Discrete Structures

Recursion Practice Problems

Name: .................................................................
1. Solve the following recurrences and prove that your solutions are correct.

\[ T(n) = \begin{cases} 
2 & \text{for } n = 1 \\
T(n-1) + 7 & \text{for } n \geq 2 
\end{cases} \]

\[ T(n) = \begin{cases} 
3 & \text{for } n = 1 \\
2T(n-1) & \text{for } n \geq 2 
\end{cases} \]

\[ T(n) = \begin{cases} 
2 & \text{for } n = 1 \\
(n+1)T(n-1) & \text{for } n \geq 2 
\end{cases} \]
2. Solve the following recurrence and prove that your solutions are correct.

\[ P_n = \begin{cases} 
1 & \text{for } n = 0 \\
2 & \text{for } n = 1 \\
5P_{n-1} - 6P_{n-2} & \text{for } n \geq 2 
\end{cases} \]
3. Some facts about Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

\[ F_n = \begin{cases} 
0 & \text{for } n = 0 \\
1 & \text{for } n = 1 \\
F_{n-1} + F_{n-2} & \text{for } n \geq 2 
\end{cases} \]

What is the smallest \( n \) for which \( F_n > 100 \)? What is the smallest \( n \) for which \( F_n > 1000 \)?

Let \( A_n = (F_1 + F_2 + \cdots + F_n)/n \) be the average of the first \( n \) Fibonacci numbers. What is the smallest \( n \) for which \( A_n > 10 \)?

Find all \( n \) for which \( F_n = n \). Explain why these are the only cases.

Find all \( n \) for which \( F_n = n^2 \). Explain why these are the only cases.
4. Prove the following identity for $n \geq 2$:

$$F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2}$$
5. Prove the following identity for \( n \geq 1 \):

\[
F_{2n+1} = F_{n+1}^2 + F_n^2
\]
6. Define the following (almost Fibonacci) recurrence

\[
G_n = \begin{cases} 
0 & \text{for } n = 0 \\
1 & \text{for } n = 1 \\
G_{n-1} + G_{n-2} + 1 & \text{for } n \geq 2 
\end{cases}
\]

Find the values of \(G_0, G_1, \ldots, G_{10}\).

Express \(G_n\) as a function of Fibonacci numbers.

Prove that your expression for \(G_n\) is correct for all \(n \geq 0\).
7. Find at least three interesting and fascinating facts or properties about the Fibonacci numbers and/or the Golden Ratio that do not appear in the class presentation.
   You do not need to provide proofs.
   Support your findings with pointers to their resources.