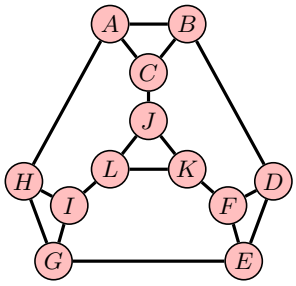
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



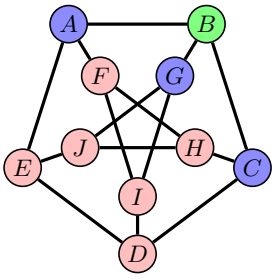
Grötzsch



FourTriangles

(a) What is the size of the **smallest** dominating set in the Petersen graph? Give an example of such a dominating set and prove that no smaller dominating set exists.

Answer: 3. See below an example. The blue vertices form a dominating set. They are the three neighbors of the green vertex B.

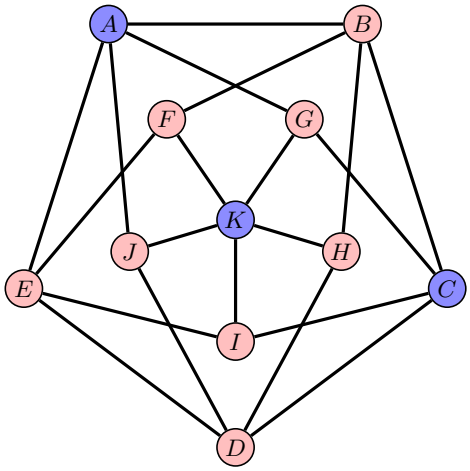


Proposition (Petersen graph): No dominating set in the Petersen graph has fewer than 3 vertices.

Proof: Each vertex dominates itself and its three neighbors. Therefore, two vertices dominate at most eight of the ten vertices of the Petersen graph. As a result, a dominating set in the Petersen graph must contain at least 3 vertices.

(b) What is the size of the **smallest** dominating set in the Grötzsch graph? Give an example of such a dominating set and prove that no smaller dominating set exists.

Answer: 3. See below an example. The blue vertices form a dominating set: K dominates F, G, H, I, J, and A and C dominate the outer cycle.

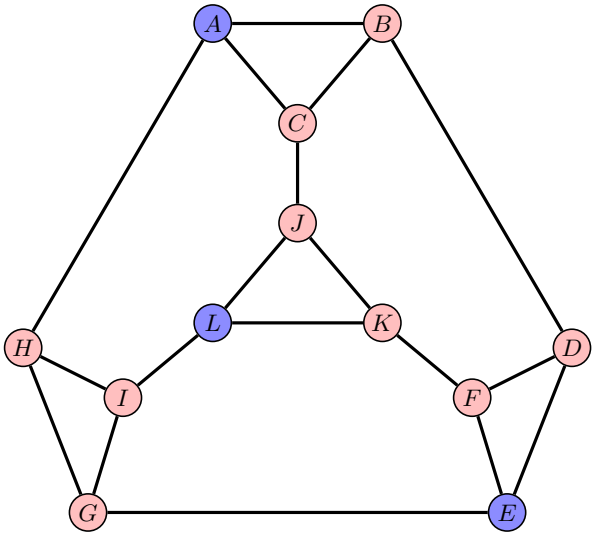


Proposition (Grötzsch graph): No dominating set in the Grötzsch graph has fewer than 3 vertices.

Proof: A vertex dominates itself and its neighbors. The middle vertex K has degree 5, and every other vertex has degree at most 4. Hence, two vertices other than K dominate at most $5 + 5 = 10$ of the 11 vertices. Two vertices that include K dominate K and F, G, H, I, J, and the second vertex must dominate all five outer-cycle vertices. However, an outer-cycle vertex dominates only three outer-cycle vertices, and each of F, G, H, I, J dominates only two. Hence, a dominating set has at least 3 vertices.

(c) What is the size of the **smallest** dominating set in the FourTriangles graph? Give an example of such a dominating set and prove that no smaller dominating set exists.

Answer: 3. See below an example. The blue vertices form a dominating set: A dominates A, B, C, H, E dominates D, E, F, G, and L dominates I, J, K, L, so every vertex is dominated exactly once.



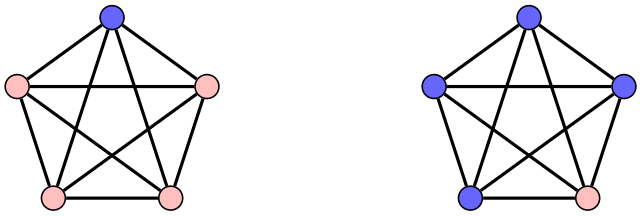
Proposition (FourTriangles graph): No dominating set in the FourTriangles graph has fewer than 3 vertices.

Proof: Every vertex has degree 3, so it dominates four vertices: itself and its three neighbors. Hence, two vertices dominate at most eight of the twelve vertices.

(d) A **vertex cover** is a set of vertices that contains at least one endpoint of every edge. Prove that in a graph without isolated vertices (vertices with no neighbors), every vertex cover is a dominating set. Show that a dominating set is not necessarily a vertex cover. Moreover, show that for every $n \geq 2$, there is a graph with n vertices whose smallest vertex cover is $n-1$ times as large as its smallest dominating set.

Answer: Let VC be a vertex cover, and let v be a vertex that is not in VC. Since v is not isolated, it has a neighbor u, and since the edge (u, v) is covered, $u \in VC$. Hence, v has a neighbor in VC, and VC is a dominating set. However, a dominating set is not necessarily a vertex cover. In the complete graph K_n, a single vertex dominates all the other vertices, so the smallest dominating set has size 1, but for $n \geq 3$ it is not a vertex cover. In fact, a vertex cover of K_n misses at most one vertex, since if it misses two vertices, the edge between them is not covered. Hence, for every $n \geq 2$, the smallest vertex cover of K_n has size $n-1$, which is $n-1$ times as large as the smallest dominating set. This ratio is the largest possible, since all the vertices except one always form a vertex cover, and only K_n attains it, since if u and v are not adjacent, then all the vertices except u and v form a vertex cover.

For example, in K_5 below, the single blue vertex on the left is a dominating set, while a vertex cover needs at least the four blue vertices on the right.



Symmetry and counting: All three graphs are very symmetric, but the Petersen graph is the most symmetric. It can be mapped onto itself in 120 ways, so that any vertex can go to any other vertex and any edge to any other edge. The FourTriangles graph has 24 such mappings: any vertex can go to any other vertex, but an edge of a triangle can never go to an edge between triangles. The Grötzsch graph has only 10: the rotations and reflections of its drawing, all of which keep K in place. The details of these mappings are omitted. Below, the optimal solutions are counted in the labeled graphs, in which every vertex has a distinct name, and up to "symmetry".

The Petersen graph: The labeled Petersen graph has exactly ten different dominating sets of size 3: the three neighbors of each of its ten vertices. For example, the blue vertices in the answer to part (a) are the three neighbors of the green vertex. The three neighbors of a vertex v form a dominating set since they dominate v and themselves, and each of the remaining six vertices is at distance 2 from v,[65] and thus is adjacent to one of them. Conversely, let DS be a dominating set of size 3. Each vertex of DS dominates four vertices: itself and its three neighbors. These are $3 \cdot 4 = 12$ dominations of only ten vertices, so exactly two of them are *extra*: they dominate a vertex that is also dominated by another vertex of DS. No two vertices of DS are adjacent. Otherwise, these two vertices dominate each other, which gives two extra dominations, and the third vertex of DS gives a third extra domination, since it is either adjacent to one of them or has a common neighbor with it. Hence, every two vertices of DS are at distance 2, and thus have a common neighbor, which is dominated twice. Since there are only two extra dominations, all three pairs have the same common neighbor v. Thus, DS consists of the three neighbors of v.

Up to "symmetry", there is only one dominating set of size 3 in the Petersen graph. The reason is that the Petersen graph can be mapped onto itself so that any vertex is mapped to any other vertex, and such a mapping maps the three neighbors of the first vertex to the three neighbors of the second vertex. The details of these mappings are omitted.

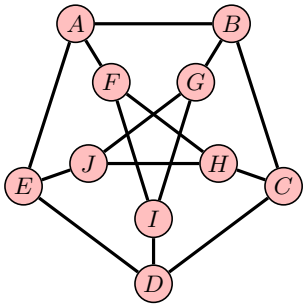
The other two graphs: The labeled Grötzsch graph has 5 dominating sets of size 3: the example in part (b) and its rotations, so up to "symmetry" there is only one. The labeled FourTriangles graph has 4 dominating sets of size 3: $\{A, E, L\}$, $\{B, G, K\}$, $\{C, F, I\}$, and $\{D, H, J\}$. Together, they split the twelve vertices into four groups, and up to "symmetry" there is only one. The details are omitted.

[65] The Petersen graph has diameter 2, and every two non-adjacent vertices have exactly one common neighbor; see the solutions of Problems 4.4(a) and 3.1(d) on pages 80 and 55.

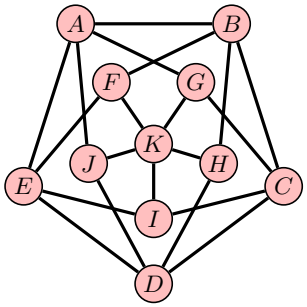
7.5. Let G be a simple undirected graph with $n \geq 1$ vertices. A **matching** in G is a set M of edges, no two of which share an endpoint. That is, every vertex of G is an endpoint of at most one edge of M.

For example, the only matching in the null graph (no edges) is the empty set. In contrast, the largest matching in the complete graph K_n (every two vertices are adjacent) has size $\lfloor n/2 \rfloor$. On the one hand, a matching with k edges has $2k$ different endpoints, so $2k \leq n$. On the other hand, since there is an edge between every pair of vertices, the vertices can be split into $\lfloor n/2 \rfloor$ pairs, and the edges between the two vertices of each pair form a matching (if n is odd, one vertex is left out).

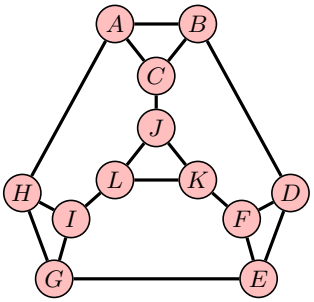
Consider the Petersen graph, the Grötzsch graph, and the FourTriangles graph below.



Petersen



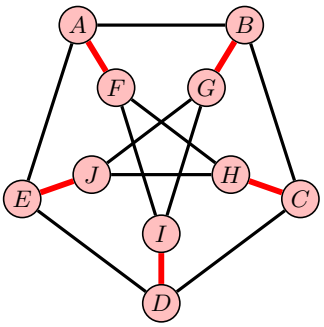
Grötzsch



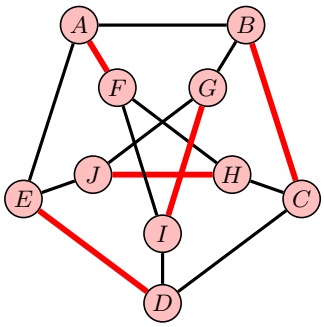
FourTriangles

(a) What is the size of the **largest** matching in the Petersen graph? Give an example of such a matching and prove that no larger matching exists.

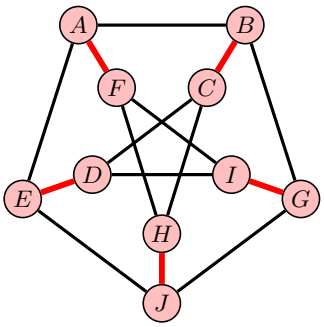
Answer: 5. See below two examples, M_1 and M_2. In each example, the red edges form a matching. The right drawing shows M_2 on another drawing of the Petersen graph (see Symmetry and counting at the end of this solution).



M_1



M_2



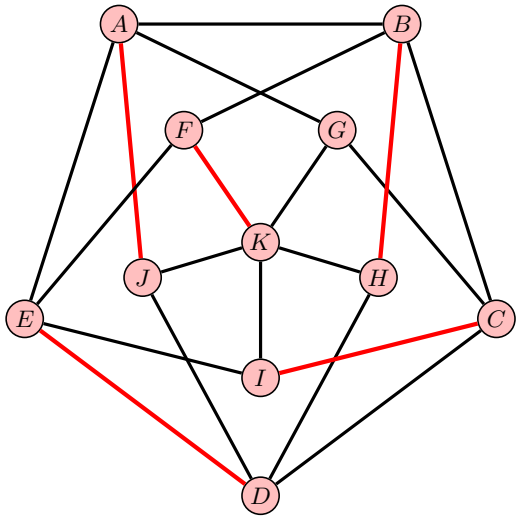
M_2 redrawn

Proposition (Petersen graph): No matching in the Petersen graph has more than 5 edges.

Proof: The edges of a matching do not share endpoints. Hence, a matching with k edges has $2k$ different endpoints. Since the Petersen graph has 10 vertices, $2k \leq 10$, and therefore $k \leq 5$.

(b) What is the size of the **largest** matching in the Grötzsch graph? Give an example of such a matching and prove that no larger matching exists.

Answer: 5. See below an example. The red edges (K, F), (A, J), (B, H), (C, I), and (D, E) form a matching, and only G is unmatched.

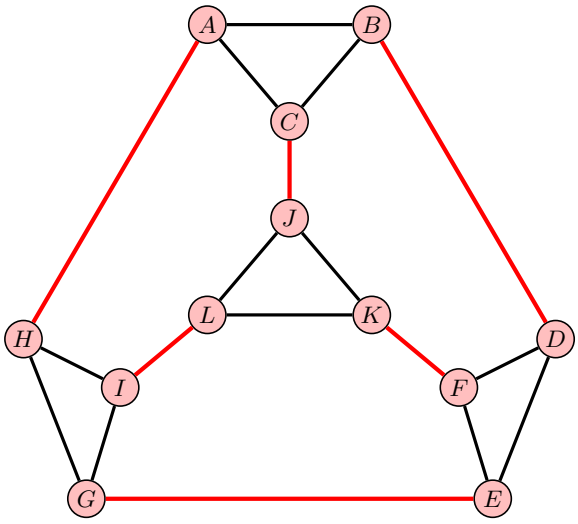


Proposition (Grötzsch graph): No matching in the Grötzsch graph has more than 5 edges.

Proof: A matching with k edges has $2k$ different endpoints. Since the Grötzsch graph has 11 vertices, $2k \leq 11$, and therefore $k \leq 5$.

(c) What is the size of the **largest** matching in the FourTriangles graph? Give an example of such a matching and prove that no larger matching exists.

Answer: 6. See below an example. Every vertex has exactly one neighbor outside its own triangle, so the six red edges between the triangles, (B, D), (E, G), (A, H), (C, J), (F, K), and (I, L), form a matching of size 6 that contains all the vertices.



Proposition (FourTriangles graph): No matching in the FourTriangles graph has more than 6 edges.

Proof: A matching with k edges has $2k$ different endpoints. Since the FourTriangles graph has 12 vertices, $2k \leq 12$, and therefore $k \leq 6$.

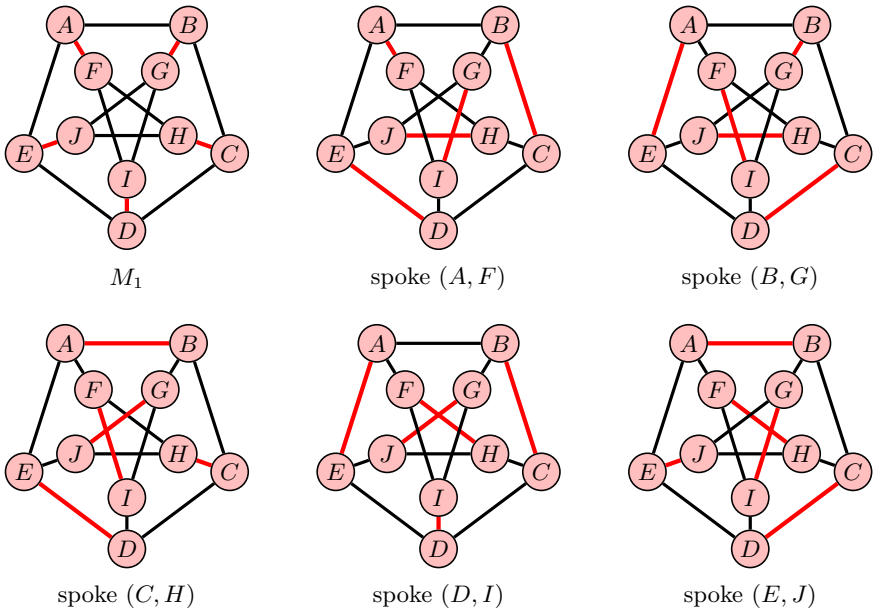
(d) The Petersen graph has exactly six matchings of size 5. Draw them, prove that every two of them share an edge, and conclude that the edges of the Petersen graph cannot be split into three matchings of size 5.

Answer: A matching of size 5 has 10 endpoints, so every vertex is an endpoint of exactly one of its edges. Such a matching is called a **perfect matching**. The labeled Petersen graph, in which every vertex has a distinct name, has exactly six different matchings of size 5. To see this, call the five edges between the outer cycle and the inner cycle **spokes**. In a matching of size 5, each outer-cycle vertex is matched either by its spoke or to one of its outer-cycle neighbors. The outer-cycle vertices of the second kind come in pairs, and the outer cycle has five vertices. Hence, the number of spokes in the matching is odd.

- *Five spokes:* This is the matching M_1.
- *Three spokes:* The two remaining outer-cycle vertices must be matched to each other, so they are adjacent. Their two inner neighbors must also be matched to each other. However, the inner neighbors of two adjacent outer-cycle vertices are never adjacent (for example, F and G). By rotation, the same holds for the other four pairs. Hence, this case is impossible.
- *One spoke:* Removing the two endpoints of the spoke leaves a path of four vertices from the outer cycle and a path of four vertices from the inner cycle. In each path, the only way to match all four vertices is to take the first and the last edges of the path. Hence, each spoke belongs to exactly one such matching. For example, M_2 is the matching that contains the spoke (A, F).

Thus, there are exactly $1 + 5 = 6$ matchings of size 5.

See below the six matchings of size 5 (red): M_1, which consists of the five spokes, and the five one-spoke matchings, each labeled by its spoke. As labeled matchings they are different, but up to "symmetry" they are all the same (see Symmetry and counting at the end of this solution).

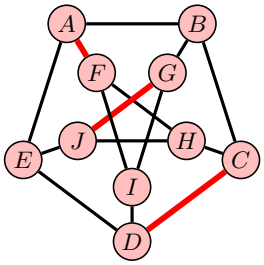


It remains to show that every two of these matchings share an edge. The matching M_1 shares with each one-spoke matching its spoke. By the rotation of the drawing, it remains to compare the one-spoke matching $M_2 = \{(A, F), (B, C), (D, E), (H, J), (G, I)\}$ of the spoke (A, F) with the one-spoke matchings of the spokes (B, G) and (C, H). These are $\{(B, G), (C, D), (E, A), (I, F), (H, J)\}$, which shares (H, J) with M_2, and $\{(C, H), (D, E), (A, B), (J, G), (I, F)\}$, which shares (D, E) with M_2. Hence, every two matchings of size 5 share an edge. In fact, they share exactly one edge, as shown in Symmetry and counting at the end of this solution. If the 15 edges were split into three matchings of size 5, these matchings would share no edge, a contradiction.[66]

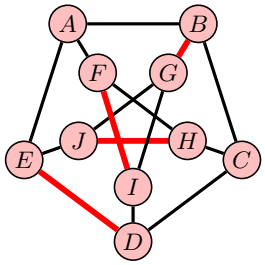
[66] Equivalently, the edges of the Petersen graph cannot be colored with 3 colors; see the solution of Problem 6.5 on page 110.

(e) A matching is **maximal** if no edge can be added to it so that it remains a matching, and **maximum** if it has the largest possible size. Show that every maximum matching is maximal. Find maximal matchings of sizes 3 and 4 in the Petersen graph, and prove that the Petersen graph has no maximal matching with fewer than 3 edges.

Answer: Every maximum matching is maximal, since adding an edge to it would give a larger matching, but not vice versa. By part (a), every matching of size 5 is maximum, and hence also maximal. However, maximal matchings of size 4 also exist. In the right example below, the two unmatched vertices A and C are not adjacent, so every edge has an endpoint in the matching, and no edge can be added. Even maximal matchings of size 3 exist. In the left example below, the four unmatched vertices B, E, H, and I are not adjacent to each other, so again no edge can be added.



size 3



size 4

Finally, no maximal matching of size 2, 1, or 0 exists. Two edges have four endpoints, each of degree 3, so they touch at most $4 \cdot 3 - 2 = 10$ edges (the two matching edges are counted twice), while the graph has 15 edges. A single edge touches even fewer edges, and the empty matching touches none.

Up to "symmetry", there is only one maximal matching of size 3 and only one maximal matching of size 4: the two examples above. The labeled Petersen graph has exactly 5 maximal matchings of size 3 and exactly 60 maximal matchings of size 4.

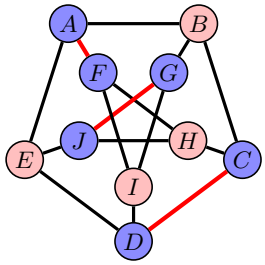
Size 3. Up to "symmetry", there is only one maximal matching of size 3. The four unmatched vertices of such a matching must form an independent set, since otherwise the edge between two of them could be added. Each of these four vertices has three edges, all going to the six matched vertices. These are 12 different edges, so exactly $15 - 12 = 3$ edges join two matched vertices, and the matching must consist of exactly these three edges. Hence, a maximal matching of size 3 is determined by its four unmatched vertices. The labeled Petersen graph has exactly five independent sets of size 4, and each is obtained from the previous one by a rotation of 72°. For $\{B, E, H, I\}$, the three remaining edges (A, F), (C, D), and (G, J) form a matching, and therefore so do the three remaining edges of each rotation. Thus, there are exactly five maximal matchings of size 3, all rotations of the left example.

Size 4. Up to "symmetry", there is also only one maximal matching of size 4, although the labeled Petersen graph has 60 of them. Let u and v be the two unmatched vertices. They are not adjacent, since otherwise the edge (u, v) could be added. Two non-adjacent vertices of the Petersen graph have exactly one common neighbor w (two common neighbors would form a cycle of length 4). The only neighbor of w other than u and v, call it x, must be matched to w. Each of the six remaining vertices loses exactly one neighbor among u, v, and x (a vertex adjacent to two of them would create a second common neighbor of u and v or a cycle of length 4), so these six vertices and the edges between them form a graph in which every vertex has degree 2. Since the Petersen graph has no cycles of length 3 or 4, this graph is a single cycle of length 6. Such a cycle has exactly two matchings that contain all of its vertices. Hence, each of the $45 - 15 = 30$ pairs of non-adjacent vertices is the unmatched pair of exactly two maximal matchings of size 4, which gives 60. In the right example, $u = A$, $v = C$, $w = B$, $x = G$, and the cycle is (D, E, J, H, F, I). The Petersen graph can be mapped onto itself so that these two examples are mapped to each other and any pair of non-adjacent vertices is mapped to any other such pair; the details of these mappings are omitted. Note also that the other 30 matchings of size 4 are not maximal: each is obtained by removing one edge from one of the six matchings of size 5.

(f) A **vertex cover** is a set of vertices that contains at least one endpoint of every edge. Prove that in every graph, every vertex cover is at least as large as every matching, and that the endpoints of a maximal matching form a vertex cover. Conclude that these endpoints form a vertex cover that is at most twice as large as the smallest vertex cover.

Proof: A vertex cover must contain an endpoint of each edge of a matching, and these endpoints are all different, since the edges of a matching do not share endpoints. Hence, in any graph, every vertex cover is at least as large as every matching. In the Petersen graph, the largest matching has size 5, while the smallest vertex cover has size 6.[67] Hence, the two sizes may differ. In the other direction, the endpoints of a maximal matching always form a vertex cover. Otherwise, some edge has no endpoint in the matching, and this edge could be added to the matching, contradicting its maximality. Combining the two observations, the $2|M|$ endpoints of a maximal matching M form a vertex cover whose size is at most twice the size of the smallest vertex cover. In the Petersen graph, the left maximal matching above, of size 3, gives the vertex cover $\{A, C, D, F, G, J\}$ of size 6, which is the smallest possible, while M_1 gives all 10 vertices, which is still at most $2 \cdot 6$.

See below the maximal matching of part (e) (red) and the vertex cover formed by its endpoints (blue).



Symmetry and counting: All three graphs are very symmetric, but the Petersen graph is the most symmetric. It can be mapped onto itself in 120 ways, so that any vertex can go to any other vertex and any edge to any other edge. The FourTriangles graph has 24 such mappings: any vertex can go to any other vertex, but an edge of a triangle can never go to an edge between triangles. The Grötzsch graph has only 10: the rotations and reflections of its drawing, all of which keep K in place. The details of these mappings are omitted. Below, the optimal solutions are counted in the labeled graphs, in which every vertex has a distinct name, and up to "symmetry".

The Petersen graph: Up to "symmetry", there is only one matching of size 5 in the Petersen graph. Rotating the drawing by 72° maps the Petersen graph onto itself and maps each of the five one-spoke matchings to the next one. Hence, it is enough to show that the Petersen graph can be redrawn so that M_2 becomes the five spokes. Removing the edges of M_2 leaves two cycles of length 5: $A-B-G-J-E-A$ and $C-D-I-F-H-C$. Draw the first cycle as the outer cycle and place the M_2-partner of each of its vertices inside, next to it. Then the edges of M_2 become the spokes. Moreover, the second cycle becomes the inner cycle: its five edges (C, D), (D, I), (I, F), (F, H), and (H, C) join the partners of B and E, of E and G, of G and A, of A and J, and of J and B, respectively. These are exactly the five pairs of non-adjacent vertices of the outer cycle $A-B-G-J-E-A$, and in the traditional drawing the inner cycle joins the inner partners of exactly these pairs. The right drawing in the answer to part (a) shows the result.

By the description of the six matchings in the answer to part (d), each edge of the Petersen graph belongs to exactly two of the six matchings of size 5. Here, the spoke at a vertex v is the spoke that touches v. Each spoke belongs to M_1 and to its one-spoke matching; for example, (A, F) belongs to M_1 and M_2. The one-spoke matching of the spoke at an outer-cycle vertex v contains the two outer-cycle edges that touch a neighbor of v but not v itself (the first and the last edges of the remaining path). Hence, each outer-cycle edge (u, w) belongs to the one-spoke matchings of the spokes at the other outer-cycle neighbor of u and at the other outer-cycle neighbor of w, and to no other matching of size 5. For example, (B, C) belongs to the one-spoke matchings of the spokes at A and at D. Similarly, each inner-cycle edge belongs to the one-spoke matchings of the spokes at the other inner-cycle neighbors of its two endpoints. For example, (H, J) belongs to M_2 (the one-spoke matching of the spoke at F) and to the one-spoke matching of the spoke at G. As a check, $6 \cdot 5 = 30 = 2 \cdot 15$.

[67] See the solution of Problem 7.3 on page 121.

Consequently, every two of the six matchings share exactly one edge. Indeed, there are $6 \cdot 5/2 = 15$ pairs of matchings, every pair shares at least one edge by part (d), and each of the 15 edges is shared by exactly one pair. Thus, the edges of the Petersen graph correspond one to one to the pairs of its matchings of size 5.

The other two graphs: The labeled Grötzsch graph has 87 matchings of size 5, and up to "symmetry" there are 10 of them (the details are omitted). The labeled FourTriangles graph has 8 matchings of size 6, and up to "symmetry" there are 3 of them. In such a matching, each triangle has 1 or 3 vertices that are matched by edges between triangles, since its other vertices are matched in pairs by its own edges. If two triangles have 3 such vertices, then each of them is matched to every other triangle, so all four triangles have 3, and the matching consists of the 6 edges between triangles (1 matching). If exactly one triangle has 3, it is matched to the other three triangles, and in each of them the remaining two vertices are matched to each other (4 matchings, one for each triangle). If every triangle has 1, the edges between triangles in the matching join the four triangles in two pairs (3 matchings). In total, there are $1 + 4 + 3 = 8$ matchings.

Matchings and Hamiltonian cycles: A **Hamiltonian cycle** is a simple cycle that visits all the vertices of the graph. The Petersen graph does not have a Hamiltonian cycle.[68] Assume to the contrary that it has a Hamiltonian cycle. Each vertex has degree 3, and the cycle uses two of its edges. Hence, the 5 edges that are not in the cycle form a matching of size 5, and removing this matching leaves a single cycle of length 10. However, as shown in Symmetry and counting above, up to "symmetry", M_1 is the only matching of size 5: the Petersen graph can be mapped onto itself so that any matching of size 5 is mapped to M_1. Such a mapping maps cycles to cycles. Since removing M_1 leaves two cycles of length 5, the outer cycle and the inner cycle, removing any matching of size 5 also leaves two cycles of length 5 rather than one cycle of length 10, which is a contradiction.

[68] Another proof appears in the solution of Problem 4.2(b) on page 77.