Algorithms: Growth of Functions

Amotz Bar-Noy

CUNY
A finite set of precise instructions for performing a computation or for solving a problem.

A specific set of instructions for carrying out a procedure or solving a problem, usually with the requirement that the procedure terminates at some point.

A procedure for solving a mathematical problem in a finite number of steps that frequently involves repetition of an operation.

A step-by-step procedure for solving a problem or accomplishing some end especially by a computer.

A logical arithmetical or computational procedure that if correctly applied ensures the solution of a problem.

A finite set of unambiguous instructions performed in a prescribed sequence to achieve a goal, especially a mathematical rule or procedure used to compute a desired result.
Algorithm: Definitions

- A word used by programmers when they do not want to explain what they did.
- A word used by those whose program failed to justify what they did.
Algorithm

**Synonym?**
- Method, Procedure, Program, Process, Recipe, Routine, Solution, Technique, Mechanism, Scheme, Way, Design, Plan, Strategy, Construction, ...

**Etymology**
- Alteration of Middle English algorisme,
- from Old French & Medieval Latin;
- from Medieval Latin algorismus,
- from Arabic al-khuwarizmi,
- from the name of the 9th-century Persian Mathematician Al-Khowârizmi who was the "first" to formalize the rules for the four basic arithmetic operations.
The Ultimate Algorithmic Problem!? 

Question

- What do we need to design "great" algorithms?

Answer

Apply some combination of these five attributes!!!
The Ultimate Algorithmic Problem!? 

**Question**
- What do we need to design ”great” algorithms?

**Attributes**
1. Talent?
2. Intuition?
3. Luck?
4. Experience?
5. Hard work?

Apply some combination of these five attributes!!!
The Ultimate Algorithmic Problem!? 

Question

What do we need to design "great" algorithms?

Attributes

1. Talent?
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Answer

Apply some combination of these five attributes!!!
What is an Algorithm?

Online Videos

- https://www.youtube.com/watch?v=Da5TOXCwLSg
- https://www.youtube.com/watch?v=6hfOvs8pY1k
- https://www.youtube.com/watch?v=CvSOaYi89B4&feature=youtu.be
Three Ancient Algorithms

The Babylonian Multiplication Algorithm
- Introduced around 3700 years ago.

The Euclid’s Greatest Common Divisor Algorithm
- Introduced around 2300 years ago.

The Sieve of Eratosthenes to Find Prime Numbers Algorithm
- Introduced around 2200 years ago.
Although there are some evidences of early multiplication algorithms in Egypt (around 1700-2000 BC) the oldest algorithm is widely accepted to have been found on a set of Babylonian clay tablets that date to around 1600-1800 BC.

Their true significance only came to light in 1972 when computer scientist & mathematician Donald E. Knuth published the first English translations of various Cuneiform mathematical tablets.

The Babylonians had developed a nice way to explain an algorithm by examples as the algorithm itself was being defined.

The tablets also appear to have been an early form of instruction manual.
The Euclid’s Greatest Common Divisor Algorithm

- The Euclidian algorithm is a procedure used to find the greatest common divisors (GCD) of two positive integers.

- It was first described by Euclid in his manuscript the Elements written around 300 BC.

- It is a very efficient computation that is still used today by computers in some form or other.
The Sieve of Eratosthenes Algorithm

- The Sieve of Eratosthenes is an ancient algorithm for finding all prime numbers up to any given limit.

- It is attributed to the Greek mathematician Eratosthenes of Cyrene and was “invented” around 200 BC.

- The algorithm iteratively marks as composite (i.e., not prime) the multiples of each prime, starting with the first prime number, 2.

- The “less efficient” method sequentially tests each candidate number for divisibility by previously found prime.

Video

https://www.youtube.com/watch?v=klcIklsWzrY&feature=youtu.be
# Algorithms — Properties

## Correctness
- For all valid inputs.

## Termination
- Does not run forever on some inputs.

## Complexity – Efficiency
- As a function of the input size.
- Worst-Case and/or Average-Case.

## Scalability
- “Similar” structure and efficiency for any input size.

## Limitations
- For the algorithm and for the problem.

## Optimality
- Optimal or near-optimal or approximately optimal solutions.
Cost and Complexity

Cost
- How much resources does the algorithm require?
  - Usually time and space (memory).

Complexity
- As a function of the input size.
  - Usually an integer $n > 0$.
  - Usually a monotonic non-decreasing function.

Terminology
- Complexity is often called running-time because time is the dominating cost.
Worst Case and Average Case Complexity

Worst case
- $T(n)$ is the **worst case complexity**:
  - If for all inputs of size $n$ the complexity is $T(n)$.

Average case
- $T(n)$ is the **average case complexity**:
  - If the average complexity over all length $n$ inputs is $T(n)$.
  - Averaging based on some distribution of the inputs
  - Usually the uniform distribution.
Bounds

**Upper Bound**
- A function $f(n)$ such that $T(n) \leq f(n)$ for all $n$.

**Lower bound**
- A function $g(n)$ such that $T(n) \geq g(n)$ for all $n$.

**Tight bound**
- A function $h(n)$ such that $T(n) \approx h(n)$ for all $n$. 

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Performance Evaluation of Algorithms

Theoretical analysis
- All possible inputs.
- Independent of hardware/software implementation.
- High level language.

Experimental Study
- Some typical inputs.
- Depends on hardware/software implementation.
- A real program.
**Objective**

- Develop a **language** to express that Algorithm $A$ is better than or worse than or equivalent to Algorithm $B$.

**Technique**

- Define a “$\leq$” relation between functions measuring the **growth** of functions.

**Robustness**

- Being **independent** of the hardware/software environment: Turing machine, RAM machine, classroom model, today computers, and future super-computers.

**An important property**

- Constants that can be affected by changing the environment should be ignored.
### Examples of Function Growth

<table>
<thead>
<tr>
<th>Running Time</th>
<th>1 second</th>
<th>1 minute</th>
<th>1 hour</th>
</tr>
</thead>
<tbody>
<tr>
<td>$400n$</td>
<td>2,500</td>
<td>150,000</td>
<td>9,000,000</td>
</tr>
<tr>
<td>$2n^2$</td>
<td>707</td>
<td>5,477</td>
<td>42,426</td>
</tr>
<tr>
<td>$n^4$</td>
<td>31</td>
<td>88</td>
<td>244</td>
</tr>
<tr>
<td>$2^n$</td>
<td>19</td>
<td>25</td>
<td>31</td>
</tr>
</tbody>
</table>

- Maximum size of a problem that can be solved in one second, one minute, and one hour, for various running times measured in **microseconds**.
### Examples of Function Growth

<table>
<thead>
<tr>
<th>Running Time</th>
<th>New Maximum Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$400n$</td>
<td>$256m$</td>
</tr>
<tr>
<td>$2n^2$</td>
<td>$16m$</td>
</tr>
<tr>
<td>$n^4$</td>
<td>$4m$</td>
</tr>
<tr>
<td>$2^n$</td>
<td>$m + 8$</td>
</tr>
</tbody>
</table>

- Increase in the maximum size of a problem that can be solved with a certain complexity, by using a computer that is **256 times faster** than the previous one.
- Each entry is given as a function of $m$, the previous maximum problem size.
The “$O$, $\Omega$, $\Theta$, $o$, $\omega$” Notation

**Big-Oh**
\[ f(n) = O(g(n)) \text{ if } f(n) \text{ asymptotically less than or equal to } g(n). \]

**Big-Omega**
\[ f(n) = \Omega(g(n)) \text{ if } f(n) \text{ asymptotically greater than or equal to } g(n). \]

**Big-Theta**
\[ f(n) = \Theta(g(n)) \text{ if } f(n) \text{ asymptotically equal to } g(n). \]

**Little-o**
\[ f(n) = o(g(n)) \text{ if } f(n) \text{ asymptotically strictly less than } g(n). \]

**Little-omega**
\[ f(n) = \omega(g(n)) \text{ if } f(n) \text{ asymptotically strictly greater than } g(n). \]
Big-Oh, Big-Omega, and Big-Theta

\[ f(n) = O(g(n)) \]
- **There exist** a real constant \( c > 0 \) and an integer constant \( n_0 > 0 \) such that \( f(n) \leq cg(n) \) for every integer \( n \geq n_0 \).

\[ f(n) = \Omega(g(n)) \]
- **There exist** a real constant \( c > 0 \) and an integer constant \( n_0 > 0 \) such that \( f(n) \geq cg(n) \) for every integer \( n \geq n_0 \).

\[ f(n) = \Theta(g(n)) \]
- **There exist** two real constants \( c' \geq c'' > 0 \) and an integer constant \( n_0 > 0 \) such that \( c''g(n) \leq f(n) \leq c'g(n) \) for every integer \( n \geq n_0 \).
# Growth of Functions

## Big-Oh and Big-Omega

<table>
<thead>
<tr>
<th></th>
<th>$f(n) = O(g(n))$</th>
<th>$g(n) = O(f(n))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(n)$ grows faster</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>$f(n)$ grows faster</td>
<td>NO</td>
<td>YES</td>
</tr>
<tr>
<td>same growth</td>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$f(n) = \Omega(g(n))$</th>
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<tbody>
<tr>
<td>$g(n)$ grows faster</td>
<td>NO</td>
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</tr>
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<td>NO</td>
</tr>
<tr>
<td>same growth</td>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>
\[ f(n) = O(g(n)) \iff g(n) = \Omega(f(n)) \]

**Assume \( f(n) = O(g(n)) \)**

- By the definition of \( O \), there exist \( c > 0 \) and \( n_0 > 0 \) such that \( f(n) \leq cg(n) \) for every \( n \geq n_0 \).
- It follows that \( g(n) \geq (1/c)f(n) \) for every \( n \geq n_0 \).
- Since \( 1/c > 0 \), by the definition of \( \Omega \), \( g(n) = \Omega(f(n)) \).

**Assume \( g(n) = \Omega(f(n)) \)**

- By the definition of \( \Omega \), there exist \( c > 0 \) and \( n_0 > 0 \) such that \( g(n) \geq cf(n) \) for every \( n \geq n_0 \).
- It follows that \( f(n) \leq (1/c)g(n) \) for every \( n \geq n_0 \).
- Since \( 1/c > 0 \), by the definition of \( O \), \( f(n) = O(g(n)) \).
\[ f(n) = \Theta(g(n)) \iff (f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))) \]

**Assume** \( f(n) = \Theta(g(n)) \)

- By the definition of \( \Theta \), there exist \( c', c'' > 0 \) and \( n_0 > 0 \) such that \( c'' g(n) \leq f(n) \leq c' g(n) \) for every \( n \geq n_0 \).
- By the definition of \( O \), \( f(n) = O(g(n)) \) for \( c = c' \) and \( n_0 \).
- By the definition of \( \Omega \), \( f(n) = \Omega(g(n)) \) for \( c = c'' \) and \( n_0 \).

**Assume** \( f(n) = O(g(n)) \) and \( f(n) = \Omega(g(n)) \)

- By the definition of \( O \), there exist \( c_1 > 0 \) and \( n_1 > 0 \) such that \( f(n) \leq c_1 g(n) \) for every \( n \geq n_1 \).
- By the definition of \( \Omega \), there exist \( c_2 > 0 \) and \( n_2 > 0 \) such that \( f(n) \geq c_2 g(n) \) for every integer \( n \geq n_2 \).
- Therefore, for \( n_0 \geq \max \{ n_1, n_2 \} \), it follows that \( c_2 g(n) \leq f(n) \leq c_1 g(n) \) for every \( n \geq n_0 \).
- By the definition of \( \Theta \), \( f(n) = \Theta(g(n)) \) for \( c' = c_1, c'' = c_2 \), and \( n_0 \).
\(\Theta\) is an equivalence relation
- **Reflexive:** \(f(n) = \Theta(f(n))\).
- **Symmetric:** \((f(n) = \Theta(g(n))) \iff (g(n) = \Theta(f(n)))\).
- **Transitive:** \(f(n) = \Theta(g(n)) \land g(n) = \Theta(h(n)) \implies (f(n) = \Theta(h(n)))\).

\(O\) and \(\Omega\), are reflexive relations
- \(f(n) = O(f(n))\).
- \(f(n) = \Omega(f(n))\).

\(O\) and \(\Omega\), are not symmetric relations
- \(f(n) = O(g(n))\) does not imply that \(g(n) = O(f(n))\).
- \(f(n) = \Omega(g(n))\) does not imply that \(g(n) = \Omega(f(n))\).

\(O\) and \(\Omega\), are transitive relations
- \(f(n) = O(g(n)) \land g(n) = O(h(n)) \implies f(n) = O(h(n))\).
- \(f(n) = \Omega(g(n)) \land g(n) = \Omega(h(n)) \implies f(n) = \Omega(h(n))\).
Growth of Functions

$n^2$ vs. $n$

$n = \mathcal{O}(n^2)$ and $n^2 = \Omega(n)$

- Observe that $n \leq n^2$ for integer $n \geq 1$ ($n < n^2$ for integer $n > 1$).
- Therefore, for $c = 1$ and $n_0 = 1$, the definition of $\mathcal{O}$ implies that $n = \mathcal{O}(n^2)$ and the definition of $\Omega$ implies that $n^2 = \Omega(n)$.

$n^2 \neq \mathcal{O}(n)$ and $n \neq \Omega(n^2)$

- Observe that if $(1/c) < n$ for a constant $c > 0$, then by multiplying both sides of the inequality by $cn$, it follows that $n < cn^2$.
- Therefore, $n < cn^2$ for every real constant $c > 0$ and integer $n \geq n_1 > (1/c)$.
- As a result, there are no real constant $c > 0$ and integer $n_0$ such that $n \geq cn^2$ for every integer $n \geq n_0$.
- Consequently, the definitions of $\mathcal{O}$ and $\Omega$ cannot be applied to get $n^2 = \mathcal{O}(n)$ or $n = \Omega(n^2)$. 

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Examples

\( \Theta \)

- \( 3n = \Theta(n/2) \)
- \( 1000000n = \Theta(n/100000) \)
- \( \log_2(n) = \Theta(\log_{10}(n)) \)

\( O \) and \( \Omega \)

- \( 5n = O(n \log_2(n/2)) \)
- \( n \log_2(n)/100000 = \Omega(10000000n) \).

Polynomials

- \( a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0 = \Theta(n^d) \)
  
  * for constants \( a_0, a_1, \ldots, a_d \) and \( a_d > 0 \).
Observations

Eliminating constants

For any real constant $c$ and $\Psi \in \{O, \Omega, \Theta\}$:

- $\Psi(cf(n)) = \Psi(f(n))$.
- $\Psi(f(n)/c) = \Psi(f(n))$.
- $\Psi(c) = \Psi(1)$.

Addition and multiplication rules

For $\Psi \in \{O, \Omega, \Theta\}$:

- $\Psi(f(n)) + \Psi(g(n)) = \Psi(f(n) + g(n))$.
- $\Psi(f(n)) \times \Psi(g(n)) = \Psi(f(n) \times g(n))$.

max rules

- $\Psi(f(n) + g(n)) = \Psi(\max \{f(n), g(n)\})$. 
Little-oh and Little-omega

\( f(n) = o(g(n)) \)

- If \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \):
  - For any constant \( c > 0 \) there exists an integer constant \( n_0 > 0 \) such that \( f(n) \leq cg(n) \) for every integer \( n \geq n_0 \).

\( f(n) = \omega(g(n)) \)

- If \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \):
  - For any constant \( c > 0 \) there exists an integer constant \( n_0 > 0 \) such that \( f(n) \geq cg(n) \) for every integer \( n \geq n_0 \).
Growth of Functions

Propositions

\( o \) and \( \omega \)
- \( f(n) = o(g(n)) \iff g(n) = \omega(f(n)) \).
- \( f(n) = o(g(n)) \land g(n) = o(h(n)) \Rightarrow f(n) = o(h(n)) \).
- \( f(n) = \omega(g(n)) \land g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n)) \).

\( o \) vs. \( \Theta \)
- \( f(n) = o(g(n)) \Rightarrow f(n) = \Theta(g(n)) \).
- \( f(n) = \Theta(g(n)) \nRightarrow f(n) = o(g(n)) \).

\( \omega \) vs \( \Omega \)
- \( f(n) = \omega(g(n)) \Rightarrow f(n) = \Omega(g(n)) \).
- \( f(n) = \Omega(g(n)) \nRightarrow f(n) = \omega(g(n)) \).
Examples

- \( \log_2 n = o(\sqrt{n}) \).
- \( n = \omega(\sqrt{n}) \).
- \( n^3 = \omega(n^2) \).
- \( 10^{100}n = o(n^2/10^{100}) \).
## Hierarchy of Functions

<table>
<thead>
<tr>
<th>Category</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>$1$</td>
</tr>
<tr>
<td>Log star</td>
<td>$\log^* n$</td>
</tr>
<tr>
<td>Loglog</td>
<td>$\log \log n$</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>$\log n$</td>
</tr>
<tr>
<td>Poly-logarithmic</td>
<td>$\log^k n$</td>
</tr>
<tr>
<td>Sub-linear</td>
<td>$n^\epsilon$</td>
</tr>
<tr>
<td>Linear</td>
<td>$n$</td>
</tr>
<tr>
<td>Above-linear</td>
<td>$n \log n$</td>
</tr>
<tr>
<td>Quadratic</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Cubic</td>
<td>$n^3$</td>
</tr>
<tr>
<td>Polynomial</td>
<td>$n^k$</td>
</tr>
<tr>
<td>Super-polynomial</td>
<td>$n^{\log n}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$2^n$</td>
</tr>
<tr>
<td>Factorial</td>
<td>$n!$</td>
</tr>
<tr>
<td>Super-exponential</td>
<td>$n^n$</td>
</tr>
<tr>
<td>Exponential tower</td>
<td>$2^{2^{\ldots}}$</td>
</tr>
</tbody>
</table>

*constant integer $k > 1$*
Recursion

Definition

- Recursion occurs when a “thing” is defined in terms of itself or of its type.

Focus

- Recursive formulas in mathematics.
- Recursive programs in computer science.

Illustrations

- [Image 1](https://res.cloudinary.com/indepth-dev/image/fetch/w_1000,f_auto/https://admin.indepth.dev/content/images/2020/02/1_appBwh6_RtvocVxwqpp1HA.jpeg)
- [Image 2](https://upload.wikimedia.org/wikipedia/commons/thumb/8/80/SierpinskiTriangle.svg/887px-SierpinskiTriangle.svg.png)
Recursive Formulas

Definition

- A recursive formula is defined on the set of integers greater than or equal to some number $m$ (usually 0 or 1).
- The formula computes the $n$th value based on some or all of the previous $n - 1$ values.

Goal

- Given initial values and a recursive formula, find its closed-form expression that does not depend on previous values.

Recursion and induction

- Usually proving the correctness of a solution (a closed-form expression) to a recursive formula is done by induction.
The Non-Negative Integers

The recursive formula:

\[
N(n) = \begin{cases} 
0 & \text{for } n = 0 \\
N(n - 1) + 1 & \text{for } n > 0 
\end{cases}
\]

The solution

\[N(n) = n\]

Proof by induction

- Induction base: \(N(0) = 0\)
- Induction hypothesis: \(N(k) = k\) for all \(0 \leq k < n\)
- Inductive step for \(n \geq 1\):

\[
N(n) = N(n - 1) + 1 = (n - 1) + 1 = n
\]
A Generalization

The recursive formula

\[ T(n) = \begin{cases} 
  b & \text{for } n = 0 \text{ and a real number } b \\
  T(n - 1) + a & \text{for } n > 0 \text{ and a real number } a 
\end{cases} \]

The solution

\[ T(n) = b + an \]

Proof by induction

- Induction base: \( T(0) = b = b + a \cdot 0 \)
- Induction hypothesis: \( T(k) = b + ak \) for all \( 0 \leq k < n \)
- Inductive step for \( n \geq 1 \):

\[
T(n) = T(n - 1) + a \\
= b + a(n - 1) + a \\
= b + an
\]
Special Cases

The recursive formula

\[
T(n) = \begin{cases} 
  b & \text{for } n = 0 \text{ and a real number } b \\
  T(n - 1) + a & \text{for } n > 0 \text{ and a real number } a 
\end{cases}
\]

The solution

\[
T(n) = b + an
\]

Examples

- **All integers:** \( b = 0 \) and \( a = 1 \) \( \implies T(n) = n \)
- **Even integers:** \( b = 0 \) and \( a = 2 \) \( \implies T(n) = 2n \)
- **Odd integers:** \( b = 1 \) and \( a = 2 \) \( \implies T(n) = 2n + 1 \)

An infinite arithmetic progression

- \((X(0), X(1), \ldots, X(n), \ldots) = (x, x + d, x + 2d, \ldots, x + (n - 1)d, \ldots)\)
- \( b = x \) and \( a = d \) \( \implies X(n) = x + dn \)
Powers of integers

The recursive formula

\[ P(n) = \begin{cases} 
1 & \text{for } n = 0 \\
 d \cdot P(n - 1) & \text{for } n \geq 1 
\end{cases} \]

The solution

\[ P(n) = d^n \]

Proof by induction

- Induction base: \( P(0) = 1 = d^0 \)
- Induction hypothesis: \( P(k) = d^k \) for all \( 0 \leq k < n \)
- Inductive step for \( n \geq 1 \): \( P(n) = d \cdot P(n - 1) = d \cdot d^{n-1} = d^n \)
Factorials

The recursive formula

\[ F(n) = \begin{cases} 
1 & \text{for } n = 0 \\
F(n-1) & \text{for } n \geq 1 
\end{cases} \]

The solution

\[ F(n) = n! \]

Proof by induction

- Induction base: \( F(0) = 1 = 0! \)
- Induction hypothesis: \( F(k) = k! \) for all \( 0 \leq k < n \)
- Inductive step for \( n \geq 1 \): \( F(n) = nF(n-1) = n(n-1)! = n! \)
The $\log_2$ Function

The recursive formula

$$L(n) = \begin{cases} 
0 & \text{for } n = 1 \\
L(n/2) + 1 & \text{for } n = 2^h \text{ and } h > 0
\end{cases}$$

The solution

$$L(n) = \log_2(n)$$

$$L(2^h) = \log_2(2^h) = h$$
The $\log_2$ Function

Proof by induction

- Induction base: $L(1) = 0 = \log_2(1)$
- Induction hypothesis: $L(2^k) = k$ for all $0 \leq k < \log_2(n)$
- Inductive step: Assume $n = 2^h$ for $h > 0$ and therefore $n/2 = 2^{h-1}$

\[
L(n) = L(2^h) \\
= L(2^{h-1}) + 1 \\
= \log_2(2^{h-1}) + 1 \\
= (h - 1) + 1 \\
= h \\
= \log_2(n)
\]
The Sum $1 + 2 + \cdots + n$

The recursive formula

$$S(n) = \begin{cases} 
0 & \text{for } n = 0 \\
S(n - 1) + n & \text{for } n \geq 1 
\end{cases}$$

Small values of $n$

- $S(0) = 0$.
- $S(1) = S(0) + 1 = 0 + 1 = 1$
- $S(2) = S(1) + 2 = 1 + 2 = 3$
- $S(3) = S(2) + 3 = 3 + 3 = 6$
- $S(4) = S(3) + 4 = 6 + 4 = 10$

The solution

$$S(n) = \frac{n(n + 1)}{2}$$
The Sum $1 + 2 + \cdots + n$

Proof by induction

- Induction base: $S(0) = 0 = \frac{0 \cdot 1}{2}$
- Induction hypothesis: $S(k) = \frac{k(k+1)}{2}$ for all $0 \leq k < n$
- Inductive step for $n \geq 1$:

\[
S(n) = S(n - 1) + n = \frac{(n - 1)n}{2} + n = \frac{n^2 - n}{2} + \frac{2n}{2} = \frac{n^2 + n}{2} = \frac{n(n + 1)}{2}
\]
The Sum $1 + 3 + \cdots + (2k - 1)$

The recursive formula

$$S(n) = \begin{cases} 
1 & \text{for } n = 1 \\
S(n - 2) + n & \text{for odd } n > 1
\end{cases}$$

Small values of $n$

- $S(1) = 1$
- $S(3) = S(1) + 3 = 1 + 3 = 4$
- $S(5) = S(3) + 5 = 4 + 5 = 9$
- $S(7) = S(5) + 7 = 9 + 7 = 16$

The solution

Assume $n = 2h - 1$,

$$S(n) = S(2h - 1) = h^2 = \left(\frac{n + 1}{2}\right)^2$$
The Sum $1 + 3 + \cdots + (2k - 1)$

Proof by induction

- Induction base: $S(1) = S(2 \cdot 1 - 1) = 1 = 1^2$
- Induction hypothesis: $S(2k - 1) = k^2$ for all $1 \leq k < h$
- Inductive step for $n = 2h - 1$ and $h > 1$:

$$S(2h - 1) = S(2h - 3) + (2h - 1)$$
$$= S(2(h - 1) - 1) + (2h - 1)$$
$$= (h - 1)^2 + (2h - 1)$$
$$= (h^2 - 2h + 1) + (2h - 1)$$
$$= h^2$$
A General Recursive Formula

**Theorem**

- Let $a$ and $b$ be real numbers.
- Let $r \neq 1$ be a positive real number.
- Assume

\[
T(n) = \begin{cases} 
  b & \text{for } n = 0 \\
  rT(n - 1) + a & \text{for } n \geq 1 
\end{cases}
\]

- Then

\[
T(n) = br^n + a\frac{r^n - 1}{r - 1}
\]
A General Recursive Formula

The recursive formula

\[ T(n) = \begin{cases} 
  b & \text{for } n = 0 \\
  rT(n - 1) + a & \text{for } n \geq 1 
\end{cases} \]

Top-Down evaluation

\[ T(n) = rT(n - 1) + a \\
= r^2 T(n - 2) + ar + a \\
= r^3 T(n - 3) + ar^2 + ar + a \\
\vdots \\
= r^n T(0) + ar^{n-1} + ar^{n-2} + \cdots + ar + a \\
= br^n + a \sum_{i=0}^{n-1} r^i \\
= br^n + a \frac{r^n - 1}{r - 1} \]
A General Recursive Formula

The recursive formula

\[ T(n) = \begin{cases} 
  b & \text{for } n = 0 \\
  rT(n-1) + a & \text{for } n \geq 1 
\end{cases} \]

Bottom-Up evaluation

\[
\begin{align*}
T(0) &= b \\
T(1) &= rT(0) + a = br + a \\
T(2) &= rT(1) + a = br^2 + ar + a \\
T(3) &= rT(2) + a = br^3 + ar^2 + ar + a \\
& \vdots \\
T(n) &= br^n + a \sum_{i=0}^{n-1} r^i \\
T(n) &= br^n + a \frac{r^n - 1}{r - 1}
\end{align*}
\]
Growth of Functions

First Special Case of the General Formula

**Theorem**
- Recursive formula: \( T(0) = b \) and \( T(n) = rT(n-1) + a \)
- Closed-form formula: \( T(n) = br^n + a \frac{r^n-1}{r-1} \)

\( b = 1, \ a = 0, \text{ and } \ r \neq 1 \)
- Recursive formula: \( T(0) = 1 \) and \( T(n) = rT(n-1) \)
- Closed-form formula:
  \[
  T(n) = 1 \cdot r^n + 0 \cdot \frac{r^n - 1}{r - 1}
  = r^n
  \]
Second Special Case of the General Formula

**Theorem**

- Recursive formula: \( T(0) = b \) and \( T(n) = rT(n - 1) + a \)
- Closed-form formula: \( T(n) = br^n + a \frac{r^n - 1}{r - 1} \)

\( b = 0, \ a = 1, \text{ and } r = 2 \)

- Recursive formula: \( T(0) = 0 \) and \( T(n) = 2T(n - 1) + 1 \)
- Closed-form formula:
  
  \[
  T(n) = 0 \cdot r^n + 1 \cdot \frac{2^n - 1}{2 - 1} = 2^n - 1
  \]
Third Special Case of the General Formula

**Theorem**
- Recursive formula: $T(0) = b$ and $T(n) = rT(n - 1) + a$
- Closed-form formula: $T(n) = br^n + a\frac{r^n-1}{r-1}$

$b = 1$, $a = -1$, and $r = 2$
- Recursive formula: $T(0) = 1$ and $T(n) = 2T(n - 1) - 1$
- Closed-form formula:

$$T(n) = 1 \cdot 2^n - 1 \cdot \frac{2^n - 1}{2 - 1} = 1$$

**Intuition**
- $T(n) = 1$ because $2 \cdot 1 - 1 = 1$
Fourth Special Case of the General Formula

**Theorem**

- Recursive formula: \( T(0) = b \) and \( T(n) = rT(n - 1) + a \)
- Closed-form formula: \( T(n) = br^n + a\frac{r^n - 1}{r - 1} \)

\( b = 0, a = 1/2, \text{ and } r = 1/2 \)

- Recursive formula: \( T(0) = 0 \) and \( T(n) = (1/2)T(n - 1) + 1/2 \)
- Closed-form formula:

\[
T(n) = 0 \cdot r^n + \frac{1}{2} \cdot \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} = \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}}
\]

\[
= 1 - \left(\frac{1}{2}\right)^n = 1 - 2^{-n}
\]
Tower of Hanoi

Definition by example

https://www.youtube.com/watch?v=5Wn4EboLrMM

General Definition

- There are three pegs (rods) called $A$, $B$, and $C$ and $n \geq 1$ disks of different sizes.
- Initially all the disks are located on peg $A$ ordered from the largest at the bottom to the smallest at the top.
- A **legal move** takes any top disk and moves it to another peg as long as it is not placed on top of a smaller disk.
- **Goal:** Move the $n$ disks from $A$ to $B$ using only legal moves.
- **Efficiency:** Move the disks with as few as possible legal moves.
Tower of Hanoi

Demo
- https://www.mathsisfun.com/games/towerofhanoi.html

Recursive solution for four disks
- https://www.youtube.com/watch?v=YstLjLCGmgg

General Recursive solution
- **Initial call**: Move \( n \geq 1 \) disks from A to B.
- **Recursive call**: Move \( 1 \leq k \leq n \) disks from peg \( X \) to peg \( Y \) for \( X \neq Y \) and \( X, Y \in \{ A, B, C \} \).
- **Recursive base**: If \( k = 1 \): Move the top disk from \( X \) to \( Y \).
- **Recursive step**: If \( k > 1 \):
  - Move the top \( k - 1 \) disks from \( X \) to \( Z \notin \{ X, Y \} \).
  - Move the top disk from \( X \) to \( Y \).
  - Move the top \( k - 1 \) disks from \( Z \) to \( Y \).
**Tower of Hanoi**

**Correctness: proof by induction outline**

- **Induction base:** When $n = 1$, a disk can be legally moved from any peg to any other peg.

- **Induction hypothesis:** The smallest $1 \leq k < n$ disks can be legally moved from any peg to any peg.

- **Inductive step:**
  - The $n - 1$ smaller disks are legally moved from peg $A$ to peg $C$ by the induction hypothesis.
  - The largest disk is legally moved from peg $A$ to the empty peg $B$.
  - The $n - 1$ smaller disks are legally moved from peg $C$ to peg $B$ by the induction hypothesis.
Number of moves:

- Let $M(n)$ be the number of moves made by the recursive solution for $n \geq 1$ disks.

- Trivially, $M(1) = 1$.

- For $n > 1$, recursively, $M(n) = 2M(n - 1) + 1$.

- By the generalized formula (second special case)
  $$M(n) = 2^n - 1$$
A Divide-and-Conquer Recursive Formula

The recursive formula

\[ T(n) = \begin{cases} 
0 & \text{for } n = 1 \\
2T(n/2) + n & \text{for } n > 1 \text{ a power of 2} 
\end{cases} \]

Small values of \( n = 2^h \)

- \( T(2^0) = T(1) = 0 \)
- \( T(2^1) = T(2) = 2T(1) + 2 = 2 \)
- \( T(2^2) = T(4) = 2T(2) + 4 = 8 \)
- \( T(2^3) = T(8) = 2T(4) + 8 = 24 \)
- \( T(2^4) = T(16) = 2T(8) + 16 = 64 \)
- \( T(2^5) = T(32) = 2T(16) + 32 = 160 \)
A Divide-and-Conquer Recursive Formula

Guessing the solution

- For $n = 2^h$ guess:
  \[ T(n) = n \log_2 n \]

Verify the guess for small numbers

- $1 \log_2 1 = 0 = T(1)$
- $2 \log_2 2 = 2 = T(2)$
- $4 \log_2 4 = 8 = T(4)$
- $8 \log_2 8 = 24 = T(8)$
- $16 \log_2 16 = 64 = T(16)$
- $32 \log_2 32 = 160 = T(32)$
A Divide-and-Conquer Recursive Formula

Proof by induction

\[
T(n) = 2T\left(\frac{n}{2}\right) + n
\]
\[
= 2\left(\frac{n}{2}\right) \log_2 \left(\frac{n}{2}\right) + n
\]
\[
= n \log_2 n - 1 + n
\]
\[
= n \log_2 n
\]
A Generalized Divide-and-Conquer Recursive Formula

The recursive formula

For real numbers $a$ and $b$ (independent of $n$)

$$T(n) = \begin{cases} 
    a & \text{for } n = 1 \\
    2T(n/2) + bn & \text{for } n > 1 \text{ a power of 2}
\end{cases}$$

Small values of $n = 2^h$

- $T(2^0) = T(1) = a$
- $T(2^1) = T(2) = 2T(1) + 2b = 2b + 2a$
- $T(2^2) = T(4) = 2T(2) + 4b = 8b + 4a$
- $T(2^3) = T(8) = 2T(4) + 8b = 24b + 8a$
- $T(2^4) = T(16) = 2T(8) + 16b = 64b + 16a$
- $T(2^5) = T(32) = 2T(16) + 32b = 160b + 32a$
A Generalized Divide-and-Conquer Recursive Formula

Guessing the solution

For $n = 2^h$ guess:

$$T(n) = bn \log_2 n + an$$

Verify the guess for small numbers

- $b \cdot 1 \log_2 1 + a \cdot 1 = a = T(1)$
- $b \cdot 2 \log_2 2 + a \cdot 2 = 2b + 2a = T(2)$
- $b \cdot 4 \log_2 4 + a \cdot 4 = 8b + 4a = T(4)$
- $b \cdot 8 \log_2 8 + a \cdot 8 = 24b + 8a = T(8)$
- $b \cdot 16 \log_2 16 + a \cdot 16 = 64b + 16a = T(16)$
- $b \cdot 32 \log_2 32 + a \cdot 32 = 160b + 32a = T(32)$
A Generalized Divide-and-Conquer Recursive Formula

Proof by induction

\[ T(n) = 2T(n/2) + bn \]
\[ = 2(b(n/2) \log_2(n/2) + a(n/2)) + bn \]
\[ = (bn(\log_2 n - 1) + an) + bn \]
\[ = bn \log_2 n - bn + an + bn \]
\[ = bn \log_2 n + an \]
Another Divide-and-Conquer Recursive Formula

The recursive formula

For real numbers $a$ and $b$ (independent of $n$)

$$T(n) = \begin{cases} 
  a & \text{for } n = 1 \\
  T(n/2) + b & \text{for } n > 1 \text{ a power of 2}
\end{cases}$$

Small values of $n = 2^h$

- $T(2^0) = T(1) = a$
- $T(2^1) = T(2) = T(1) + b = b + a$
- $T(2^2) = T(4) = T(2) + b = 2b + a$
- $T(2^3) = T(8) = T(4) + b = 3b + a$
- $T(2^4) = T(16) = T(8) + b = 4b + a$
- $T(2^5) = T(32) = T(16) + b = 5b + a$
Another Divide-and-Conquer Recursive Formula

Guessing the solution

- For $n = 2^h$ guess:

\[ T(n) = b \log_2 n + a \]

Verify the guess for small numbers

- $b \log_2 1 + a = a = T(1)$
- $b \log_2 2 + a = b + a = T(2)$
- $b \log_2 4 + a = 2b + a = T(4)$
- $b \log_2 8 + a = 3b + a = T(8)$
- $b \log_2 16 + a = 4b + a = T(16)$
- $b \log_2 32 + a = 5b + a = T(32)$
Another Divide-and-Conquer Recursive Formula

Proof by induction

\[
T(n) = T(n/2) + b \\
= (b \log_2(n/2) + a) + b \\
= b(\log_2 n - 1) + a + b \\
= b \log_2 n - b + a + b \\
= b \log_2 n + a
\]
A Third Divide-and-Conquer Recursive Formula

The recursive formula

\[ T(n) = \begin{cases} 
1 & \text{for } n = 1 \\
4T(n/2) & \text{for } n > 1 \text{ a power of 2}
\end{cases} \]

Small values of \( n = 2^h \)

- \( T(2^0) = T(1) = 1 \)
- \( T(2^1) = T(2) = 4T(1) = 4 \)
- \( T(2^2) = T(4) = 4T(2) = 16 \)
- \( T(2^3) = T(8) = 4T(4) = 64 \)
- \( T(2^4) = T(16) = 4T(8) = 256 \)
- \( T(2^5) = T(32) = 4T(16) = 1024 \)
A Third Divide-and-Conquer Recursive Formula

Guessing the solution

- For $n = 2^h$ guess:

  $$T(n) = n^2$$

Verify the guess for small numbers

- $1^2 = 1 = T(1)$
- $2^2 = 4 = T(2)$
- $4^2 = 16 = T(4)$
- $8^2 = 64 = T(8)$
- $16^2 = 256 = T(16)$
- $32^2 = 1024 = T(32)$
A Third Divide-and-Conquer Recursive Formula

Proof by induction

\[ T(n) = 4T(n/2) = 4(n/2)^2 = 4(n^2/4) = n^2 \]
The Master Theorem

Assumptions

- For real numbers $a > 0$, $b > 1$, and $d \geq 0$ (independent of $n$):
  
  \[
  T(n) = \begin{cases} 
  \Theta(1) & \text{for } n = 1 \\
  aT(n/b) + \Theta(n^d) & \text{for } n > 1 
  \end{cases}
  \]

- $n/b$ can be either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$.

Theorem

- **Case I:** If $d < \log_b a$
  - then $T(n) = \Theta(n^{\log_b a})$.

- **Case II:** If $d = \log_b a$
  - then $T(n) = \Theta(n^{\log_b a \log n}) = \Theta(n^d \log n)$.

- **Case III:** If $d > \log_b a$
  - then $T(n) = \Theta(n^d)$. 

Example I

\[
T(1) = 1 \\
T(n) = 9T(n/3) + n
\]

- \(a = 9\).
- \(b = 3\).
- \(d = 1\).
- \(\log_b a = \log_3 9 = 2 > 1 = d\).

\(\implies \text{Case I: } T(n) = \Theta(n^2).\)
Example II

\[ T(1) = 1 \]
\[ T(n) = T(2n/3) + 1 \]

- \( a = 1 \).
- \( b = 3/2 \).
- \( d = 0 \).
- \( \log_b a = \log_{3/2} 1 = 0 = d \).

\[ \implies \text{Case II: } T(n) = \Theta(n^0 \log n) = \Theta(\log n). \]
Example III

\[ T(1) = 1 \]
\[ T(n) = 3T(n/4) + n \]

- \( a = 3 \).
- \( b = 4 \).
- \( d = 1 \).
- \( \log_b a = \log_4 3 \approx 0.793 < 1 = d \).

\( \Longrightarrow \) **Case III:** \( T(n) = \Theta(n) \).
Proof Outline for the Master Theorem

- Assume that $n$ is a power of $b$.
- There are $\log_b(n)$ levels to the recursion.
- The $k$th level is made up of $a^k$ subproblems.
- Each subproblem at level $k$ is of size $n/b^k$.
- The total work done at level $k$ is:

$$w(k) = a^k \cdot \Theta \left( \frac{n}{b^k} \right)^d = \Theta(n^d) \cdot \left( \frac{a}{b^d} \right)^k$$
Proof Outline for the Master Theorem

- The numbers \(w(0), w(1), \ldots, w(\log_b(n))\) form a geometric series with ratio \(a/b^d\).
- \(w(0) = \Theta(n^d)\).
- \(w(\log_b(n)) = \Theta(a^{\log_b(n)}) = \Theta(n^{\log_b(a)})\).

\[T(n) = \sum_{k=0}^{\log_b(n)} w(k) = \Theta(n^d) \sum_{k=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^k.\]

- The sum depends on the ratio \(a/b^d\).
Proof Outline for the Master Theorem

- If $a/b^d < 1$ then the sum is dominated by the first term.
  
  $T(n) = \Theta(w(0)) = \Theta(n^d)$.

- If $a/b^d = 1$ then all $\Theta(\log(n))$ terms are equal to $\Theta(n^d)$.
  
  $T(n) = \Theta(n^d \log(n))$.

- If $a/b^d > 1$ then the sum is dominated by the last term.
  
  $T(n) = \Theta(w(\log_b(n))) = \Theta(n^{\log_b(a)})$.

Comparing $a/b^d$ to 1 is equivalent to comparing $a$ to $b^d$ which is equivalent to comparing $\log_b(a)$ to $d$. 
Algorithm $\mathcal{A}$ has a worst case complexity $T(n)$

- To prove that $T(n) = O(f(n))$,
  - show this for all inputs of size $n$ for all $n \geq n_0$ for some integer $n_0$.

- To prove that $T(n) = \Omega(f(n))$,
  - show this for one input of size $n$ for infinitely many $n$.

- To prove that $T(n) = \Theta(f(n))$,
  - show that $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$. 
Algorithm $\mathcal{A}$ has an average case complexity $T(n)$ for a given distribution

- To prove that $T(n) = O(f(n))$,
  - show this by averaging over all inputs of size $n$ for all $n \geq n_0$ for some integer $n_0$

- To prove that $T(n) = \Omega(f(n))$,
  - show this by averaging over all inputs of size $n$ for infinitely many $n$.

- To prove that $T(n) = \Theta(f(n))$,
  - show that $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$. 