Quiz 2: Solution

Input: An array \(A = [A_1, A_2, \ldots, A_n]\) of \(n \geq 2\) positive integers.

Definitions: Denote the sum of all the integers in the array by \(S\):
\[
S = A_1 + \cdots + A_n
\]
and for \(1 \leq i \leq n\), denote the sum of all the integers in the array except \(A_i\) by \(S_i\):
\[
S_i = S - A_i = A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n
\]

Example: Let \(A = [16, 2, 128, 64, 1, 8, 32, 4]\). Then \(S = 255\) and
\[
S_1 = 239; \quad S_2 = 253; \quad S_3 = 127; \quad S_4 = 191; \quad S_5 = 254; \quad S_6 = 247; \quad S_7 = 223; \quad S_8 = 251
\]

Task: Design an algorithm that computes \(S_1, S_2, \ldots, S_n\) only with plus operations (you are not allowed to use the minus operation).

Complexity: What is the number of plus operations used by your algorithm?

\(\Theta(n^2)\)-Algorithm:
- For \(1 \leq i \leq n\), compute \(S_i = A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n\).

Correctness: By definition.

Complexity: For \(1 \leq i \leq n\), computing \(S_i\) is done by exactly \(n - 2\) plus operations. Therefore, the total number of plus operations in this algorithm is
\[
n(n - 2) = n^2 - 2n = \Theta(n^2)
\]

\(\Theta(n)\)-Algorithm:
- Compute the prefix-sum of the first \(n - 1\) integers in \(A\). For \(1 \leq i \leq n - 1\), let \(P_i = \sum_{j=1}^{i} A_i\).
- Compute the suffix-sum of the last \(n - 1\) integers in \(A\). For \(n \geq i \geq 2\) let \(Q_i = \sum_{j=i}^{n} A_i\).
- Compute the array \(S\)
\[
S_i = \begin{cases} 
Q_2 & \text{for } i = 1 \\
P_{n-1} & \text{for } i = n \\
P_{i-1} + Q_{i+1} & \text{for } 2 \leq i \leq n - 1
\end{cases}
\]

Correctness: By definition, \(S_1 = Q_2\) and \(S_n = P_{n-1}\). Fix \(2 \leq i \leq n - 1\). Then \(P_{i-1} = A_1 + \cdots + A_{i-1}\) and \(Q_{i+1} = A_{i+1} + \cdots + A_n\). Therefore,
\[
S_i = P_{i-1} + Q_{i+1} = A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n
\]

Remark: Note that there is no need to compute the last values of the prefix-sum \((P_n)\) and the last value of the suffix-sum \((Q_1)\) because they are not required for the computations of \(S_1, S_2, \ldots, S_n\).

Complexity: The \(n - 1\) prefix-sum values can be computed with \(n - 2\) plus operations and so are the \(n - 1\) suffix-sum values. Then, for \(2 \leq i \leq n - 1\), all the \(S_i\) values are computed with \(n - 2\) plus operations. The total number of plus operations in this algorithm is
\[
(n - 2) + (n - 2) + (n - 2) = 3n - 6 = \Theta(n)
\]
Example

\[ A = [16, 2, 128, 64, 1, 8, 32, 4] \]
\[ P = [16, 18, 146, 210, 211, 219, 251, \ast] \]
\[ Q = [\ast, 239, 237, 109, 45, 44, 36, 4] \]
\[ S = [239, 253, 127, 191, 254, 247, 223, 251] \]