Algorithm: Definitions

1. A finite set of precise instructions for performing a computation or for solving a problem.

2. A specific set of instructions for carrying out a procedure or solving a problem, usually with the requirement that the procedure terminates at some point.

3. A procedure for solving a mathematical problem in a finite number of steps that frequently involves repetition of an operation.

4. A step-by-step procedure for solving a problem or accomplishing some end especially by a computer.

5. A logical arithmetical or computational procedure that if correctly applied ensures the solution of a problem.

6. A finite set of unambiguous instructions performed in a prescribed sequence to achieve a goal, especially a mathematical rule or procedure used to compute a desired result.
Algorithm: Definitions

- A word used by programmers when they do not want to explain what they did.

- A word used by those whose program failed to justify what they did.
**Algorithm**

- **Synonym:** Method, Procedure, Program, Recipe, Routine, Solution, Technique . . .

- **Etymology:** Alteration of Middle English *algorisme*, from Old French & Medieval Latin; from Medieval Latin *algorismus*, from Arabic *al-khuwarizmi*, from the name of the Persian Mathematician Al-Khowârizmi who was the first to formalize the rules for the 4 basic arithmetic operations.
Question: how do we solve problems?
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1. Talent?
2. Intuition?
3. Luck?
4. Experience?
5. Hard work?

Answer: use a combination of these 5 factors!!!
The Ultimate Algorithmic Problem!? 

Question: how do we solve problems?

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3. Luck?
4. Experience?
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Answer: use a combination of these 5 factors!!!
How to solve a problem? Some Heuristics

1. Search for a pattern.
2. Draw a figure.
3. Formulate an equivalent problem.
4. Modify the problem.
5. Choose effective notation.
7. Divide into cases.
8. Work backward.
10. Pursue parity.
11. Consider extreme cases.
Three Ancient Algorithms

1. The Babylonian Multiplication Algorithm:
   - Around 3700 years ago

2. The Euclid’s Greatest Common Divisor Algorithm:
   - Around 2300 years ago

3. The Sieve of Eratosthenes to Find Prime Numbers Algorithm:
   - Around 2200 years ago
The Babylonian Multiplication Algorithm

- Although there are some evidences of early multiplication algorithms in Egypt (around 1700-2000 BC) the oldest algorithm is widely accepted to have been found on a set of Babylonian clay tablets that date to around 1600-1800 BC.

- Their true significance only came to light in 1972 when computer scientist & mathematician Donald E. Knuth published the first English translations of various cuneiform mathematical tablets.

- The Babylonians had developed a nice way to explain an algorithm by examples as the algorithm itself was being defined.

- The tablets also appear to have been an early form of instruction manual.
The Euclid’s Greatest Common Divisor Algorithm

- The Euclidian algorithm is a procedure used to find the greatest common divisors (GCD) of two positive integers.

- It was first described by Euclid in his manuscript the Elements written around 300 BC.

- It is a very efficient computation that is still used today by computers in some form or other.
The Sieve of Eratosthenes to Find Prime Numbers

Algorithm

The Sieve of Eratosthenes is an ancient algorithm for finding all prime numbers up to any given limit.

It is attributed to the Greek mathematician Eratosthenes of Cyrene and was “invented” around 200 BC.

The algorithm iteratively marks as composite (i.e., not prime) the multiples of each prime, starting with the first prime number, 2.

The “less efficient” method sequentially tests each candidate number for divisibility by previously found prime.

[https://www.visnos.com/demos/sieve-of-eratosthenes](https://www.visnos.com/demos/sieve-of-eratosthenes)
**Correctness:** for all valid inputs.

**Complexity – Efficiency:** as a function of the input size.
- Worst-Case and/or Average-Case.

**Scalability:** “similar” efficiency for any input size.

**Limitations:** for the algorithm and for the problem.

**Optimality:** optimal or near-optimal or approximately optimal solutions.
How much of resources does an algorithm require?
- Usually: time and space (memory).

**Complexity**: as a function of the input length.
- Usually an integer $n > 0$.
- Usually a monotonic non-decreasing function.
Worst Case and Average Case Complexity

- $T(n)$ is a **worst case complexity**: If for all inputs of length $n$ the complexity is $T(n)$.

- $T(n)$ is an **average case complexity**: If the average complexity over all length $n$ inputs is $T(n)$. Averaging based on some distribution of the inputs (usually the uniform distribution).
**Bounds**

- **An Upper bound:** A function $f(n)$ such that $T(n) \leq f(n)$ for all $n$.

- **A Lower bound:** A function $g(n)$ such that $T(n) \geq g(n)$ for all $n$.

- **A Tight bound:** A function $h(n)$ such that $T(n) \approx h(n)$ for all $n$.
Performance Evaluation of Algorithms

**Theoretical analysis:**
- All possible inputs.
- Independent of hardware/software implementation.
- High level language.

**Experimental Study:**
- Some typical inputs.
- Depends on hardware/software implementation.
- A real program.
Objective: A language to express that Algorithm $A$ is better than or worse than or equivalent to Algorithm $B$.

Need to define a “≤” relation between functions measuring the growth of functions.


Ignore constants that can be affected by changing the environment.
Examples of Function Growth

<table>
<thead>
<tr>
<th>Running Time</th>
<th>1 second</th>
<th>1 minute</th>
<th>1 hour</th>
</tr>
</thead>
<tbody>
<tr>
<td>$400n$</td>
<td>2,500</td>
<td>150,000</td>
<td>9,000,000</td>
</tr>
<tr>
<td>$2n^2$</td>
<td>707</td>
<td>5,477</td>
<td>42,426</td>
</tr>
<tr>
<td>$n^4$</td>
<td>31</td>
<td>88</td>
<td>244</td>
</tr>
<tr>
<td>$2^n$</td>
<td>19</td>
<td>25</td>
<td>31</td>
</tr>
</tbody>
</table>

- Maximum size of a problem that can be solved in one second, one minute, and one hour, for various running times measured in microseconds.
Examples of Function Growth

<table>
<thead>
<tr>
<th>Running Time</th>
<th>New Maximum Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$400n$</td>
<td>$256m$</td>
</tr>
<tr>
<td>$2n^2$</td>
<td>$16m$</td>
</tr>
<tr>
<td>$n^4$</td>
<td>$4m$</td>
</tr>
<tr>
<td>$2^n$</td>
<td>$m + 8$</td>
</tr>
</tbody>
</table>

- Increase in the maximum size of a problem that can be solved with a certain complexity, by using a computer that is 256 times faster than the previous one.
- Each entry is given as a function of $m$, the previous maximum problem size.
The “$O$, $\Omega$, $\Theta$, $o$, $\omega$” Notation

- **Big-Oh:** \( f(n) = O(g(n)) \) if \( f(n) \) asymptotically less than or equal to \( g(n) \).

- **Big-Omega:** \( f(n) = \Omega(g(n)) \) if \( f(n) \) asymptotically greater than or equal to \( g(n) \).

- **Big-Theta:** \( f(n) = \Theta(g(n)) \) if \( f(n) \) asymptotically equal to \( g(n) \).

- **Little-o:** \( f(n) = o(g(n)) \) if \( f(n) \) asymptotically strictly less than \( g(n) \).

- **Little-omega:** \( f(n) = \omega(g(n)) \) if \( f(n) \) asymptotically strictly greater than \( g(n) \).
Big-Oh, Big-Omega, and Big-Theta

- \( f(n) = O(g(n)) \):
  - There exists a real constant \( c > 0 \) and an integer constant \( n_0 > 0 \) such that \( f(n) \leq cg(n) \) for every integer \( n \geq n_0 \).

- \( f(n) = \Omega(g(n)) \):
  - There exists a real constant \( c > 0 \) and an integer constant \( n_0 > 0 \) such that \( f(n) \geq cg(n) \) for every integer \( n \geq n_0 \).

- \( f(n) = \Theta(g(n)) \):
  - There exist two real constants \( c', c'' > 0 \) and an integer constant \( n_0 > 0 \) such that \( c''g(n) \leq f(n) \leq c'g(n) \) for every integer \( n \geq n_0 \).
\[ f(n) = O(g(n)) \text{ iff } g(n) = \Omega(f(n)) \]

**Assume \( f(n) = O(g(n)) \)**
- By the definition of \( O \), there exist \( c > 0 \) and \( n_0 > 0 \) such that \( f(n) \leq cg(n) \) for every \( n \geq n_0 \).
- It follows that \( g(n) \geq (1/c)f(n) \) for every \( n \geq n_0 \).
- Since \( 1/c > 0 \), by the definition of \( \Omega \), \( g(n) = \Omega(f(n)) \).

**Assume \( g(n) = \Omega(f(n)) \)**
- By the definition of \( \Omega \), there exist \( c > 0 \) and \( n_0 > 0 \) such that \( g(n) \geq cf(n) \) for every \( n \geq n_0 \).
- It follows that \( f(n) \leq (1/c)g(n) \) for every \( n \geq n_0 \).
- Since \( 1/c > 0 \), by the definition of \( O \), \( f(n) = O(g(n)) \).
\( f(n) = \Theta(g(n)) \) iff \( f(n) = O(g(n)) \) and \( f(n) = \Omega(g(n)) \)

Assume \( f(n) = \Theta(g(n)) \)
- By the definition of \( \Theta \), there exist \( c', c'' > 0 \) and \( n_0 > 0 \) such that \( c''g(n) \leq f(n) \leq c'g(n) \) for every \( n \geq n_0 \).
- By the definition of \( O \), \( f = O(g) \) for \( c = c' \) and \( n_0 \).
- By the definition of \( \Omega \), \( f = \Omega(g) \) for \( c = c'' \) and \( n_0 \).

Assume \( f(n) = O(g(n)) \) and \( f(n) = \Omega(g(n)) \)
- By the definition of \( O \), there exist \( c_1 > 0 \) and \( n_1 > 0 \) such that \( f(n) \leq c_1 g(n) \) for every \( n \geq n_1 \).
- By the definition of \( \Omega \), there exist \( c_2 > 0 \) and \( n_2 > 0 \) such that \( f(n) \geq c_2 g(n) \) for every integer \( n \geq n_2 \).
- Therefore, for \( n_0 \geq \max\{n_1, n_2\} \), it follows that \( c_2 g(n) \leq f(n) \leq c_1 g(n) \) for every \( n \geq n_0 \).
- By the definition of \( \Theta \), \( f = \Theta(g) \) for \( c' = c_1, c'' = c_2 \), and \( n_0 \).
More Propositions

- \( f(n) = \Theta(g(n)) \) iff \( g(n) = \Theta(f(n)) \).

- \( f(n) = O(g(n)) \) and \( g(n) = O(h(n)) \) \( \Rightarrow \) \( f(n) = O(h(n)) \).

- \( f(n) = \Omega(g(n)) \) and \( g(n) = \Omega(h(n)) \) \( \Rightarrow \) \( f(n) = \Omega(h(n)) \).

- \( f(n) = \Theta(g(n)) \) and \( g(n) = \Theta(h(n)) \) \( \Rightarrow \) \( f(n) = \Theta(h(n)) \).
\( n^2 \) vs. \( n \)

\[ n = O(n^2) \text{ and } n^2 = \Omega(n) \]

- Observe that \( n < n^2 \) for every integer \( n \geq 1 \).
- Therefore, for \( c = 1 \) and \( n_0 = 1 \), the definition of \( O \) implies that \( n = O(n^2) \) and the definition of \( \Omega \) implies that \( n^2 = \Omega(n) \).

\[ n^2 \neq O(n) \text{ and } n \neq \Omega(n^2) \]

- Observe that if \( (1/c) < n \) for a constant \( c > 0 \), then by multiplying both sides of the inequality by \( cn \), it follows that \( n < cn^2 \).
- Therefore, \( n < cn^2 \) for every real constant \( c > 0 \) and integer \( n \geq n_1 > (1/c) \).
- As a result, there are no real constant \( c > 0 \) and integer \( n_0 \) such that \( n \geq cn^2 \) for every integer \( n \geq n_0 \).
- Consequently, the definitions of \( O \) and \( \Omega \) cannot be applied to get \( n^2 = O(n) \) or \( n = \Omega(n^2) \).
More Examples

- \(3n = \Theta(n/2)\).
- \(1000000n = \Theta(n/100000)\).
- \(n \log_2 n/100000 = \Omega(100000000n)\).
- \(\log_2(n) = \Theta(\log_{10}(n))\).
- \(a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0 = \Theta(n^d)\)
  - for constants \(a_0, a_1, \ldots, a_d\) and \(a_d > 0\).
More Propositions

- For any real constant $c$:
  - $O(cf(n)) = O(f(n))$.  
  - $O(f(n)/c) = O(f(n))$.  
  - $O(c) = O(1)$.  

- $O(f(n)) + O(g(n)) = O(f(n) + g(n)) = O(\max \{ f(n), g(n) \})$.  

- $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$.  

Little-oh and Little-omega

\( f(n) = o(g(n)) \) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \):

For any constant \( c > 0 \) there exists an integer constant \( n_0 > 0 \) such that \( f(n) \leq cg(n) \) for every integer \( n \geq n_0 \).

\( f(n) = \omega(g(n)) \) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \):

For any constant \( c > 0 \) there exists an integer constant \( n_0 > 0 \) such that \( f(n) \geq cg(n) \) for every integer \( n \geq n_0 \).
Examples

- $\log_2 n = o(\sqrt{n})$.
- $n^3 = \omega(n^2)$.
- $10^{100} n = o(n^2/10^{100})$.

Propositions

- $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$.
- $f(n) = o(g(n))$ and $g(n) = o(h(n)) \implies f(n) = o(h(n))$.
- $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n)) \implies f(n) = \omega(h(n))$. 
## Hierarchy of Functions

<table>
<thead>
<tr>
<th>Function Type</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>$1$</td>
</tr>
<tr>
<td>Log star</td>
<td>$\log^* n$</td>
</tr>
<tr>
<td>Loglog</td>
<td>$\log \log n$</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>$\log n$</td>
</tr>
<tr>
<td>Poly-logarithmic</td>
<td>$\log^k n$ for constant integer $k &gt; 1$</td>
</tr>
<tr>
<td>Sub-linear</td>
<td>$n^\varepsilon$ for constant $0 &lt; \varepsilon &lt; 1$</td>
</tr>
<tr>
<td>Linear</td>
<td>$n$</td>
</tr>
<tr>
<td>Above-linear</td>
<td>$n \log n$</td>
</tr>
<tr>
<td>Quadratic</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Cubic</td>
<td>$n^3$</td>
</tr>
<tr>
<td>Polynomial</td>
<td>$n^k$ for constant integer $k &gt; 1$</td>
</tr>
<tr>
<td>Super-polynomial</td>
<td>$n^{\log n}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$2^n$</td>
</tr>
<tr>
<td>Factorial</td>
<td>$n!$</td>
</tr>
<tr>
<td>Super-exponential</td>
<td>$n^n$</td>
</tr>
<tr>
<td>Exponential tower</td>
<td>$2^2 \cdots n$ powers</td>
</tr>
</tbody>
</table>

Amotz Bar-Noy (CUNY)  
Analysis of Algorithms

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Consider the following recursive formula:

\[
T(1) = 0.
\]

\[
T(n) = 2T(n/2) + n.
\]

Compute the solution for small powers of 2:

\[
T(2) = 2T(1) + 2 = 2.
\]

\[
T(4) = 2T(2) + 4 = 8.
\]

\[
T(8) = 2T(4) + 8 = 24.
\]

\[
T(16) = 2T(8) + 16 = 64.
\]
Guessing the Solution

For $n = 2^k$ (power of 2), guess:

$$T(n) = n \log_2 n.$$ 

Verify the guess for small numbers:

- $1 \log_2 1 = 0$.
- $2 \log_2 2 = 2$.
- $4 \log_2 4 = 8$.
- $8 \log_2 8 = 24$.
- $16 \log_2 16 = 64$. 
A Proof by Induction

\[ T(n) = 2T(n/2) + n \]
\[ = 2(n/2) \log_2(n/2) + n \]
\[ = n(\log_2 n - 1) + n \]
\[ = n \log_2 n \]
A More Complicated Recursive Formula

Consider the following recursive formula:

- \( T(1) = a. \)
- \( T(n) = 2T(n/2) + bn. \)

For some constants \( a, b \) (independent of \( n \)).

Compute the solution for small powers of 2:

- \( T(2) = 2T(1) + 2b = 2b + 2a. \)
- \( T(4) = 2T(2) + 4b = 8b + 4a. \)
- \( T(8) = 2T(4) + 8b = 24b + 8a. \)
- \( T(16) = 2T(8) + 16b = 64b + 16a. \)
Guessing the Solution

For \( n = 2^k \) (power of 2), guess:

\[ T(n) = bn \log_2 n + an. \]

Verify the guess for small numbers:

- \( b \cdot 1 \log_2 1 + a \cdot 1 = a. \)
- \( b \cdot 2 \log_2 2 + a \cdot 2 = 2b + 2a. \)
- \( b \cdot 4 \log_2 4 + a \cdot 4 = 8b + 4a. \)
- \( b \cdot 8 \log_2 8 + a \cdot 8 = 24b + 8a. \)
- \( b \cdot 16 \log_2 16 + a \cdot 16 = 64b + 16a. \)
A Proof by Induction

\[ T(n) = 2T(n/2) + bn \]
\[ = 2(b(n/2) \log_2(n/2) + a(n/2)) + bn \]
\[ = bn(\log_2 n - 1) + an + bn \]
\[ = bn \log_2 n + an \]
Another Recursive Formula

Consider the following recursive formula:

- $T(1) = a$.
- $T(n) = T(n/2) + b$.

For some constants $a, b$ (independent of $n$).

Compute the solution for small powers of 2:

- $T(2) = T(1) + b = b + a$.
- $T(4) = T(2) + b = 2b + a$.
- $T(8) = T(4) + b = 3b + a$.
- $T(16) = T(8) + b = 4b + a$. 
Guessing the Solution

For $n = 2^k$ (power of 2), guess:

$$T(n) = b \log_2 n + a.$$ 

Verify the guess for small numbers:

- $b \cdot \log_2 1 + a = a.$
- $b \cdot \log_2 2 + a = b + a.$
- $b \cdot \log_2 4 + a = 2b + a.$
- $b \cdot \log_2 8 + a = 3b + a.$
- $b \cdot \log_2 16 + a = 4b + a.$
A Proof by Induction

\[ T(n) = T(n/2) + b \]
\[ = (b \log_2(n/2) + a) + b \]
\[ = b(\log_2 n - 1) + a + b \]
\[ = b \log_2 n + a \]
Another Recursive Formula

Consider the following recursive formula:

- \( T(1) = 0 \).
- \( T(n) = 4T(n/2) + n \).

Compute the solution for small powers of 2:

- \( T(2) = 4T(1) + 2 = 2 \).
- \( T(4) = 4T(2) + 4 = 12 \).
- \( T(8) = 4T(4) + 8 = 56 \).
- \( T(16) = 4T(8) + 16 = 240 \).
Guessing the Solution

For $n = 2^k$ (power of 2), guess:

$$T(n) = n^2 - n.$$ 

Verify the guess for small numbers:

- $1^2 - 1 = 0$.
- $2^2 - 2 = 2$.
- $4^2 - 4 = 12$.
- $8^2 - 8 = 56$.
- $16^2 - 16 = 240$. 
A Proof by Induction

\[
T(n) = 4T(n/2) + n \\
= 4((n/2)^2 - (n/2)) + n \\
= 4(n^2/4) - 2n + n \\
= n^2 - n
\]
The Master Theorem

**Assumptions:**
- Let \( a > 0 \) and \( b > 1 \) and \( d \geq 0 \) be constants (Independent of \( n \)).
- Let \( T(1) = \Theta(1) \).
- Let \( T(n) = aT(n/b) + \Theta(n^d) \) for \( n > 1 \).
  - \( n/b \) can be either \( \lfloor n/b \rfloor \) or \( \lceil n/b \rceil \).

**Theorem:**

- **Case I:** If \( d < \log_b a \),
  - then \( T(n) = \Theta(n^{\log_b a}) \).
- **Case II:** If \( d = \log_b a \),
  - then \( T(n) = \Theta(n^{\log_b a \log n}) = \Theta(n^d \log n) \).
- **Case III:** If \( d > \log_b a \),
  - then \( T(n) = \Theta(n^d) \).
Example 1

\[
T(1) = 1 \\
T(n) = 9T(n/3) + n
\]

- \(a = 9\).
- \(b = 3\).
- \(d = 1\).
- \(\log_b a = \log_3 9 = 2 > 1 = d\).

\[\implies \textbf{Case I: } T(n) = \Theta(n^2).\]
Example II

\[ T(1) = 1 \]
\[ T(n) = T(2n/3) + 1 \]

- \(a = 1\).
- \(b = 3/2\).
- \(d = 0\).
- \(\log_b a = \log_{3/2} 1 = 0 = d\).

\[ \Rightarrow \textbf{Case II: } T(n) = \Theta(\log n). \]
Example III

\[ T(1) = 1 \]
\[ T(n) = 3T(n/4) + n \]

- \( a = 3 \).
- \( b = 4 \).
- \( d = 1 \).
- \( \log_b a = \log_4 3 \approx 0.793 < 1 = d \).

\[ \implies \text{Case III: } T(n) = \Theta(n). \]
Proof Outline for the Master Theorem

- Assume that \( n \) is a power of \( b \).
- There are \( \log_b(n) \) levels to the recursion.
- The \( k \)th level is made up of \( a^k \) subproblems.
- Each subproblem at level \( k \) is of size \( n/b^k \).
- The total work done at level \( k \) is:

\[
w(k) = a^k \cdot \Theta \left( \frac{n}{b^k} \right)^d = \Theta(n^d) \cdot \left( \frac{a}{b^d} \right)^k
\]
The numbers $w(0), w(1), \ldots, w(\log_b(n))$ form a geometric series with ratio $a/b^d$.

- $w(0) = \Theta(n^d)$.
- $w(\log_b(n)) = \Theta(a^{\log_b(n)}) = \Theta(n^{\log_b(a)})$.

$T(n) = \sum_{k=0}^{\log_b(n)} w(k) = \Theta(n^d) \sum_{k=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^k$.

The sum depends on the ratio $a/b^d$. 

Proof Outline for the Master Theorem

Amotz Bar-Noy (CUNY)
Proof Outline for the Master Theorem

- If $a/b^d < 1$ then the sum is dominated by the first term.
  - $T(n) = \Theta(w(0)) = \Theta(n^d)$.

- If $a/b^d = 1$ then all $\Theta(\log(n))$ terms are equal to $\Theta(n^d)$.
  - $T(n) = \Theta(n^d \log(n))$.

- If $a/b^d > 1$ then the sum is dominated by the last term.
  - $T(n) = \Theta(w(\log_b(n))) = \Theta(n^{\log_b(a)})$.

Comparing $a/b^d$ to 1 is equivalent to comparing $a$ to $b^d$ which is equivalent to comparing $\log_b(a)$ to $d$. 
Algorithm $A$ has a **worst case** complexity $T(n)$:

- To prove that $T(n) = O(f(n))$, show this for all inputs of size $n$ for all $n$.
- To prove that $T(n) = \Omega(f(n))$, show this for one input of size $n$ for infinitely many $n$.
- To prove that $T(n) = \Theta(f(n))$, show that $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$. 
Algorithm $\mathcal{A}$ has an *average case* complexity $T(n)$ for a given distribution:

- To prove that $T(n) = O(f(n))$,
  - show this by *averaging over all* inputs of size $n$ for all $n$.

- To prove that $T(n) = \Omega(f(n))$,
  - show this by *averaging over all* inputs of size $n$ for infinitely many $n$.

- To prove that $T(n) = \Theta(f(n))$,
  - show that $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$. 

The Prefix-Sum Problem


- **Output:** An array $S$ of size $n$ such that for $1 \leq i \leq n$,

$$S[i] = \sum_{j=1}^{i} A[j].$$
The Prefix-Sum Problem


**Output:** An array $S$ of size $n$ such that for $1 \leq i \leq n$,

$$S[i] = \sum_{j=1}^{i} A[j].$$

**Example:**
- $A = [3, 1, 2, 3, 18, 100, \ldots]$
- $S = [3, 4, 6, 9, 27, 127, \ldots]$
Algorithm I

prefix-sum($A$)
for $i = 1$ to $n$ do
    $S[i] := 0$
for $i = 1$ to $n$ do
    for $j = 1$ to $i$ do

Correctness:
By definition.
Algorithm 1

- **prefix-sum**($A$)
  
  ```
  for i = 1 to n do
    S[i] := 0
  
  for i = 1 to n do
    for j = 1 to i do
  ```

- **Correctness**: By definition.
Algorithm I – Complexity

- $\Theta(n)$ time for the first loop.

- $\Theta(n^2)$ time for the second loop.
  - $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ iterations of the inner loop.
  - $\Theta(1)$ time for each iteration.

- $\Theta(n) + \Theta(n^2) = \Theta(n^2)$ time complexity.
Algorithm II

prefix-sum(A)

\[
\text{for } i = 2 \text{ to } n \text{ do} \\
S[i] := S[i - 1] + A[i]
\]

Correctness: By Induction.
Algorithm II

- **prefix-sum**($A$)
  - for $i = 2$ to $n$ do
    - $S[i] := S[i - 1] + A[i]$

- **Correctness:** By Induction.
Algorithm II – Complexity

- $n - 1$ iterations of the only loop.
- $\Theta(1)$ time for each iteration.
- $\Theta(n)$ time complexity.
Evaluating a Polynomial

- **Input:** Real numbers $a_0, a_1, \ldots, a_n$ and $c$.
- **Output:** The value of the polynomial $P(x)$ for $x = c$:
  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$
**Evaluating a Polynomial**

- **Input:** Real numbers $a_0, a_1, \ldots, a_n$ and $c$.

- **Output:** The value of the polynomial $P(x)$ for $x = c$:
  \[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0. \]

- **Example:**
  - $a_3 = 5$, $a_2 = 7$, $a_1 = 3$, $a_0 = 11$, and $c = 2$.
  - $P(x) = 5x^3 + 7x^2 + 3x + 11$.
  - $P(2) = 5 \cdot 2^3 + 7 \cdot 2^2 + 3 \cdot 2 + 11 = 85.$
Evaluating a Polynomial

**Input:** Real numbers $a_0, a_1, \ldots, a_n$ and $c$.

**Output:** The value of the polynomial $P(x)$ for $x = c$:

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0. \]

**Example:**

- $a_3 = 5, a_2 = 7, a_1 = 3, a_0 = 11$, and $c = 2$.
- $P(x) = 5x^3 + 7x^2 + 3x + 11$.
- $P(2) = 5 \cdot 2^3 + 7 \cdot 2^2 + 3 \cdot 2 + 11 = 85$.

**Optimization goal:** Minimize the number of operations (multiplications and additions) between real numbers.
Algorithm I: A Direct Approach

**Polynomial-Evaluation** \( P(x), c \)

\[
P(c) = a_0
\]

for \( i = 1 \) to \( n \) do

\[
a = a_i
\]

for \( j = 1 \) to \( i \) do

\[
a = a \cdot c
\]

\((* \ a = a_i c^i *)\)

\[
P(c) = P(c) + a
\]

\((* \ P(c) = a_i c^i + a_{i-1} c^{i-1} + \cdots + a_1 c + a_0 *)\)
Algorithm I: A Direct Approach

**Polynomial-Evaluation** \((P(x), c)\)

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P(c) = a_0
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\[
a = a \cdot c
\]

(* \(a = a_i c^i\) *)

\[
P(c) = P(c) + a
\]

(* \(P(c) = a_i c^i + a_{i-1} c^{i-1} + \cdots + a_1 c + a_0\) *)

**Correctness:** By definition.
Algorithm I – Complexity

- \( i \) multiplications in the \( i \)th iteration of the inner loop.

- \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \) multiplications overall.

- \( n \) additions in the outer loop.

- Total of \( \frac{1}{2} n^2 + \frac{3}{2} n \) operations.

- \( \Theta(n^2) \) time complexity.
Algorithm II: A Prefix-Sum Approach

**Idea:** Compute $c, c^2, c^3, \ldots, c^n$ all the powers of $c$ using the efficient prefix-sum method.
Algorithm II: A Prefix-Sum Approach

**Idea:** Compute $c, c^2, c^3, \ldots, c^n$ all the powers of $c$ using the efficient prefix-sum method.

**Polynomial-Evaluation** $(P(x), c)$

$$
P(c) = a_0 \\
cc = 1 \\
\text{for } i = 1 \text{ to } n \text{ do} \\
\hspace{1em} cc = cc \cdot c \\
\hspace{1em} (* cc = c^i *) \\
P(c) = P(c) + a_i \cdot cc \\
\hspace{1em} (* P(c) = a_i c^i + a_{i-1} c^{i-1} + \cdots + a_1 c + a_0 *)
$$

**Correctness:** By induction.
Algorithm II: A Prefix-Sum Approach

**Idea:** Compute \( c, c^2, c^3, \ldots, c^n \) all the powers of \( c \) using the efficient *prefix-sum* method.

**Polynomial-Evaluation** \((P(x), c)\)

\[
P(c) = a_0 \\
cc = 1 \\
\text{for } i = 1 \text{ to } n \text{ do} \\
\quad cc = cc \cdot c \\
\quad (* \ cc = c^i *) \\
P(c) = P(c) + a_i \cdot cc \\
\quad (* \ P(c) = a_i c^i + a_{i-1} c^{i-1} + \cdots + a_1 c + a_0 *)
\]

**Correctness:** By induction.
Algorithm II – Complexity

- 2 multiplications in the \(i\)th iteration of the loop.
- 1 addition in the \(i\)th iteration of the loop.

Total of 3\(n\) operations: 2\(n\) multiplications and \(n\) additions.

\(\Theta(n)\) time complexity.
Algorithm III: A Sophisticated Method

Idea and proof of correctness:

\[ P(x) = (\cdots ((a_n x + a_{n-1}) x + a_{n-2}) x + \cdots ) x + a_0. \]
Algorithm III: A Sophisticated Method

- **Idea and proof of correctness:**
  \[ P(x) = (\cdots ((a_n x + a_{n-1}) x + a_{n-2}) x + \cdots ) x + a_0. \]

- **Example:** \( 4x^3 + 3x^2 + 2x + 1 = ((4x + 3)x + 2)x + 1. \)
Algorithm III: A Sophisticated Method

Idea and proof of correctness:

\[ P(x) = (\cdots (a_n x + a_{n-1}) x + a_{n-2}) x + \cdots ) x + a_0. \]

Example: \( 4x^3 + 3x^2 + 2x + 1 = ((4x + 3)x + 2)x + 1. \)

Polynomial-Evaluation \((P(x), c)\)

\[ P(c) = a_n \]

\[ \text{for } i = n - 1 \text{ downto } 0 \text{ do} \]

\[ P(c) = P(c) \cdot c + a_i \]

\[ (* P(c) = a_n c^{n-i} + a_{n-1} c^{n-i-1} + \cdots + a_{i+1} c = a_i *) \]
Algorithm III: A Sophisticated Method

- **Idea and proof of correctness:**
  \[ P(x) = (\cdots (((a_n x + a_{n-1}) x + a_{n-2}) x + \cdots ) x + a_0. \]

- **Example:** \( 4x^3 + 3x^2 + 2x + 1 = ((4x + 3)x + 2)x + 1. \)

- **Polynomial-Evaluation** \((P(x), c)\)
  \[ P(c) = a_n \]
  \[ \text{for } i = n - 1 \text{ downto } 0 \text{ do} \]
  \[ P(c) = P(c) \cdot c + a_i \]
  (* \( P(c) = a_n c^{n-i} + a_{n-1} c^{n-i-1} + \cdots + a_{i+1} c + a_i \) *)

- **Correctness:** By Induction.
Algorithm III: Example

**Input:** $P(x) = 16x^4 + 8x^3 + 4x^2 + 2x + 1$
Algorithm III: Example

- **Input:** \( P(x) = 16x^4 + 8x^3 + 4x^2 + 2x + 1 \)

- **Algorithm:**

...
Algorithm III: Example

Input: \( P(x) = 16x^4 + 8x^3 + 4x^2 + 2x + 1 \)

Algorithm:
- \( P_4(x) = a_4 = 16 \)
Algorithm III: Example

**Input:** $P(x) = 16x^4 + 8x^3 + 4x^2 + 2x + 1$

**Algorithm:**
- $P_4(x) = a_4 = 16$
- $P_3(x) = P_4(x)x + a_3 = 16x + 8$
Algorithm III: Example

**Input:** $P(x) = 16x^4 + 8x^3 + 4x^2 + 2x + 1$

**Algorithm:**

- $P_4(x) = a_4 = 16$
- $P_3(x) = P_4(x)x + a_3 = 16x + 8$
- $P_2(x) = P_3(x)x + a_2 = 16x^2 + 8x + 4$
Algorithm III: Example

**Input:** $P(x) = 16x^4 + 8x^3 + 4x^2 + 2x + 1$

**Algorithm:**
- $P_4(x) = a_4 = 16$
- $P_3(x) = P_4(x)x + a_3 = 16x + 8$
- $P_2(x) = P_3(x)x + a_2 = 16x^2 + 8x + 4$
- $P_1(x) = P_2(x)x + a_1 = 16x^3 + 8x^2 + 4x + 2$
Algorithm III: Example

**Input:** \( P(x) = 16x^4 + 8x^3 + 4x^2 + 2x + 1 \)

**Algorithm:**

- \( P_4(x) = a_4 = 16 \)
- \( P_3(x) = P_4(x)x + a_3 = 16x + 8 \)
- \( P_2(x) = P_3(x)x + a_2 = 16x^2 + 8x + 4 \)
- \( P_1(x) = P_2(x)x + a_1 = 16x^3 + 8x^2 + 4x + 2 \)
- \( P(x) = P_1(x)x + a_0 = 16x^4 + 8x^3 + 4x^2 + 2x + 1 \)
1. **multiplication** in the $i$th iteration of the loop.

2. **addition** in the $i$th iteration of the loop.

Total of $2n$ operations: $n$ multiplications and $n$ additions.

$\Theta(n)$ time complexity.
The Josephus Problem

**Story:** \( n \) People are standing in a circle waiting to be executed. Counting begins at a specified point in the circle and proceeds around the circle clockwise. After \( k \) people are skipped, the next person is executed. The procedure is repeated with the remaining people starting with the next person until only one person remains, and is freed.

**Problem:** Given the starting point \( n \) and \( k \) find the position in the initial circle to avoid execution.

**A video lecture for** \( k = 0 \): https://www.youtube.com/watch?v=uCsD3ZGzMgE