1. Compute \((1001 \mod d)\) for \(d = 2, 3, \ldots, 10\).

\[
\begin{align*}
1001 &= 500 \cdot 2 + 1 \implies (1001 \mod 2) = 1 \\
1001 &= 333 \cdot 3 + 2 \implies (1001 \mod 3) = 2 \\
1001 &= 250 \cdot 4 + 1 \implies (1001 \mod 4) = 1 \\
1001 &= 200 \cdot 5 + 1 \implies (1001 \mod 5) = 1 \\
1001 &= 166 \cdot 6 + 5 \implies (1001 \mod 6) = 5 \\
1001 &= 143 \cdot 7 + 0 \implies (1001 \mod 7) = 0 \\
1001 &= 125 \cdot 8 + 1 \implies (1001 \mod 8) = 1 \\
1001 &= 111 \cdot 9 + 2 \implies (1001 \mod 9) = 2 \\
1001 &= 100 \cdot 10 + 1 \implies (1001 \mod 10) = 1
\end{align*}
\]

Compute \((1001^2 \mod d)\) for \(d = 2, 3, \ldots, 10\).

\[
\begin{align*}
(1001^2 \mod 2) &= ((1001 \mod 2)^2 \mod 2) = (1^2 \mod 2) = 1 \\
(1001^2 \mod 3) &= ((1001 \mod 3)^2 \mod 3) = (2^2 \mod 3) = 1 \\
(1001^2 \mod 4) &= ((1001 \mod 4)^2 \mod 4) = (1^2 \mod 4) = 1 \\
(1001^2 \mod 5) &= ((1001 \mod 5)^2 \mod 5) = (1^2 \mod 5) = 1 \\
(1001^2 \mod 6) &= ((1001 \mod 6)^2 \mod 6) = (5^2 \mod 6) = 1 \\
(1001^2 \mod 7) &= ((1001 \mod 7)^2 \mod 7) = (0^2 \mod 7) = 0 \\
(1001^2 \mod 8) &= ((1001 \mod 8)^2 \mod 8) = (1^2 \mod 8) = 1 \\
(1001^2 \mod 9) &= ((1001 \mod 9)^2 \mod 9) = (2^2 \mod 9) = 4 \\
(1001^2 \mod 10) &= ((1001 \mod 10)^2 \mod 10) = (1^2 \mod 10) = 1
\end{align*}
\]
2. Definition: $m$ is the inverse of $n$ modulo $d$ if $(nm \mod d) = 1$.

Find the inverse of $n = 2, 3, \ldots, 10$ modulo 11 if exist.

\[
\begin{align*}
2 \cdot 6 &= 12 = 1 \cdot 11 + 1 \implies (2^{-1} \mod 11) = 6 \\
3 \cdot 4 &= 12 = 1 \cdot 11 + 1 \implies (3^{-1} \mod 11) = 4 \\
4 \cdot 3 &= 12 = 1 \cdot 11 + 1 \implies (4^{-1} \mod 11) = 3 \\
5 \cdot 9 &= 45 = 4 \cdot 11 + 1 \implies (5^{-1} \mod 11) = 9 \\
6 \cdot 2 &= 12 = 1 \cdot 11 + 1 \implies (6^{-1} \mod 11) = 2 \\
7 \cdot 8 &= 56 = 5 \cdot 11 + 1 \implies (7^{-1} \mod 11) = 8 \\
8 \cdot 7 &= 56 = 5 \cdot 11 + 1 \implies (8^{-1} \mod 11) = 7 \\
9 \cdot 5 &= 45 = 4 \cdot 11 + 1 \implies (9^{-1} \mod 11) = 5 \\
10 \cdot 10 &= 12 = 1 \cdot 11 + 1 \implies (10^{-1} \mod 11) = 10 \\
\end{align*}
\]

Find the inverse of $n = 2, 3, \ldots, 8$ modulo 9 if exist.

\[
\begin{align*}
2 \cdot 5 &= 10 = 1 \cdot 9 + 1 \implies (2^{-1} \mod 9) = 5 \\
4 \cdot 7 &= 28 = 3 \cdot 9 + 1 \implies (4^{-1} \mod 9) = 7 \\
5 \cdot 2 &= 10 = 1 \cdot 9 + 1 \implies (5^{-1} \mod 9) = 2 \\
7 \cdot 4 &= 28 = 3 \cdot 9 + 1 \implies (7^{-1} \mod 9) = 4 \\
8 \cdot 8 &= 64 = 7 \cdot 9 + 1 \implies (8^{-1} \mod 9) = 8 \\
\end{align*}
\]

- Both $(3n \mod 9)$ and $(6n \mod 9)$ are either $(0 \mod 9)$ or $(3 \mod 9)$ or $(6 \mod 9)$ for any integer $n$. Therefore, neither 3 nor 9 have an inverse modulo 9.
- In general, if $\gcd(n, d) \neq 1$ then $n$ does not have an inverse modulo $d$. Therefore, since $\gcd(9, 6) = 3$ and $\gcd(6, 3) = 3$, it follows that both 6 and 3 do not have an inverse modulo 9.
3. Euler’s Tution function: \( \varphi(n) \) is the number of positive integers less than \( n \) that are relatively prime to \( n \).

**Proposition I:** \( \varphi(p) = p - 1 \) for any prime number \( p \).

**Proposition II:** \( \varphi(p^k) = p^k - p^{k-1} \) for any positive integer \( k \) and a prime number \( p \).

**Proposition III:** \( \varphi(nm) = \varphi(n)\varphi(m) \) for any two relatively prime \( n \) and \( m \) (\( \gcd(n, m) = 1 \)).

- 127 is a prime number. Therefore, by Proposition I,
  \[
  \varphi(127) = 127 - 1 = 126
  \]

- 625 = 5^4 and 5 is a prime number. Therefore, by Proposition II,
  \[
  \varphi(625) = \varphi(5^4) \\
  = 5^4 - 5^3 \\
  = 625 - 125 \\
  = 500
  \]

- 713 = 31 · 23 and \( \gcd(31, 23) = 1 \) because both 31 and 23 are prime numbers. Therefore, by Propositions III and Proposition I,
  \[
  \varphi(713) = \varphi(31 \cdot 23) \\
  = \varphi(31) \cdot \varphi(23) \\
  = (31 - 1)(23 - 1) \\
  = 30 \cdot 22 \\
  = 660
  \]

- 360 = 2^3 · 3^2 · 5 and 2, 3, and 5 are prime numbers. Therefore, by Propositions III and Proposition II,
  \[
  \varphi(360) = \varphi(2^3 \cdot 3^2 \cdot 5) \\
  = \varphi(2^3) \cdot \varphi(3^2) \cdot \varphi(5) \\
  = (2^3 - 2^2) \cdot (3^2 - 3^1) \cdot 4 \\
  = 4 \cdot 6 \cdot 4 \\
  = 96
  \]
4. Modular exponentiation.

- Since \(2^{100} = (2^2)^{50} = 4^{50}\) and \((4 \mod 3) = 1\), it follows that

\[
(2^{100} \mod 3) = (4^{50} \mod 3) = ((4 \mod 3)^{50}) \mod 3 = (1^{50} \mod 3) = (1 \mod 3) = 1
\]

- Since \(2^{100} = (2^4)^{25} = 16^{50}\) and \((16 \mod 5) = 1\), it follows that

\[
(2^{100} \mod 5) = (16^{25} \mod 5) = ((16 \mod 5)^{25}) \mod 5 = (1^{25} \mod 5) = (1 \mod 5) = 1
\]

- Since \(2^{100} = 2 \cdot (2^3)^{33} = 2 \cdot 8^{33}\) and \((8 \mod 7) = 1\), it follows that

\[
(2^{100} \mod 7) = ((2 \cdot 8^{33}) \mod 7) = ((2 \mod 7) \cdot (8 \mod 7)^{33}) \mod 7 = ((2 \cdot 1^{33}) \mod 7) = (2 \mod 7) = 2
\]

- \(31\) is a prime number, therefore \((19^{30} \mod 31) = 1\) by Fermat’s Little Theorem.

\[
(19^{90} \mod 31) = ((19^{30})^3 \mod 31) = ((19^{30} \mod 31)^3 \mod 31) = (1^3 \mod 31) = (1 \mod 31) = 1
\]

- \(\gcd(47, 77) = 1\) and \(\varphi(77) = \varphi(7) \cdot \varphi(11) = 6 \cdot 10 = 60\)

Therefore, Euler’s Theorem implies that \((47^{60} \mod 77) = 1\).

\[
(47^{61} \mod 77) = ((47 \cdot 47^{60}) \mod 77) = (((47 \mod 77) \cdot (47^{60} \mod 77)) \mod 77) = ((47 \cdot 1) \mod 77) = (47 \mod 77) = 47
\]
5. Definition: \( \gcd(n, m) \) is the largest positive integer that divides both \( n \) and \( m \).

- Let \( p \neq q \) be two different prime numbers. What is \( \gcd(p, q) \)?
  **Answer:** Let \( g = \gcd(p, q) \). Since \( g \mid p \) and \( g \mid q \) and both \( p \) and \( q \) are prime numbers, it follows that the only candidates for the greatest common divisor of \( p \) and \( q \) are 1, \( p \), and \( q \). But \( p \nmid q \) and \( q \nmid p \). Therefore, \( \gcd(p, q) = 1 \).

- Let \( k \) and \( h \) be two positive integers. What is \( \gcd(2^k, 3^h) \)?
  **Answer:** The only divisors of \( 2^k \) are powers of 2 and the only divisors of \( 3^h \) are powers of 3. As a result, \( 2^k \) and \( 3^h \) do not have a common divisor greater than 1. that is, \( \gcd(2^k, 3^h) = 1 \).

- Find \( \gcd(1001, 4433) \) using the Euclid Algorithm.
  **Answer:** Euclid’s algorithm finds the \( \gcd(4433, 1001) \) in three rounds.
  (a) The pair \( (4433, 1001) \) is replaced by the pair \( (1001, 429) \) since \( 4433 = 4 \cdot 1001 + 429 \).
  (b) The pair \( (1001, 429) \) is replaced by the pair \( (429, 143) \) since \( 1001 = 2 \cdot 429 + 143 \).
  (c) The algorithm terminates because \( 429 = 3 \cdot 143 \).

Indeed, The prime factorizations of both numbers are

\[
\begin{align*}
1001 & = 7 \cdot 11 \cdot 13 \\
4433 & = 11 \cdot 13 \cdot 31
\end{align*}
\]

Therefore, \( \gcd(1001, 4433) = 11 \cdot 13 = 143 \).

- Find \( \gcd(60, 84, 140) \).
  **Answer:** The prime factors of the three numbers are

\[
\begin{align*}
60 & = 2^2 \cdot 3 \cdot 5 \\
84 & = 2^2 \cdot 3 \cdot 7 \\
140 & = 2^2 \cdot 5 \cdot 7
\end{align*}
\]

As a result, only \( 4 = 2^2 \) divides all three numbers. Therefore, \( \gcd(60, 84, 140) = 4 \).

Note that \( \gcd(60, 84) = 12 \), \( \gcd(60, 140) = 20 \), and \( \gcd(84, 140) = 28 \). But the greatest common divisor of all three numbers is only 4.
6. **Definition**: lcm\((n, m)\) is the least positive integer that is a multiple of both \(n\) and \(m\).

- Let \(p \neq q\) be two different prime numbers. What is lcm\((p, q)\)?
  
  **Answer**: Let \(\ell = \text{lcm}(p, q)\). Since \(p | \ell\) and \(q | \ell\) and both \(p\) and \(q\) are prime numbers, it follows that the only candidates for the least common multiple of \(p\) and \(q\) are \(p, q, \) and \(p \cdot q\). But \(p \nmid q\) and \(q \nmid p\). Therefore, \(\text{lcm}(p, q) = p \cdot q\).

- What is lcm\((35, 55, 65)\)?
  
  **Answer**: \(35 = 5 \cdot 7, 55 = 5 \cdot 11, \) and \(65 = 5 \cdot 13\). Therefore,
  
  \[
  \text{lcm}(35, 55, 65) = 5 \cdot 7 \cdot 11 \cdot 13 = 5005
  \]

- What is the smallest positive integer \(n > 1\) for which \((n \mod 10) = (n \mod 14) = 1\)?
  
  **Answer**: Both 10 and 14 must divide \(n - 1\). Therefore, the smallest positive integer is \(\text{lcm}(10, 14) + 1\). The answer is \(n = 71\) since
  
  \[
  \text{lcm}(10, 14) = \text{lcm}(2 \cdot 5, 2 \cdot 7) = 2 \cdot 5 \cdot 7 = 70
  \]

- Find the smallest positive integer \(n > 1\) for which \((n \mod d) = 1\) for all \(2 \leq d \leq 10\).
  
  **Answer**: It can be shown that
  
  \[
  \text{lcm}(2, 3, 4, 5, 6, 7, 8, 9, 10) = 2520
  \]
  Therefore, similarly to the previous problem, the answer is 2521.