1. Solve the following recurrences and prove that your solutions are correct.

**Recurrence I:**

\[
T(n) = \begin{cases} 
2 & \text{for } n = 1 \\
T(n-1) + 7 & \text{for } n \geq 2 
\end{cases}
\]

**Bottom-Up evaluation:**

\[
egin{align*}
T(1) &= 2 = 7 \cdot 1 - 5 \\
T(2) &= T(1) + 7 = 9 = 7 \cdot 2 - 5 \\
T(3) &= T(2) + 7 = 16 = 7 \cdot 3 - 5 \\
T(4) &= T(3) + 7 = 23 = 7 \cdot 4 - 5 \\
&\vdots \\
T(n) &= \quad = 7 \cdot n - 5
\end{align*}
\]

**Top-Down evaluation:**

\[
egin{align*}
T(n) &= T(n-1) + 7 = T(n-1) + 1 \cdot 7 \\
&= (T(n-2) + 7) + 1 \cdot 7 = T(n-2) + 2 \cdot 7 \\
&= (T(n-3) + 7) + 2 \cdot 7 = T(n-3) + 3 \cdot 7 \\
&= (T(n-4) + 7) + 3 \cdot 7 = T(n-4) + 4 \cdot 7 \\
&\vdots \\
&= T(n - (n-1)) + (n-1)7 \\
&= T(1) + 7n - 7 \\
&= 2 + 7n - 7 \\
&= 7n - 5
\end{align*}
\]

**Solution:** \(T(n) = 7n - 5\) for \(n \geq 1\).

**Proof by induction:**

- **Induction base.** \(T(1) = 2 = 7 \cdot 1 - 5\).
- **Induction hypothesis.** \(T(n-1) = 7(n-1) - 5 = 7n - 12\) for \(n > 1\).
- **Inductive step.** For \(n > 1\),

\[
egin{align*}
T(n) &= T(n-1) + 7 \quad \text{(* definition of } T(n) *) \\
&= (7n - 12) + 7 \quad \text{(* induction hypothesis *)} \\
&= 7n - 5 \quad \text{(* rearranging terms *)}
\end{align*}
\]
Recurrence II:

\[
T(n) = \begin{cases} 
3 & \text{for } n = 1 \\
2T(n-1) & \text{for } n \geq 2
\end{cases}
\]

Bottom-Up evaluation:

\[
\begin{align*}
T(1) &= 3 = 3 \cdot 1 = 3 \cdot 2^0 \\
T(2) &= 2T(1) = 6 = 3 \cdot 2 = 3 \cdot 2^1 \\
T(3) &= 2T(2) = 12 = 3 \cdot 4 = 3 \cdot 2^2 \\
T(4) &= 2T(3) = 24 = 3 \cdot 8 = 3 \cdot 2^3 \\
\vdots & \quad \vdots \\
T(n) &= 3 \cdot 2^{n-1}
\end{align*}
\]

Top-Down evaluation:

\[
\begin{align*}
T(n) &= 2 \cdot T(n-1) = 2^1 \cdot T(n-1) \\
&= 2^1 \cdot (2 \cdot T(n-2)) = 2^2 \cdot T(n-2) \\
&= 2^2 \cdot (2 \cdot T(n-3)) = 2^3 \cdot T(n-3) \\
&= 2^3 \cdot (2 \cdot T(n-2)) = 2^4 \cdot T(n-4) \\
\vdots & \quad \vdots \\
&= 2^{n-1} \cdot T(n - (n - 1)) \\
&= 2^{n-1} \cdot T(1) \\
&= 2^{n-1} \cdot 3 \\
&= 3 \cdot 2^{n-1}
\end{align*}
\]

Solution: \( T(n) = 3 \cdot 2^{n-1} \) for \( n \geq 1 \).

Proof by induction:

- **Induction base.** \( T(1) = 3 = 3 \cdot 1 = 3 \cdot 2^0 = 3 \cdot 2^{1-1} \).
- **Induction hypothesis.** \( T(n-1) = 3 \cdot 2^{n-2} \) for \( n > 1 \).
- **Inductive step.** For \( n > 1 \),

\[
\begin{align*}
T(n) &= 2 \cdot T(n-1) \quad \text{(* definition of } T(n) *) \\
&= 2 \cdot (3 \cdot 2^{n-2}) \quad \text{(* induction hypothesis *)} \\
&= 3 \cdot (2 \cdot 2^{n-2}) \quad \text{(* rearranging terms *)} \\
&= 3 \cdot 2^{n-1} \quad \text{(* definition of the power function *)}
\end{align*}
\]
Recurrence III:

\[
T(n) = \begin{cases} 
2 & \text{for } n = 1 \\
(n + 1)T(n - 1) & \text{for } n \geq 2 
\end{cases}
\]

Bottom-Up evaluation:

\[
\begin{align*}
T(1) &= 2 = 2 \cdot 1 \\
T(2) &= 3T(1) = 6 = 3 \cdot 2 \cdot 1 \\
T(3) &= 4T(2) = 24 = 4 \cdot 3 \cdot 2 \cdot 1 \\
T(4) &= 5T(3) = 120 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
&\vdots \\
T(n) &= (n + 1) \cdot n \cdot (n - 1) \cdots 2 \cdot 1 \\
T(n) &= (n + 1)! 
\end{align*}
\]

Top-Down evaluation:

\[
\begin{align*}
T(n) &= (n + 1) \cdot T(n - 1) \\
&= (n + 1) \cdot (n \cdot T(n - 2)) \\
&= (n + 1) \cdot n \cdot ((n - 1) \cdot T(n - 3)) \\
&= (n + 1) \cdot n \cdot (n - 1) \cdot ((n - 2) \cdot T(n - 4)) \\
&\vdots \\
&= (n + 1) \cdot n \cdot (n - 1) \cdots 3 \cdot T(1) \\
&= (n + 1) \cdot n \cdot (n - 1) \cdots 3 \cdot 2 \\
&= (n + 1) \cdot n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1 \\
&= (n + 1)! 
\end{align*}
\]

Solution:  \( T(n) = (n + 1)! \) for \( n \geq 1 \).

Proof by induction:

- **Induction base.** \( T(1) = 2 = 2 \cdot 1 = 2! = (1 + 1)! \).
- **Induction hypothesis.** \( T(n - 1) = n! \) for \( n > 1 \).
- **Inductive step.** For \( n \geq 1 \),

\[
\begin{align*}
T(n) &= (n + 1)T(n - 1) \quad (* \text{ definition of } T(n) *) \\
&= (n + 1)n! \quad (* \text{ induction hypothesis } *) \\
&= (n + 1)! \quad (* \text{ definition of factorial } *)
\end{align*}
\]
2. Solve the following recurrence and prove that your solution is correct.

\[ P_n = \begin{cases} 
1 & \text{for } n = 0 \\
2 & \text{for } n = 1 \\
5P_{n-1} - 6P_{n-2} & \text{for } n \geq 2
\end{cases} \]

Small values of \( n \):

\[
\begin{align*}
P(0) &= 1 \\
P(1) &= 2 \\
P(2) &= 5 \cdot 2 - 6 \cdot 1 = 10 - 6 = 4 \\
P(3) &= 5 \cdot 4 - 6 \cdot 2 = 20 - 12 = 8 \\
P(4) &= 5 \cdot 8 - 6 \cdot 4 = 40 - 24 = 16 \\
P(5) &= 5 \cdot 16 - 6 \cdot 8 = 80 - 48 = 32
\end{align*}
\]

Solution: \( P(n) = 2^n \) for \( n \geq 0 \).

Proof by induction:

- **Induction base.** \( P(0) = 1 = 2^0 \) and \( P(1) = 2 = 2^1 \).
- **Induction hypothesis.** \( P(n - 1) = 2^{n-1} \) and \( P(n - 2) = 2^{n-2} \) for \( n \geq 2 \).
- **Inductive step.** For \( n \geq 2 \),

\[
\begin{align*}
P(n) &= 5P(n - 1) - 6P(n - 2) \quad (\ast \text{ definition of } P(n) \ast) \\
&= 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} \quad (\ast \text{ induction hypothesis } \ast) \\
&= 5 \cdot 2^{n-1} - 3 \cdot 2^{n-1} \quad (\ast \text{ algebra } \ast) \\
&= 2 \cdot 2^{n-1} \quad (\ast \text{ algebra } \ast) \\
&= 2^n \quad (\ast \text{ algebra } \ast)
\end{align*}
\]
3. Some facts about the Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

$$F_n = \begin{cases} 
0 & \text{for } n = 0 \\
1 & \text{for } n = 1 \\
F_{n-1} + F_{n-2} & \text{for } n \geq 2 
\end{cases}$$

- What is the smallest $n$ for which $F_n > 100$?
  \textbf{Answer:} $F_{11} = 89$ and $F_{12} = 144$, therefore 12 is the smallest $n$ for which $F_n > 100$.

- What is the smallest $n$ for which $F_n > 1000$?
  \textbf{Answer:} $F_{16} = 987$ and $F_{17} = 1597$, therefore 17 is the smallest $n$ for which $F_n > 1000$.

- Let $A_n = (F_1 + F_2 + \cdots + F_n)/n$ be the average of the first $n$ Fibonacci numbers. What is the smallest $n$ for which $A_n > 10$?
  \textbf{Answer:} Recall that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$. Therefore the sequence $A_1, A_2, \ldots$ is

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(n)$</td>
<td>1</td>
<td>1</td>
<td>4/3</td>
<td>7/4</td>
<td>12/5</td>
<td>20/6</td>
<td>33/7</td>
<td>54/8</td>
<td>88/9</td>
<td>143/10</td>
</tr>
</tbody>
</table>

Since $A_9 = 88/9 < 10$ and $A_{10} = 143/10 > 10$, it follows that 10 is the smallest $n$ for which $A_n > 10$.

- Find all $n$ for which $F_n = n$. Explain why these are the only cases.
  \textbf{Answer:} By inspection, $F_0 = 0$, $F_1 = 1$, and $F_5 = 5$ while $F_2 \neq 2$, $F_3 \neq 3$, and $F_4 \neq 4$. Since $F_n$ as a function of $n$ grows faster than the function $n$ for $n > 5$ (see the remark below), it follows that $F_n > n$ for $n > 5$.

- Find all $n$ for which $F_n = n^2$. Explain why these are the only cases.
  \textbf{Answer:} By inspection, $F_0 = 0 = 0^2$, $F_1 = 1 = 1^2$, and $F_{12} = 144 = 12^2$ while $F_n \neq n$ for $n \in \{2, 3, \ldots, 11\}$. Since $F_n$ as a function of $n$ grows faster than the function $n^2$ for $n > 12$ (see the remark below), it follows that $F_n > n^2$ for $n > 12$.

\textbf{Remark:} $F_{n+1}/F_n \approx \phi = 1.618 \ldots$. Therefore $F_n$ as a function of $n$ grows faster than the function $n^2$ for which $(n + 1)^2/n^2$ approaches 1 as $n$ tends to infinity. In particular after $F_{12} = 144$ for which $F_n = n^2$, it is always the case that $F_n > n^2$ for $n > 12$. This can be proven by induction. Similarly, after $F_5 = 5$ for which $F_n = n$, it is always the case that $F_n > n$ for $n > 5$. 

6
4. Prove the following identity for $n \geq 2$:

$$F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2}$$

**The cases** $n = 2, 3, 4, 5, 6$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$F_n + F_1$</th>
<th>$F_n + F_2$</th>
<th>$F_n + F_3$</th>
<th>$F_n + F_4$</th>
<th>$F_n + F_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2 + 1</td>
<td>3 + 1</td>
<td>5 + 2</td>
<td>8 + 3</td>
<td>13 + 5</td>
</tr>
<tr>
<td>3</td>
<td>3 - 0</td>
<td>5 - 1</td>
<td>8 - 1</td>
<td>13 - 2</td>
<td>21 - 3</td>
</tr>
<tr>
<td>4</td>
<td>$F_4 - F_0$</td>
<td>$F_5 - F_1$</td>
<td>$F_6 - F_2$</td>
<td>$F_7 - F_3$</td>
<td>$F_8 - F_4$</td>
</tr>
</tbody>
</table>

**Proof:** For $n \geq 2$,

$$F_{n+1} + F_{n-1} = F_{n+1} + (F_n - F_{n-2}) \quad (\ast F_{n-1} = F_n - F_{n-2} \ast)$$

$$= (F_{n+1} + F_n) - F_{n-2} \quad (\ast \text{rearranging parenthesis} \ast)$$

$$= F_{n+2} - F_{n-2} \quad (\ast F_{n+2} = F_{n+1} + F_n \ast)$$
5. Prove the following identity for \( n \geq 1 \):

\[
F_{2n+1} = F_{n+1}^2 + F_n^2
\]

**Proof by induction:**

- **Induction base.** Verify correctness for \( n = 1, 2, 3, 4 \):

\[
\begin{align*}
F_3 &= 2 = 1^2 + 1^2 = F_1^2 + F_2^2 \\
F_5 &= 5 = 2^2 + 1^2 = F_2^2 + F_3^2 \\
F_7 &= 13 = 3^2 + 2^2 = F_3^2 + F_4^2 \\
F_9 &= 34 = 5^2 + 3^2 = F_4^2 + F_5^2
\end{align*}
\]

- **Induction hypothesis.** Assume that for \( n \geq 3 \):

\[
\begin{align*}
F_{2n-1} &= F_n^2 + F_{n-1}^2 \\
F_{2n-3} &= F_{n-1}^2 + F_{n-2}^2
\end{align*}
\]

- **Inductive step.** By replacing \( F_{2n+1} \) with \( F_{2n} + F_{2n-1} \), then replacing \( F_{2n} \) with \( F_{2n-1} + F_{2n-2} \) and combining terms, and then replacing \( F_{2n-2} \) with \( F_{2n-3} - F_{2n-3} \) and combining terms, \( F_{2n+1} \) becomes a function of \( F_{2n-1} \) and \( F_{2n-3} \).

\[
F_{2n+1} = F_{2n} + F_{2n-1} \\
= (F_{2n-1} + F_{2n-2}) + F_{2n-1} \\
= 2F_{2n-1} + F_{2n-2} \\
= 2F_{2n-1} + (F_{2n-1} - F_{2n-3}) \\
= 3F_{2n-1} - F_{2n-3}
\]

After applying the induction hypothesis for \( F_{2n-1} \) and \( F_{2n-3} \) and simplifying, \( F_{2n+1} \) becomes a function of the squares of \( F_n, F_{n-1}, \) and \( F_{n-2} \).

\[
F_{2n+1} = 3F_{2n-1} - F_{2n-3} \\
= 3(F_n^2 + F_{n-1}^2) - (F_{n-1}^2 + F_{n-2}^2) \\
= 3F_n^2 + 2F_{n-1}^2 - F_{n-2}^2
\]

By replacing \( F_{n-2} \) with \( F_n - F_{n-1} \) and simplifying, \( F_{2n+1} \) becomes a function of the squares of \( F_n \) and \( F_{n-1} \).

\[
F_{2n+1} = 3F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \\
= 3F_n^2 + 2F_{n-1}^2 - (F_n - F_{n-1})^2 \\
= 3F_n^2 + 2F_{n-1}^2 - (F_n^2 - 2F_{n-1}F_n + F_{n-1}^2) \\
= 2F_n^2 + F_{n-1}^2 + 2F_{n-1}F_n \\
= (F_n^2 + 2F_{n-1}F_n + F_{n-1}^2) + F_n^2 \\
= (F_n + F_{n-1})^2 + F_n^2
\]

The proof is completed by replacing \( F_n + F_{n-1} \) with \( F_{n+1} \).

\[
F_{2n+1} = (F_n + F_{n-1})^2 + F_n^2 \\
= F_{n+1}^2 + F_n^2
\]
6. Define the following (almost Fibonacci) recurrence

\[
G_n = \begin{cases} 
0 & \text{for } n = 0 \\
1 & \text{for } n = 1 \\
G_{n-1} + G_{n-2} + 1 & \text{for } n \geq 2
\end{cases}
\]

The first 11 values in the sequence \(G_0, G_1, \ldots, G_{10}\) are:

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_n)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>20</td>
<td>33</td>
<td>54</td>
<td>88</td>
<td>143</td>
</tr>
</tbody>
</table>

There are two “natural” guesses for \(G_n\) as a function of fibonacci numbers based on the first 13 numbers in the Fibonacci sequence:

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_n)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
</tr>
</tbody>
</table>

\[
G_n = F_{n+2} - 1
\]

\[
G_n = F_0 + F_1 + \cdots + F_n
\]

**Remark:** Recall that \(F_0 + F_1 + \cdots F_n = F_{n+2} - 1\) is an identity that can be proven by induction.

**Proposition 1:** \(G_n = F_{n+2} - 1\) for \(n \geq 0\).

**Proof by induction:**

- **Induction base.** Prove that \(G_0 = F_2 - 1\) and that \(G_1 = F_3 - 1\):

  \[
  G_0 = 0 = 1 - 0 = F_2 - 1
  \]
  \[
  G_1 = 1 = 2 - 1 = F_3 - 1
  \]

- **Induction hypothesis.** Assume that for \(n \geq 2\):

  \[
  G_{n-1} = F_{n+1} - 1
  \]
  \[
  G_{n-2} = F_n - 1
  \]

- **Inductive step.** Prove that \(G_n = F_{n+2} - 1\) for \(n \geq 2\):

  \[
  G_n = G_{n-1} + G_{n-2} + 1 \quad \text{(* definition of } G_n \text{ *)}
  \]
  \[
  = (F_{n+1} - 1) + (F_n - 1) + 1 \quad \text{(* induction hypothesis *)}
  \]
  \[
  = (F_{n+1} + F_n) - 1 \quad \text{(* simplifying *)}
  \]
  \[
  = F_{n+2} - 1 \quad \text{(* } F_{n+2} = F_{n+1} + F_n \text{ *)}
  \]
Proposition 2: \( G_n = F_0 + F_1 + \cdots + F_n \) for \( n \geq 0 \).

Proof by induction:

- **Induction base.** Prove that \( G_0 = F_0 \) and that \( G_1 = F_0 + F_1 \):
  \[
  G_0 = 0 = F_0 \\
  G_1 = 1 = 0 + 1 = F_0 + F_1
  \]

- **Induction hypothesis.** Assume that for \( n \geq 2 \):
  \[
  G_{n-1} = F_0 + F_1 + F_2 + \cdots + F_{n-1} \\
  G_{n-2} = F_0 + F_1 + \cdots + F_{n-2}
  \]

- **Inductive step.** Prove that \( G_n = F_0 + F_1 + \cdots + F_n \) for \( n \geq 2 \):
  \[
  G_n = G_{n-1} + G_{n-2} + 1 \\
  = (F_0 + F_1 + F_2 + \cdots + F_{n-1}) + (F_0 + F_1 + \cdots + F_{n-2}) + 1 \\
  = (F_0 + 1) + (F_1 + F_2 + \cdots + F_{n-1}) + (F_0 + F_1 + \cdots + F_{n-2}) \\
  = F_1 + (F_1 + F_2 + \cdots + F_{n-1}) + (F_0 + F_1 + \cdots + F_{n-2}) \\
  = F_1 + (F_1 + F_0) + (F_2 + F_1) + \cdots + (F_{n-1} + F_{n-2}) \\
  = F_1 + F_2 + F_3 + \cdots + F_n \\
  = F_0 + F_1 + F_2 + F_3 + \cdots + F_n
  \]
  (\* definition of \( G_n \) \*)
  (\* induction hypothesis \*)
  (\* rearranging terms \*)
  (\* \( F_0 + 1 = F_1 \) \*)
  (\* rearranging terms \*)
  (\* Fibonacci recurrence \*)
  (\* \( F_0 = 0 \) \*)
7. For \( n \geq 1 \), there are \( F_{n+1} \) permutations \( \pi = (\pi(1), \pi(2), \ldots, \pi(n)) \) of the numbers \( \{1, 2, \ldots, n\} \) in which the value of \( \pi(i) \) is either \( i-1 \), or \( i \), or \( i+1 \) for all \( 1 \leq i \leq n \) where \( F_n \) is the \( n^{\text{th}} \) Fibonacci number.

**Proof:** For \( n \geq 1 \), a permutation \( \pi \) is **good** if \( \pi(i) \) is either \( i-1 \), or \( i \), or \( i+1 \) for all \( 1 \leq i \leq n \). Otherwise, \( \pi \) is a **bad** permutation. Observe that if \( \pi \) is a bad permutation then there exists at least one index \( i \) for which \( |\pi(i) - i| \geq 2 \) while if \( \pi \) is a good permutation then \( |\pi(i) - i| \leq 1 \) for all \( 1 \leq i \leq n \).

\( n = 1 \): The only permutation \( (1) \) is a good permutation.

\( n = 2 \): The two permutations \( (1, 2) \) and \( (2, 1) \) are good permutations.

\( n = 3 \): Out of the six permutations, the three permutations \( (1, 2, 3), (2, 1, 3), \) and \( (1, 3, 2) \) are good permutations while the three permutations \( (3, 1, 2), (2, 3, 1), \) and \( (3, 2, 1) \) are bad permutations since either \( \pi(1) = 3 \) or \( \pi(3) = 1 \).

\( n = 4 \): Out of the 24 permutations, only the five permutations \( (1, 2, 3, 4), (2, 1, 3, 4), (1, 3, 2, 4), (1, 2, 4, 3), \) and \( (2, 1, 4, 3) \) are good permutations.

For \( n \geq 1 \), let \( G_n \) be the number of good permutations. The above shows that \( G_1 = 1 = F_2, G_2 = 2 = F_3, G_3 = 3 = F_4, \) and \( G_4 = 5 = F_5 \).

For \( n \geq 3 \), let \( \pi \) be a good permutation. Therefore, \( \pi(n) = n \) or \( \pi(n) = n-1 \) because otherwise \( |\pi(n) - n| \geq 2 \).

- Assume \( \pi(n) = n \). Then \( \pi' = (\pi(1), \pi(2), \ldots, \pi(n-1)) \) must be a good permutation for the numbers \( \{1, 2, \ldots, n-1\} \). It follows that there are \( G_{n-1} \) good permutations \( \pi \) in which \( \pi(n) = n \).
- Assume \( \pi(n) = n-1 \). Then \( \pi(n-1) = n \) because otherwise \( \pi(i) = n \) for some \( i \leq n-2 \) which contradicts the goodness of \( \pi \). Moreover, \( \pi'' = (\pi(1), \pi(2), \ldots, \pi(n-2)) \) must be a good permutation for the numbers \( \{1, 2, \ldots, n-2\} \). It follows that there are \( G_{n-2} \) good permutations \( \pi \) in which \( \pi(n) = n-1 \) and \( \pi(n-1) = n \).

The above two cases imply that \( G_n = G_{n-1} + G_{n-2} \) for \( n \geq 3 \). Since \( G_1 = 1 = F_2 \) and \( G_2 = 2 = F_3 \), it follows that the number of good permutations for \( n \geq 1 \) is the Fibonacci number \( F_{n+1} \).

**Example:** For \( n = 5 \), there are \( F_6 = 8 \) good permutations. In \( F_5 = 5 \) of them \( \pi(5) = 5 \):

\[
(1, 2, 3, 4, 5) \quad (2, 1, 3, 4, 5) \quad (1, 3, 2, 4, 5) \quad (1, 2, 4, 3, 5) \quad (2, 1, 4, 3, 5)
\]

and in \( F_4 = 3 \) of them \( \pi(4) = 5 \) and \( \pi(5) = 4 \):

\[
(1, 2, 3, 5, 4) \quad (2, 1, 3, 5, 4) \quad (1, 3, 2, 5, 4)
\]