**Greedy Algorithms**

- **Greedy algorithms** make decisions that “seem” to be the best following some greedy criteria.

- In **Off-Line** problems:
  - The whole input is known in advance.
  - Possible to do some preprocessing of the input.
  - Decisions are irrevocable.

- In **Real-Time** and **On-Line** problems:
  - The present cannot change the past.
  - The present cannot rely on the un-known future.
How and When to use Greedy Algorithms?

- **Initial solution:** Establish trivial solutions for a problem of a small size. Usually $n = 0$ or $n = 1$.

- **Top bottom procedure:** For a problem of size $n$, look for a greedy decision that reduces the size of the problem to some $k < n$ and then, apply **recursion**.

- **Bottom up procedure:** Construct the solution for a problem of size $n$ based on some greedy criteria applied on the solutions to the problems of size $k = 1, \ldots, n - 1$. 
The Coin Changing Problem

**Input:**
- Integer coin denominations $d_n > \cdots > d_2 > d_1 = 1$.
- An integer amount to pay: $A$.

**Output:** Number of coins $n_i$ for each denomination $d_i$ to get the exact amount.
- $A = n_n d_n + n_{n-1} d_{n-1} + n_2 d_2 + n_1 d_1$.

**Goal:** Minimize total number of coins.
- $N = n_n + \cdots + n_2 + n_1$.

**Remark:** There is always a solution with $N = A$ since $d_1 = 1$. 
Examples

- **USA:** $d_6 = 100$, $d_5 = 50$, $d_4 = 25$, $d_3 = 10$, $d_2 = 5$, $d_1 = 1$.
  - $A = 73 = 2 \cdot 25 + 2 \cdot 10 + 3 \cdot 1$.
  - $N = 2 + 2 + 3 = 7$.

- **Old British:** $d_3 = 240$, $d_2 = 20$, $d_1 = 1$.
  - $A = 307 = 1 \cdot 240 + 3 \cdot 20 + 7 \cdot 1$.
  - $N = 1 + 3 + 7 = 11$. 
Greedy Solution

- **Idea:** Use the largest possible denomination and update $A$.

- **Implementation:**

  **Coin-Changing** ($d_n > \cdots > d_2 > d_1 = 1$)

  ```
  for i = n downto 1
      n_i = \lfloor A/d_i \rfloor
      A = A \mod d_i = A - n_i d_i
  Return(N = n_n + \cdots + n_2 + n_1)
  ```

- **Correctness:** $A = n_n d_n + n_{n-1} d_{n-1} + n_2 d_2 + n_1 d_1$.

- **Complexity:** $\Theta(n)$ division and mod integer operations.
Optimality

- **Greedy** is optimal for the USA system.

A coin system for which Greedy is not optimal:

\[d_3 = 4, \quad d_2 = 3, \quad d_1 = 1\]

**Greedy:**

\[6 = 1 \cdot 4 + 2 \cdot 1 \Rightarrow N = 3.\]

**Optimal:**

\[6 = 2 \cdot 3 \Rightarrow N = 2.\]

A coin system for which Greedy is very "bad":

\[d_3 = x + 1, \quad d_2 = x, \quad d_1 = 1\]

\[\text{Greedy:}\]

\[2x = 1 \cdot (x + 1) + (x - 1) \cdot 1 \Rightarrow N = x.\]

**Optimal:**

\[2x = 2 \cdot x \Rightarrow N = 2.\]
**Optimality**

- **Greedy** is optimal for the USA system.

- A coin system for which **Greedy** is not optimal:
  - $d_3 = 4$, $d_2 = 3$, $d_1 = 1$ and $A = 6$:
    - **Greedy**: $6 = 1 \cdot 4 + 2 \cdot 1 \Rightarrow N = 3$.
    - **Optimal**: $6 = 2 \cdot 3 \Rightarrow N = 2$. 
Greedy is optimal for the USA system.

A coin system for which Greedy is not optimal:
- \(d_3 = 4, d_2 = 3, d_1 = 1\) and \(A = 6\):
  - Greedy: \(6 = 1 \cdot 4 + 2 \cdot 1 \Rightarrow N = 3\).
  - Optimal: \(6 = 2 \cdot 3 \Rightarrow N = 2\).

A coin system for which Greedy is very “bad”:
- \(d_3 = x + 1, d_2 = x, d_1 = 1\) and \(A = 2x\):
  - Greedy: \(2x = 1 \cdot (x + 1) + (x - 1) \cdot 1 \Rightarrow N = x\).
  - Optimal: \(2x = 2 \cdot x \Rightarrow N = 2\).
Efficiency

- **Optimal solution:** Check all possible combinations.
  - Not a polynomial time algorithm.

- **Another optimal solution:** Polynomial in both $n$ and $A$.
  - Not a strongly polynomial time algorithm.

- **Objective:**
  - Find a solution that is polynomial only in $n$.
  - Probably impossible!?
The Knapsack Problem

**Input:**

- A thief enters a store and finds $n$ items $I_1, \ldots, I_n$.
- The value of item $I_i$ is $v(I_i)$ and its weight is $w(I_i)$.
  - Both are positive integers.
- The thief can carry at most weight $W$.
- The thief either takes all of item $I_i$ or doesn’t take item $I_i$. 
The Knapsack Problem

**Input:**
- A thief enters a store and finds $n$ items $I_1, \ldots, I_n$.
- The value of item $I_i$ is $v(I_i)$ and its weight is $w(I_i)$.
  - Both are positive integers.
- The thief can carry at most weight $W$.
- The thief either takes all of item $I_i$ or doesn’t take item $I_i$.

**Goal:** Carry items with maximum total value.
- Which are these items?
- What is their total value?
A General Greedy Scheme

- **Order** the items according to some **greedy criterion**.
  - Assume this order is $J_1, J_2, \ldots, J_n$.
  - Assume $J_1$ is the most desired item and $J_n$ is the least desired item.

If $J_1$ is not too heavy ($w(J_1) \leq W$):
- **Take** item $J_1$.
- **Continue recursively** with $J_2, J_3, \ldots, J_n$ and updated maximum weight $W - w(J_1)$.

If $J_1$ is too heavy ($w(J_1) > W$):
- **Ignore** item $J_1$.
- **Continue recursively** with $J_2, J_3, \ldots, J_n$ and the same maximum weight $W$. 
A General Greedy Scheme – Implementation

Non-Recursive Knapsack \((l_1, \ldots, l_n, w(\cdot), v(\cdot), W)\)

Let \(J_1, \ldots, J_n\) be the new order on the items.

\[ S = \emptyset \quad (* \text{the set of items the thief takes} *) \]
\[ V = 0 \quad (* \text{the value of these items} *) \]

for \(i = 1\) to \(n\)

if \(w(J_i) \leq W\) then

\[ S = S \cup \{J_i\} \]
\[ V = V + v(J_i) \]
\[ W = W - w(J_i) \]

Return \((S, V)\)
Greedy Criteria

**Greedy criterion I:** Order the items by their value from the most expensive to the cheapest.

**Greedy criterion II:** Order the items by their weight from the lightest to the heaviest.

**Greedy criterion III:** Order the items by their ratio of value over weight from the largest ratio to the smallest ratio.
The three criteria are not optimal

- **Counter example for Greedy-by-Value and Greedy-by-Ratio:**
  - 3 items and maximum weight is $W = 10$. Weights and values are: $I_1 = \langle 6, 10 \rangle$, $I_2 = \langle 5, 6 \rangle$, and $I_3 = \langle 5, 6 \rangle$.
  - **Optimal** takes items $I_2$ and $I_3$ for a profit of 12.
  - **Greedy-by-Value** or **Greedy-by-Ratio** take only item $I_1$ for a profit of 10.
The three criteria are not optimal

- **Counter example for Greedy-by-Value and Greedy-by-Ratio:**
  - 3 items and maximum weight is $W = 10$. Weights and values are: $I_1 = \langle 6, 10 \rangle$, $I_2 = \langle 5, 6 \rangle$, and $I_3 = \langle 5, 6 \rangle$.
  - **Optimal** takes items $I_2$ and $I_3$ for a profit of 12.
  - **Greedy-by-Value** or **Greedy-by-Ratio** take only item $I_1$ for a profit of 10.

- **Counter example for Greedy-by-Weight:**
  - 3 items and maximum weight is $W = 10$. Weights and values are: $I_1 = \langle 6, 13 \rangle$, $I_2 = \langle 5, 6 \rangle$, and $I_3 = \langle 5, 6 \rangle$.
  - **Optimal** takes only item $I_1$ for a profit of 13.
  - **Greedy-by-Weight** takes items $I_2$ and $I_3$ for a profit of 12.
Very Bad Counter Examples for Criteria I and II

Counter example for Greedy-by-Value:

- \( n \) items and maximum weight is \( W \). Weights and values are: \( I_1 = \langle W, 2 \rangle, I_2 = \langle 1, 1 \rangle, \ldots, I_3 = \langle 1, 1 \rangle \).
- **Optimal** takes items \( I_2, \ldots, I_n \) for a profit of \( n - 1 \).
- **Greedy-by-Value** takes only item \( I_1 \) for a profit of 2.
- The ratio is \( (n - 1)/2 \).
Very Bad Counter Examples for Criteria I and II

Counter example for Greedy-by-Value:

- \( n \) items and maximum weight is \( W \). Weights and values are: 
  \[ l_1 = \langle W, 2 \rangle, \ l_2 = \langle 1, 1 \rangle, \ldots, l_3 = \langle 1, 1 \rangle. \]
- **Optimal** takes items \( l_2, \ldots, l_n \) for a profit of \( n - 1 \).
- **Greedy-by-Value** takes only item \( l_1 \) for a profit of 2.
- The ratio is \((n - 1)/2\).

Counter example for Greedy-by-Weight:

- 2 items and maximum weight is 2. Weights and values are: 
  \[ l_1 = \langle 1, 1 \rangle \text{ and } l_2 = \langle 2, x \rangle \text{ for a very large } x. \]
- **Optimal** takes item \( l_2 \) for a profit of \( x \).
- **Greedy-by-Weight** takes item \( l_1 \) for a profit of 1.
- The ratio is \( x \).
Counter example for Greedy-by-Ratio:

- 2 items and maximum weight is $W$. Weights and values are: $I_1 = \langle 1, 2 \rangle$ and $I_2 = \langle W, W \rangle$.
- **Optimal** takes items $I_2$ for a profit of $W$.
- **Greedy-by-Ratio** takes item $I_1$ for a profit of 2.
- The ratio is almost $\frac{W}{2}$.
Counter example for Greedy-by-Ratio:

- 2 items and maximum weight is \( W \). Weights and values are: 
  \( I_1 = 1 \times 2 \) and \( I_2 = W \times W \).
- **Optimal** takes items \( I_2 \) for a profit of \( W \).
- **Greedy-by-Ratio** takes item \( I_1 \) for a profit of 2.
- The ratio is almost \( \frac{W}{2} \).

A 1/2 guaranteed approximation algorithm:

- **Greedy-by-Ratio** guarantees half of the profit of **Optimal** with a tweak.
- Select either the output of greedy or the one item with the maximum value whose weight is at most \( W \).
The Fractional Knapsack Problem

- The thief can take portions of items.
- If the thief takes a fraction $0 \leq p_i \leq 1$ of item $l_i$:
  - Its value is $p_i v(l_i)$.
  - Its weight is $p_i w(l_i)$.
The Fractional Knapsack Problem

- The thief can take portions of items.
- If the thief takes a fraction $0 \leq p_i \leq 1$ of item $I_i$:
  - Its value is $p_i v(I_i)$.
  - Its weight is $p_i w(I_i)$.

**Theorem:** Greedy-by-Ratio is optimal.
Proof

- Assume that **Greedy-by-Ratio** fails on the input $I_1, \ldots, I_n$ and the weight $W$.

- Let the portions taken by **Optimal** be $p_1, \ldots, p_n$.
  - $p_i = 1$: all of item $I_i$ is taken.
  - $p_i = 0$: none of item $I_i$ is taken.
  - $0 < p_i < 1$: some but not all of item $I_i$ is taken.

- Since **Greedy-by-Ratio** fails, there exist $I_i$ and $I_j$ such that:
  - $\frac{v(I_i)}{w(I_i)} > \frac{v(I_j)}{w(I_j)}$ and $p_i < 1$ and $p_j > 0$.

- Because each unit of weight of item $I_i$ has more value than each unit of weight of item $I_j$, it is more profitable to take more of item $I_i$ and less of item $I_j$.

- A **contradiction** to the optimality of **Optimal**.
The 0 − 1 Knapsack Problem

- **Optimal solution**: Check all possible sets of items.
  - Not a polynomial time algorithm.

- **Another optimal solution**: Polynomial in both $n$ and $W$.
  - Not a strongly polynomial time algorithm.

- **Objective**:
  - Find a solution that is polynomial only in $n$.
  - Probably impossible!?
  - However, **Greedy-by-Ratio** produces “good” solutions.
The Activity-Selection Problem

- **Input:**
  - Activities $A_1, \ldots, A_n$ that need the service of a common resource.
  - Activity $A_i$ is associated with a time interval $[s_i, f_i)$ for $s_i < f_i$.
    - $A_i$ needs the service from time $s_i$ until just before time $f_i$.

- **Mutual Exclusion:** The resource serves at most one activity at any time.

- **Definition:** $A_i$ and $A_j$ are compatible if either $f_i \leq s_j$ or $f_j \leq s_i$.

- **Goal:** Find a maximum size set of compatible activities.
**Example**

- **Input:** 3 activities $A_1 = [1, 4)$, $A_2 = [3, 6)$, $A_3 = [5, 8)$.
Example

- **Input:** 3 activities $A_1 = [1, 4)$, $A_2 = [3, 6)$, $A_3 = [5, 8)$.

- **A graphical representation:**

![Graphical representation of activities](https://via.placeholder.com/150)

The best solution:

![Best solution](https://via.placeholder.com/150)
Example

- **Input:** 3 activities $A_1 = [1, 4)$, $A_2 = [3, 6)$, $A_3 = [5, 8)$.

- **A graphical representation:**

- **The best solution:**
Static vs. Dynamic Greedy

- **Static**: The *greedy* criterion is determined in advance and cannot be changed during the execution of the algorithm.

- **Dynamic**: The *greedy* criterion may be modified during the execution of the algorithm based on prior decisions.

- **Remark**: A static criterion is also a dynamic criterion.
A General Static Greedy Scheme

- **Maintain** a set $S$ of the activities that have been selected so far.

- Initially, $S = \emptyset$ and at the end, $S$ is an optimal solution.

- **Order** the activities following some greedy criterion and **consider** the activities according to this order.

- Let $A$ be the current considered activity. If $A$ is compatible with all the activities in $S$:  
  - Then **add** $A$ to $S$.
  - Else **ignore** $A$.

- **Continue** until there are no activities to consider.
A General Dynamic Greedy Scheme

- **Maintain** two sets of activities:
  - $S$ those that have been selected so far.
  - $R$ those that can still be selected.
  - Initially, $S = \emptyset$ and $R = \{A_1, \ldots, A_n\}$.
  - At the end, $S$ is an **optimal** solution and $R = \emptyset$.

- **Select** a “good” activity $A$ from $R$, following some greedy criterion.
- **Add** $A$ to $S$.
- **Delete** from $R$ the activities that are not compatible with activity $A$.
- **Continue** until $R$ is empty.
Greedy Criteria

- **Four criteria:**
  - Prefer short activities.
  - Prefer activities intersecting few other activities.
  - Prefer activities that start earlier.
  - Prefer activities that terminate earlier.

- **Optimality:** Only the fourth criterion is optimal.

- **Remarks:**
  - All four criteria are static in their nature.
  - The second criterion has a dynamic version.
**An Optimal Greedy Solution**

### Preprocessing \((A_1, \ldots, A_n)\)
- **Sort** the activities according to their finish time
- **Let** this order be \(A_1, \ldots, A_n\) \((i < j \Rightarrow f_i \leq f_j)\)

### Greedy-Activity-Selector \((A_1, \ldots, A_n)\)
- \(S = \{A_1\}\) \((A_1\text{ terminates the earliest})\)
- \(j = 1\) \((A_j\text{ is the current selected activity})\)
  - **for** \(i = 2\) **to** \(n\) \((\text{scan all the activities})\)
    - **if** \(s_i \geq f_j\) \((\text{check compatibility})\)
      - **then** \((\text{select } A_i \text{ that is compatible with } S)\)
        - \(S = S \cup \{A_i\}\)
        - \(j = i\)
    - **else** \((A_i\text{ is not compatible})\)
  - **Return** \(S\)
Correctness and Complexity

- **Correctness:** By definition.

- **Complexity:**
  - The sorting can be done in $O(n \log n)$ time.
  - There are $O(1)$ operations per each activity.
  - All together: $O(n \log n) + n \cdot O(1) = O(n \log n)$ time.
Example - Input
Example - Output
Let $T$ be an optimal set of activities.

Transform $T$ to $S$ preserving the size of $T$.

Let $A_1, \ldots, A_n$ be ordered by their \textit{finish} time.

Let $A_i$ be the first activity that is in $T$ and not in $S$.

All the activities in $T$ that finish before $A_i$ are also in $S$. 
Optimality

- $A_i \notin S \Rightarrow \exists A_j \in S$ that is not in $\mathcal{T}$ in which $j < i$.

- $A_j$ is compatible with all the activities in $\mathcal{T}$ that finish before it since they are all in $S$.

- $A_j$ is compatible with all the activities in $\mathcal{T}$ that finish after $A_i$ since it finishes before $A_j$.

Therefore, $\mathcal{T} \cup \{A_j\} \setminus \{A_i\}$ is a solution with the same size as $\mathcal{T}$ and hence optimal.

- Continue this way until $\mathcal{T}$ becomes $S$. 
Another optimal solution with 4 activities.
A third optimal solution: after the first transformation.
Example

The greedy solution: after the second transformation.
Huffman Codes

Input:
- An alphabet of $n$ symbols $a_1, \ldots, a_n$.
- A frequency $f_i$ for each symbol $a_i$:
  - $\sum_{i=1}^{n} f_i = 1$.
- A File $\mathcal{F}$ containing $L$ symbols from the alphabet.
  - $a_i$ appears exactly $n_i = f_i \cdot L$ times in $\mathcal{F}$. 
Huffman Codes

**Input:**
- An alphabet of $n$ symbols $a_1, \ldots, a_n$.
- A frequency $f_i$ for each symbol $a_i$:
  - $\sum_{i=1}^{n} f_i = 1$.
- A File $\mathcal{F}$ containing $L$ symbols from the alphabet.
  - $a_i$ appears exactly $n_i = f_i \cdot L$ times in $\mathcal{F}$.

**Output:**
- For symbol $a_i$, $1 \leq i \leq n$: A binary codeword $w_i$ of length $\ell_i$.
- A compressed (encoded) binary file $\mathcal{F}'$ of $\mathcal{F}$. 
Huffman Codes – Goals

- $L'$ the length of $\mathcal{F}'$ should be minimal.
- An efficient algorithm to find the $n$ codewords.
  - Good polynomial running time: $O(n \log n)$.
- Efficient encoding and decoding of the file
  - Should be done in $O(B)$-time.
  - $B$ is the size of the original file in bits.
Example

- A file with the alphabet \( a, b, c, d, e, f \) containing 100 symbols.
  - \( n_a = 45, n_b = 13, n_c = 12, n_d = 16, n_e = 9, n_f = 5 \).

- **Code I:**
  - \( w_a = 000, w_b = 001, w_c = 010, w_d = 011, w_e = 100, w_f = 101 \).
  - Length of encoded file is 300.

- **Code II:**
  - \( w_a = 0, w_b = 101, w_c = 100, w_d = 111, w_e = 1101, w_f = 1100 \).
  - Length of encoded file is 224
    - \( 1 \cdot 45 + 3 \cdot 13 + 3 \cdot 12 + 3 \cdot 16 + 4 \cdot 9 + 4 \cdot 5 = 224 \).

- **Remark:** Code II is optimal, \( \approx 25\% \) better than code I.
Definition: A prefix free code is a code in which no codeword is a prefix of another codeword.

Examples: Both code I and code II are prefix free.

Proposition: A code in which the lengths of all the codewords is the same is a prefix free code.

Theorem: Always exists an optimal prefix free code.

Encoding: “Easy” using tables.

Decoding: By scanning the coded text once.
A code can be represented by a rooted and ordered binary tree with \( n \) leaves.

Each leaf stores a codeword.

The codeword corresponding to a leaf is defined by the unique path from the root to the leaf:

- 0 for going left.
- 1 for going right.
A leaf is represented by the symbol and its frequency.

An internal node is labelled by the sum of the frequencies of all the leaves in its subtree.
**Proposition:** The binary tree represents a prefix free code since a path to a leaf cannot be a prefix of any other path.

**Complexity Parameters:**
- \( f(x) \) the frequency of a leaf \( x \).
- \( \ell(x) \) the length of the path from the root to \( x \).

The cost of the tree is: \( B(T) = \sum_{\text{a leaf } x} (f(x) \cdot \ell(x)) \).
- \( B(T) \) is the average length of a codeword.

The length of the encoded file: \( \sum_{\text{a leaf } x} (n(x) \cdot \ell(x)) \).
Lemma: Let $T$ be a tree that represents an optimal code. Then each internal node in the tree has two children.
Lemma: Let $T$ be a tree that represents an optimal code. Then each internal node in the tree has two children.

Proof:

Let $z$ be an internal node with only one child $y$.

There are 2 cases:

Case I: $z$ is the root.

Case II: $z$ is not the root.
Case I

- $z$ is the root: Make $y$ the new root.
Case II

- $z$ is not a root and $p$ is its parent: Bypass $z$ by making $y$ the child of $p$. 

Diagram:

```
    p
   / \   
  z   y
 / \   /
A   B C
```

```
    p
   /   
  y   
   /   
A B C
```
Proof

- In both cases:
  - $\ell(x)$ of all the leaves in the sub-tree rooted at $z$ is reduced by 1.
  - These are the only changes.
  - As a result the cost of the tree is improved.
  - A contradiction to the optimality of the code.
Example: Code I

\[ B(T) = 300 \]
Example: Improving Code I

\[ B(T) = 3 \cdot 86 + 2 \cdot 14 = 286 \]
Huffman Algorithm

- **Construct** a coding tree bottom-up.
- **Maintain** a forest with $n$ leaves in all of its trees. Each tree is optimal for its leaves.

Initially, there are $n$ singleton trees in the forest. Each tree is a leaf.

The frequency of a tree is the sum of the frequencies of all of its leaves.

**Greedy** step:
- **Find** the two trees with the minimum frequencies.
- **Combine** them together into one tree.
  - The frequency of the new tree is the sum of the frequencies of the two combined trees.

- **Terminate** when there is only one tree in the forest.
Example
Example
Example

```
25
  c:12
  b:13

30
  14
    f:5
    e:9

55
  25
    c:12
    b:13
  30
    14
      f:5
      e:9
```

Amotz Bar-Noy (CUNY)  Greedy Algorithms  Spring 2012  51 / 62
Example
Huffman Code Animation

http://www.cs.auckland.ac.nz/~jmor159/PLDS210/huffman.html
Correctness

- Huffman algorithm generates a binary tree with $n$ leaves.
- A binary tree represents a prefix free code.
A forest of binary trees.

- Initially, the forest contains $n$ singleton trees.
- At the end, the forest contains one tree.

The frequencies of the trees in the forest are maintained in a priority queue $Q$.

- Initially, the queue contains the $n$ original frequencies.
- At the end, the queue contains one frequency which is the sum of all original frequencies.
Implementation – Procedure

\textbf{Huffman}(\langle a_1, f_1 \rangle, \ldots, \langle a_n, f_n \rangle)

\textbf{Build-Queue}(\{f_1, \ldots, f_n\}, Q)

\textbf{for } i = 1 \textbf{ to } n - 1 \quad \text{(* the combination loop *)}

\hspace{1em} z = \text{Allocate-Node}() \quad \text{(* creating a new root *)}

\hspace{1em} x = \text{left}(z) = \text{Extract-Min}(Q)

\hspace{2em} (* lightest tree is the left sub-tree *)

\hspace{1em} y = \text{right}(z) = \text{Extract-Min}(Q)

\hspace{2em} (* second lightest tree is the right sub-tree *)

\hspace{1em} f(z) = f(x) + f(y) \quad \text{(* frequency of new root *)}

\hspace{1em} \text{Insert}(Q, f(z)) \quad \text{(* inserting the new root to the queue *)}

\textbf{return} \text{Extract-Min}(Q) \quad \text{(* last tree is the Huffman code *)}
Implement the priority queue with a **Binary Heap**.

The complexity of **Build-Queue** is $O(n)$.

The complexity of **Extract-Min** and **Insert** is $O(\log n)$.

The loop is executed $O(n)$ times.

The complexity of all the **Extract-Min** and the **Insert** operations is $O(n \log n)$.

The total complexity is: $O(n \log n)$. 

Optimality - First Lemma

- Let $\mathcal{A}$ be an alphabet.

- Let $x$ and $y$ be the two symbols in $\mathcal{A}$ with the smallest frequencies.

Then, there exists an optimal tree in which:

- $x$ and $y$ are adjacent leaves (differ only in their last bit).
- $x$ and $y$ are the farthest leaves from the root.
Proof

Let \( z \) and \( w \) be adjacent leaves in an optimal tree that are the farthest from the root.

Exchanging \( z \) and \( w \) with \( x \) and \( y \) yields a tree with a smaller or equal cost.
Optimality - Second Lemma

Let $T$ be an optimal tree for the alphabet $\mathcal{A}$.

Let $x, y$ be adjacent leaves in $T$ and let $z$ be their parent.

Let $\mathcal{A}'$ be $\mathcal{A}$ with a new symbol $z$ replacing $x$ and $y$ with frequency: $f(z) = f(x) + f(y)$.

Let $T'$ be the tree $T$ without the leaves $x$ and $y$ and with $z$ as a new leaf.

Then $T'$ is an optimal tree for the alphabet $\mathcal{A}'$. 
Proof

Let $T''$ be an optimal tree with smaller cost than $T'$.

Replacing $z$ in $T''$ with the two leaves $x$ and $y$ creates a tree with a smaller cost than $T$.

A contradiction to the optimality of $T$. 
Optimality

- **Theorem:** Huffman code is optimal.

- **Proof by Induction:**
  - The first lemma implies that the first greedy step is a first step towards an optimal solution.
  - The second lemma justifies the inductive steps, applying again and again the first lemma.