Algorithms

Quiz: Shortest Paths in Graphs – Solutions

1. Draw the shortest path tree rooted at $A$ of the following graph. What are the distances from $A$ to the rest of the vertices $B, C, D, E, F$? You do not need to explain how you constructed the tree and how you determined the distances.

The shortest path tree:

The distances from $A$:

<table>
<thead>
<tr>
<th>vertex</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$E$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>distance from $A$</td>
<td>1</td>
<td>8</td>
<td>4</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>
2. Draw a connected, weighted, undirected, simple graph with 5 vertices and 6 edges that has distinct positive edge weights.

Identify one vertex as the root (start) vertex and demonstrate a running of Dijkstra’s algorithm on this graph by illustrating the 5 weighted shortest path trees: the initial tree and the 4 trees implied by the algorithm after each addition of a vertex to the tree.
3. What is the running time of Dijkstra’s algorithm when the graph is a weighted undirected cycle with \( n \geq 3 \)?

**Answer:** The running time of Dijkstra’s algorithm on a connected graph with \( n \) vertices and \( m \) edges is \( \Theta(m \log n) \). In the cycle, \( m = n \). Therefore, without inspecting the running time analysis of the algorithm and/or modifying it, the running times becomes \( \Theta(n \log n) \).

Observe that for a cycle graph the priority queue contains only two vertices whose candidate distance to the source is not infinity. Therefore, adding a vertex to the shortest path tree can be done in time \( \Theta(1) \). Updating the priority queue can also be done in time \( \Theta(1) \) because only one vertex changes its priority from infinity to a finite distance. Thus, the running time of Dijkstra’s algorithm with this modified implementation is \( \Theta(n) \).

4. Let \( G \) be a complete (all possible edges exist) weighted undirected graph with \( n \geq 3 \) vertices. Assume that the weights obey the triangle inequality. That is, for any three vertices \( x, y, z \) it follows that

\[
    w(x, z) \leq w(x, y) + w(y, z)
\]

Let \( v \) be an arbitrary vertex in \( G \). What is the structure of the shortest path tree that is rooted in \( v \)?

**Answer:** Since there are no “short-cuts”, all the \( n - 1 \) other vertices in the graph would join the shortest path tree as children of the root vertex \( v \). That is, the structure of the shortest path tree is a *star tree*. 