Discrete Math

Solutions to the Prerequisite Home Quiz

1. The five components of the famous formula $e^{\pi i} - 1 = 0$ are:
   (a) The additive identity: $0$
   (b) The square root of $-1$: $i$
   (c) The multiplicative identity: $1$
   (d) The base of the natural logarithmic: $e$
   (e) The ratio of a circle’s circumference to its diameter: $\pi$

2. $1 < \sqrt{2} = 1.414... < \phi = 1.618... < e = 2.718... < \pi = 3.141...$

3. Let $A$ be the set of all the prime numbers between 10 and 40. Let $B$ be the set of all integers between 10 and 40 that are of the form by 3$k + 2$ for some integer $k$.
   (a) $A = \{11, 13, 17, 19, 23, 29, 31, 37\}$
   (b) $B = \{11, 14, 17, 20, 23, 26, 29, 32, 35, 38\}$
   (c) $A \cup B = \{11, 13, 14, 17, 19, 20, 23, 26, 29, 31, 32, 35, 37, 38\}$
   (d) $A \cap B = \{11, 17, 23, 29\}$
   (e) $A \setminus B = \{13, 19, 31, 37\}$
   (f) $B \setminus A = \{14, 20, 26, 32, 35, 38\}$

4. There are 6 possible True and False assignments for the 4 variables $x, y, z, w$ (out of the possible 16 assignments) that satisfy the formula: $(x \lor y) \land (\bar{x} \lor \bar{z} \lor \bar{w}) \land (\bar{y} \lor w)$:

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<tr>
<th>$x$</th>
<th>$y$</th>
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<th>$(x \lor y)$</th>
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5. (a) \((x + y)^2 = x^2 + 2xy + y^2\)
    (b) \((x - y)^2 = x^2 - 2xy + y^2\)
    (c) \((x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\)
    (d) \((x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3\)
    (e) \((x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i = x^n + nx^{n-1}y + \cdots + nxy^{n-1} + y^n\)
    (f) \((x - y)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{n-i} y^i = x^n - nx^{n-1}y + \cdots \pm nxy^{n-1} \pm y^n\)

6. (a) \(x^2 - y^2 = (x - y)(x + y)\)
    (b) \(x^3 - y^3 = (x - y)(x^2 + xy + y^2)\)
    (c) \(x^3 + y^3 = (x + y)(x^2 - xy + y^2)\)
    (d) \(x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2}y + y^{n-1})\)
    (e) \(x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + x^2y^{n-3} - xy^{n-2} + y^{n-1})\)
      only for odd \(n\).

7. (a) \(\frac{(n+1)!}{(n-1)!} = n(n + 1)\)
    (b) \(\left\lfloor \frac{n}{k} \right\rfloor \times k + (n \mod k) = n\)
    (c) \(x^n \times x^m = x^{n+m}\)
    (d) \(x^n \times y^n = (xy)^n\)
    (e) \(\frac{\log_a(x^n)}{\log_a(x)} = n\)
    (f) \(2 \log_a(\sqrt{x}) = \log_a(x)\)

8. (a) If \(\log_a(y) = x\), then \(a^x = y\)
    (b) If \(\log_a(x) + \log_a(y) = \log_a(z)\), then \(z = xy\)
    (c) If \(x < 0\), then \(|x| + | -x| = -2x\)

9. Let \(x = 210 = 2 \cdot 3 \cdot 5 \cdot 7\) and \(y = 225 = 3^2 \cdot 5^2\).
    (a) The Greatest Common Divisor (GCD) of \(x\) and \(y\) is \(3 \cdot 5 = 15\)
    (b) The Least Common Multiplier (LCM) of \(x\) and \(y\) is \(3150 = 2 \cdot 3^2 \cdot 5^2 \cdot 7\)

10. The following functions are ordered from the slowest to the fastest when \(n\) tends to infinity:
    \[\log \log(n) ; \quad \log(n) ; \quad n ; \quad n^2 ; \quad 2^n ; \quad n! ; \quad n^n\]

11. For the following 2 linear equations
    \[
    \begin{align*}
    x + y &= a \\
    bx + cy &= 0
    \end{align*}
    \]
    The values of \(x\) and \(y\) as a function of the 3 constants \(a, b, c\) are: \(x = \frac{ac}{c-b}\) and \(y = \frac{-ab}{c-b}\)

12. The roots of the quadratic equations \(x^2 + bx = c\) (equivalently, \(x^2 + bx - c = 0\)) are
    \[
    x_1 = \frac{-b + \sqrt{b^2 + 4c}}{2} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 + 4c}}{2}
    \]
13. Let $0 \leq x \leq 1$ be a real number and let $f(x) = [x] + \lceil x \rceil$.

$$x = 0 \implies f(x) = 0 + 0 = 0$$

$$0 < x < 1 \implies f(x) = 0 + 1 = 1$$

$$x = 1 \implies f(x) = 1 + 1 = 2$$

14. (a) $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

(b) $1 + 2 + 4 + 8 + \cdots + 2^k = 2^{k+1} - 1$

(c) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$

15. Four fair coins are flipped.

(a) The probability that all are H or all are T is $\frac{1}{8}$

(b) The probability that there is exactly one H is $\frac{1}{4}$

16. Let $T$ be a right angle triangle with sides $a$, $b$, and $c$ where $c$ is the hypotenuse (the side opposite the right angle). Then

$$c^2 = a^2 + b^2$$

17. The sum of the degrees of all the inner angles of the following geometric shapes is

(a) Triangle: 180

(b) Square: 360

(c) Pentagon: 540

(d) Hexagon: 720

(e) $n$-gon: $180(n - 2)$

18. Let $C$ be a circle whose radius is $r$ and whose diameter is $d$.

(a) The circumference of $C$ is $2\pi r = \pi d$

(b) The area of $C$ is $\pi r^2 = \frac{\pi d^2}{4}$

19. (a) $n \geq 1$ is an integer.

$$T(1) = 1$$

$$T(n) = T(n - 1) + 1$$

$$\implies T(n) = n$$

(b) $n \geq 1$ is an integer.

$$T(1) = 1$$

$$T(n) = T(n - 1) + n$$

$$\implies T(n) = n + (n - 1) + \cdots + 1 = \frac{n(n+1)}{2}$$

(c) $n \geq 1$ is an integer.

$$T(1) = 1$$

$$T(n) = 2T(n - 1)$$

$$\implies T(n) = 2^{n-1}$$
(d) \( n \geq 1 \) is a power of 2 integer.
\[
T(1) = 0 \\
T(n) = T(n/2) + 1 \\
\iff T(n) = \log_2(n)
\]

20. (a) \( f(n) \) (* \( n > 0 \) is an integer number *)
\[
c = 0 \\
\text{for } i = 1 \text{ to } n \text{ do} \\
\quad \text{for } j = 1 \text{ to } n \text{ do} \\
\quad \quad c := c + 1 \\
\Rightarrow c = n^2
\]

(b) \( f(n) \) (* \( n > 0 \) is an integer number *)
\[
c = 0 \\
\text{for } i = 1 \text{ to } n \text{ do} \\
\quad \text{for } j = i \text{ to } n \text{ do} \\
\quad \quad c := c + 1 \\
\Rightarrow c = n + (n - 1) + (n - 2) + \cdots + 1 = \frac{n(n+1)}{2}
\]

(c) \( f(n) \) (* \( n > 0 \) is an integer number *)
\[
c = 1 \\
\text{for } i = 1 \text{ to } n \text{ do} \\
\quad c := c * 2 \\
\Rightarrow c = 2^n
\]

(d) \( f(n) \) (* \( n > 0 \) is a power of 2 number *)
\[
c = 0 \\
\text{while } n > 1 \text{ do} \\
\quad n := n/2 \\
\quad c := c + 1 \\
\Rightarrow c = \log_2(n)