Discrete Structures

Recursion Practice Problems: Solutions
1. Solve the following recurrences and prove that your solutions are correct.

**Recurrence I:**

\[
T(n) = \begin{cases} 
2 & \text{for } n = 1 \\
T(n-1) + 7 & \text{for } n \geq 2
\end{cases}
\]

**Bottom-Up evaluation:**

\[
T(1) = 2 = 7 \cdot 1 - 5 \\
T(2) = T(1) + 7 = 9 = 7 \cdot 2 - 5 \\
T(3) = T(2) + 7 = 16 = 7 \cdot 3 - 5 \\
T(4) = T(3) + 7 = 23 = 7 \cdot 4 - 5 \\
\vdots \\
T(n) = 7 \cdot n - 5
\]

**Top-Down evaluation:**

\[
T(n) = T(n-1) + 7 = T(n-1) + 1 \cdot 7 \\
= (T(n-2) + 7) + 1 \cdot 7 = T(n-2) + 2 \cdot 7 \\
= (T(n-3) + 7) + 2 \cdot 7 = T(n-3) + 3 \cdot 7 \\
= (T(n-4) + 7) + 3 \cdot 7 = T(n-4) + 4 \cdot 7 \\
\vdots \\
= T(n-(n-1)) + (n-1)7 \\
= T(1) + 7n - 7 \\
= 2 + 7n - 7 \\
= 7n - 5
\]

**Solution:** \(T(n) = 7n - 5\) for \(n \geq 1\).

**Proof by induction:**

- **Induction base.** \(T(1) = 2 = 7 \cdot 1 - 5\).
- **Induction hypothesis.** \(T(n-1) = 7(n-1) - 5 = 7n - 12\) for \(n > 1\).
- **Inductive step.** For \(n > 1\),

\[
T(n) = T(n-1) + 7 \quad (* \text{definition of } T(n) *) \\
= (7n - 12) + 7 \quad (* \text{induction hypothesis } *) \\
= 7n - 5 \quad (* \text{rearranging terms } *)
\]
Recurrence II:

\[ T(n) = \begin{cases} 
3 & \text{for } n = 1 \\
2T(n-1) & \text{for } n \geq 2 
\end{cases} \]

Bottom-Up evaluation:

\[
T(1) = 3 = 3 \cdot 1 = 3 \cdot 2^0 \\
T(2) = 2T(1) = 6 = 3 \cdot 2 = 3 \cdot 2^1 \\
T(3) = 2T(2) = 12 = 3 \cdot 4 = 3 \cdot 2^2 \\
T(4) = 2T(3) = 24 = 3 \cdot 8 = 3 \cdot 2^3 \\
\vdots \quad \vdots \\
T(n) = 3 \cdot 2^{n-1}
\]

Top-Down evaluation:

\[
T(n) = 2 \cdot T(n-1) = 2^1 \cdot T(n-1) \\
= 2^1 \cdot (2 \cdot T(n-2)) = 2^2 \cdot T(n-2) \\
= 2^2 \cdot (2 \cdot T(n-3)) = 2^3 \cdot T(n-3) \\
= 2^3 \cdot (2 \cdot T(n-2)) = 2^4 \cdot T(n-4) \\
\vdots \quad \vdots \\
= 2^{n-1} \cdot T(n-(n-1)) \\
= 2^{n-1} \cdot T(1) \\
= 2^{n-1} \cdot 3 = 3 \cdot 2^{n-1}
\]

Solution: \( T(n) = 3 \cdot 2^{n-1} \) for \( n \geq 1 \).

Proof by induction:

- Induction base. \( T(1) = 3 = 3 \cdot 1 = 3 \cdot 2^0 = 3 \cdot 2^{1-1} \).
- Induction hypothesis. \( T(n-1) = 3 \cdot 2^{n-2} \) for \( n > 1 \).
- Inductive step. For \( n > 1 \),

\[
T(n) = 2 \cdot T(n-1) \quad (* \text{definition of } T(n) *) \\
= 2 \cdot (3 \cdot 2^{n-2}) \quad (* \text{induction hypothesis } *) \\
= 3 \cdot (2 \cdot 2^{n-2}) \quad (* \text{rearranging terms } *) \\
= 3 \cdot 2^{n-1} \quad (* \text{definition of the power function } *)
\]
Recurrence III:

\[ T(n) = \begin{cases} 2 & \text{for } n = 1 \\ (n + 1)T(n - 1) & \text{for } n \geq 2 \end{cases} \]

Bottom-Up evaluation:

\[
\begin{align*}
T(1) &= 2 = 2 \cdot 1 \\
T(2) &= 3T(1) = 6 = 3 \cdot 2 \cdot 1 \\
T(3) &= 4T(2) = 24 = 4 \cdot 3 \cdot 2 \cdot 1 \\
T(4) &= 5T(3) = 120 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
&\vdots \\
T(n) &= (n + 1) \cdot n \cdot (n - 1) \cdots 2 \cdot 1 \\
T(n) &= (n + 1)!
\end{align*}
\]

Top-Down evaluation:

\[
\begin{align*}
T(n) &= (n + 1) \cdot T(n - 1) \\
&= (n + 1) \cdot (n \cdot T(n - 2)) \\
&= (n + 1) \cdot n \cdot ((n - 1) \cdot T(n - 3)) \\
&= (n + 1) \cdot n \cdot (n - 1) \cdot ((n - 2) \cdot T(n - 4)) \\
&\vdots \\
&= (n + 1) \cdot n \cdot (n - 1) \cdots 3 \cdot T(1) \\
&= (n + 1) \cdot n \cdot (n - 1) \cdots 3 \cdot 2 \\
&= (n + 1) \cdot n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1 \\
&= (n + 1)!
\end{align*}
\]

Solution: \( T(n) = (n + 1)! \) for \( n \geq 1 \).

Proof by induction:

- **Induction base.** \( T(1) = 2 = 2 \cdot 1 = 2! = (1 + 1)! \).
- **Induction hypothesis.** \( T(n - 1) = n! \) for \( n > 1 \).
- **Inductive step.** For \( n \geq 1 \),

\[
\begin{align*}
T(n) &= (n + 1)T(n - 1) \quad (\ast \text{ definition of } T(n) \ast) \\
&= (n + 1)n! \quad (\ast \text{ induction hypothesis } \ast) \\
&= (n + 1)! \quad (\ast \text{ definition of factorial } \ast)
\end{align*}
\]
2. Solve the following recurrence and prove that your solution is correct.

\[ P_n = \begin{cases} 
1 & \text{for } n = 0 \\
2 & \text{for } n = 1 \\
5P_{n-1} - 6P_{n-2} & \text{for } n \geq 2
\end{cases} \]

Small values of \( n \):

- \( P(0) = 1 \)
- \( P(1) = 2 \)
- \( P(2) = 5 \cdot 2 - 6 \cdot 1 = 10 - 6 = 4 \)
- \( P(3) = 5 \cdot 4 - 6 \cdot 2 = 20 - 12 = 8 \)
- \( P(4) = 5 \cdot 8 - 6 \cdot 4 = 40 - 24 = 16 \)
- \( P(5) = 5 \cdot 16 - 6 \cdot 8 = 80 - 48 = 32 \)

Solution: \( P(n) = 2^n \) for \( n \geq 0 \).

Proof by induction:

- **Induction base.** \( P(0) = 1 = 2^0 \) and \( P(1) = 2 = 2^1 \).
- **Induction hypothesis.** \( P(n - 1) = 2^{n-1} \) and \( P(n - 2) = 2^{n-2} \) for \( n \geq 2 \).
- **Inductive step.** For \( n \geq 2 \),

\[
\begin{align*}
P(n) &= 5P(n-1) - 6P(n-2) \quad (* \text{definition of } P(n) *) \\
&= 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} \quad (* \text{induction hypothesis } *) \\
&= 5 \cdot 2^{n-1} - 3 \cdot 2^{n-1} \quad (* \text{algebra } *) \\
&= 2 \cdot 2^{n-1} \quad (* \text{algebra } *) \\
&= 2^n \quad (* \text{algebra } *)
\end{align*}
\]
3. Some facts about the Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... 

\[ F_n = \begin{cases} 
0 & \text{for } n = 0 \\
1 & \text{for } n = 1 \\
F_{n-1} + F_{n-2} & \text{for } n \geq 2 
\end{cases} \]

- What is the smallest \( n \) for which \( F_n > 100 \)?
  **Answer:** \( F_{11} = 89 \) and \( F_{12} = 144 \), therefore 12 is the smallest \( n \) for which \( F_n > 100 \).

- What is the smallest \( n \) for which \( F_n > 1000 \)?
  **Answer:** \( F_{16} = 987 \) and \( F_{17} = 1597 \), therefore 17 is the smallest \( n \) for which \( F_n > 1000 \).

- Let \( A_n = (F_1 + F_2 + \cdots + F_n)/n \) be the average of the first \( n \) Fibonacci numbers. What is the smallest \( n \) for which \( A_n > 10 \)?
  **Answer:** Recall that \( \sum_{i=1}^{n} F_i = F_{n+2} - 1 \). Therefore the sequence \( A_1, A_2, \ldots \) is

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(n) )</td>
<td>1</td>
<td>1</td>
<td>4/3</td>
<td>7/4</td>
<td>12/5</td>
<td>20/6</td>
<td>33/7</td>
<td>54/8</td>
<td>88/9</td>
<td>143/10</td>
</tr>
</tbody>
</table>

Since \( A_9 = 88/9 < 10 \) and \( A_{10} = 143/10 > 10 \), it follows that 10 is the smallest \( n \) for which \( A_n > 10 \).

- Find all \( n \) for which \( F_n = n \).
  **Answer:** By inspection, \( F_0 = 0, F_1 = 1, \) and \( F_5 = 5 \) while \( F_2 \neq 2, F_3 \neq 3, \) and \( F_4 \neq 4 \). Since \( F_n \) as a function of \( n \) grows faster than the function \( n \) for \( n > 5 \) (see the remark below), it follows that \( F_n > n \) for \( n > 5 \).

- Find all \( n \) for which \( F_n = n^2 \).
  **Answer:** By inspection, \( F_0 = 0 = 0^2, F_1 = 1 = 1^2, \) and \( F_{12} = 144 = 12^2 \) while \( F_n \neq n^2 \) for \( n \in \{2, 3, \ldots, 11\} \). Since \( F_n \) as a function of \( n \) grows faster than the function \( n^2 \) for \( n > 12 \) (see the remark below), it follows that \( F_n > n^2 \) for \( n > 12 \).

**Remark:** \( F_{n+1}/F_n \approx \phi = 1.618 \ldots \) Therefore \( F_n \) as a function of \( n \) grows faster than the function \( n^2 \) for which \( (n+1)^2/n^2 \) approaches 1 as \( n \) tends to infinity. In particular after \( F_{12} = 144 \) for which \( F_n = n^2 \), it is always the case that \( F_n > n^2 \) for \( n > 12 \). This can be proven by induction. Similarly, after \( F_5 = 5 \) for which \( F_n = n \), it is always the case that \( F_n > n \) for \( n > 5 \).
4. Prove the following identity for $n \geq 2$:

$$F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2}$$

The cases $n = 2, 3, 4, 5, 6$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$F_n + F_{n-1}$</th>
<th>2 + 1</th>
<th>3</th>
<th>3 − 0</th>
<th>$F_4 - F_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3$</td>
<td>$F_3 + F_1$</td>
<td>3 + 1</td>
<td>4</td>
<td>5 − 1</td>
<td>$F_5 - F_1$</td>
</tr>
<tr>
<td>$4$</td>
<td>$F_4 + F_2$</td>
<td>5 + 2</td>
<td>7</td>
<td>8 − 1</td>
<td>$F_6 - F_2$</td>
</tr>
<tr>
<td>$5$</td>
<td>$F_5 + F_3$</td>
<td>8 + 3</td>
<td>11</td>
<td>13 − 2</td>
<td>$F_7 - F_3$</td>
</tr>
<tr>
<td>$6$</td>
<td>$F_6 + F_4$</td>
<td>13 + 5</td>
<td>18</td>
<td>21 − 3</td>
<td>$F_8 - F_4$</td>
</tr>
</tbody>
</table>

Proof: For $n \geq 2$,

$$F_{n+1} + F_{n-1} = F_{n+1} + (F_n - F_{n-2}) \quad (* \text{ } F_{n-1} = F_n - F_{n-2} \text{ *)}$$
$$= (F_{n+1} + F_n) - F_{n-2} \quad (* \text{ rearranging parenthesis *)}$$
$$= F_{n+2} - F_{n-2} \quad (* \text{ } F_{n+2} = F_{n+1} + F_n \text{ *)}$$
5. Prove the following identity for \( n \geq 1 \):

\[
F_{2n+1} = F_{n+1}^2 + F_n^2
\]

**Proof by induction:**

- **Induction base.** Verify correctness for \( n = 1, 2, 3, 4 \):

\[
\begin{align*}
F_3 &= 2 = 1^2 + 1^2 = F_2^2 + F_1^2 \\
F_5 &= 5 = 2^2 + 1^2 = F_3^2 + F_2^2 \\
F_7 &= 13 = 3^2 + 2^2 = F_4^2 + F_3^2 \\
F_9 &= 34 = 5^2 + 3^2 = F_5^2 + F_4^2
\end{align*}
\]

- **Induction hypothesis.** Assume that for \( n \geq 3 \):

\[
\begin{align*}
F_{2n-1} &= F_n^2 + F_{n-1}^2 \\
F_{2n-3} &= F_{n-1}^2 + F_{n-2}^2
\end{align*}
\]

- **Inductive step.** By replacing \( F_{2n+1} \) with \( F_{2n} + F_{2n-1} \), then replacing \( F_{2n} \) with \( F_{2n-1} + F_{2n-2} \) and combining terms, and then replacing \( F_{2n-2} \) with \( F_{2n-1} - F_{2n-3} \) and combining terms, \( F_{2n+1} \) becomes a function of \( F_{2n-1} \) and \( F_{2n-3} \).

\[
\begin{align*}
F_{2n+1} &= F_{2n} + F_{2n-1} \\
&= (F_{2n-1} + F_{2n-2}) + F_{2n-1} \\
&= 2F_{2n-1} + F_{2n-2} \\
&= 2F_{2n-1} + (F_{2n-1} - F_{2n-3}) \\
&= 3F_{2n-1} - F_{2n-3}
\end{align*}
\]

After applying the induction hypothesis for \( F_{2n-1} \) and \( F_{2n-3} \) and simplifying, \( F_{2n+1} \) becomes a function of the squares of \( F_n, F_{n-1}, \) and \( F_{n-2} \).

\[
\begin{align*}
F_{2n+1} &= 3F_{2n-1} - F_{2n-3} \\
&= 3(F_n^2 + F_{n-1}^2) - (F_{n-1}^2 + F_{n-2}^2) \\
&= 3F_n^2 + 2F_{n-1}^2 - F_{n-2}^2
\end{align*}
\]

By replacing \( F_{n-2} \) with \( F_n - F_{n-1} \) and simplifying, \( F_{2n+1} \) becomes a function of the squares of \( F_n \) and \( F_{n-1} \).

\[
\begin{align*}
F_{2n+1} &= 3F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \\
&= 3F_n^2 + 2F_{n-1}^2 - (F_n - F_{n-1})^2 \\
&= 3F_n^2 + 2F_{n-1}^2 - (F_n^2 - 2F_n F_{n-1} + F_{n-1}^2) \\
&= 2F_n^2 + F_{n-1}^2 + 2F_n F_{n-1} \\
&= (F_n^2 + 2F_{n-1} F_n + F_{n-1}^2) + F_n^2 \\
&= (F_n + F_{n-1})^2 + F_n^2
\end{align*}
\]

The proof is completed by replacing \( F_n + F_{n-1} \) with \( F_{n+1} \).

\[
\begin{align*}
F_{2n+1} &= (F_n + F_{n-1})^2 + F_n^2 \\
&= F_{n+1}^2 + F_n^2
\end{align*}
\]
6. Define the following (almost Fibonacci) recurrence

\[ G_n = \begin{cases} 
0 & \text{for } n = 0 \\
1 & \text{for } n = 1 \\
G_{n-1} + G_{n-2} + 1 & \text{for } n \geq 2 
\end{cases} \]

The first 11 values in the sequence \( G_0, G_1, \ldots, G_{10} \) are:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_n )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>20</td>
<td>33</td>
<td>54</td>
<td>88</td>
<td>143</td>
</tr>
</tbody>
</table>

There are two “natural” guesses for \( G_n \) as a function of Fibonacci numbers based on the first 13 numbers in the Fibonacci sequence:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_n )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
</tr>
</tbody>
</table>

\[ G_n = F_{n+2} - 1 \]
\[ G_n = F_0 + F_1 + \cdots + F_n \]

**Remark:** Recall that \( F_0 + F_1 + \cdots + F_n = F_{n+2} - 1 \) is an identity that can be proven by induction.

**Proposition 1:** \( G_n = F_{n+2} - 1 \) for \( n \geq 0 \).

**Proof by induction:**

- **Induction base.** Show that \( G_0 = F_2 - 1 \) and that \( G_1 = F_3 - 1 \):

  \[ G_0 = 0 = 1 - 0 = F_2 - 1 \]
  \[ G_1 = 1 = 2 - 1 = F_3 - 1 \]

- **Induction hypothesis.** Assume that for \( n \geq 2 \):

  \[ G_{n-1} = F_{n+1} - 1 \]
  \[ G_{n-2} = F_n - 1 \]

- **Inductive step.** Prove that \( G_n = F_{n+2} - 1 \) for \( n \geq 2 \):

  \[ G_n = G_{n-1} + G_{n-2} + 1 \quad \text{(* definition of } G_n \text{*}) \]
  \[ = (F_{n+1} - 1) + (F_n - 1) + 1 \quad \text{(* induction hypothesis *)} \]
  \[ = (F_{n+1} + F_n) - 1 \quad \text{(* simplifying *)} \]
  \[ = F_{n+2} - 1 \quad \text{(* } F_{n+2} = F_{n+1} + F_n \text{*}) \]
Proposition 2: \( G_n = F_0 + F_1 + \cdots + F_n \) for \( n \geq 0 \).

Proof by induction:

- **Induction base.** Show that \( G_0 = F_0 \) and that \( G_1 = F_0 + F_1 \):
  \[
  G_0 &= 0 = F_0 \\
  G_1 &= 1 = 0 + 1 = F_0 + F_1
  \]

- **Induction hypothesis.** Assume that for \( n \geq 2 \):
  \[
  G_{n-1} = F_0 + F_1 + F_2 + \cdots + F_{n-1} \\
  G_{n-2} = F_0 + F_1 + \cdots + F_{n-2}
  \]

- **Inductive step.** Prove that \( G_n = F_0 + F_1 + \cdots + F_n \) for \( n \geq 2 \):
  \[
  G_n &= G_{n-1} + G_{n-2} + 1 \\
  &= (F_0 + F_1 + F_2 + \cdots + F_{n-1}) + (F_0 + F_1 + \cdots + F_{n-2}) + 1 \\
  &= (F_0 + 1) + (F_1 + F_2 + \cdots + F_{n-1}) + (F_0 + F_1 + \cdots + F_{n-2}) \\
  &= F_1 + (F_1 + F_2 + \cdots + F_{n-1}) + (F_0 + F_1 + \cdots + F_{n-2}) \\
  &= F_1 + (F_1 + F_0) + (F_2 + F_1) + \cdots + (F_{n-1} + F_{n-2}) \\
  &= F_1 + F_2 + F_3 + \cdots + F_n \\
  &= F_0 + F_1 + F_2 + F_3 + \cdots + F_n
  \]
  \(* \text{Fibonacci recurrence} *\)  
  \(* \text{F}_0 = 0 *\)
7. For \( n \geq 1 \), there are \( F_{n+1} \) permutations \( \pi = (\pi(1), \pi(2), \ldots, \pi(n)) \) of the numbers \( \{1, 2, \ldots, n\} \) in which the value of \( \pi(i) \) is either \( i - 1 \), or \( i \), or \( i + 1 \) for all \( 1 \leq i \leq n \) where \( F_n \) is the \( n \)th Fibonacci number.

**Proof:** For \( n \geq 1 \), a permutation \( \pi \) is **good** if \( \pi(i) \) is either \( i - 1 \), or \( i \), or \( i + 1 \) for all \( 1 \leq i \leq n \). Otherwise, \( \pi \) is a **bad** permutation. Equivalently, if \( \pi \) is a **bad** permutation then there exists at least one index \( i \) for which \( |\pi(i) - i| \geq 2 \) while if \( \pi \) is a **good** permutation then \( |\pi(i) - i| \leq 1 \) for all \( 1 \leq i \leq n \).

**n = 1:** The only permutation \( (1) \) is a **good** permutation.

**n = 2:** The two permutations \( (1, 2) \) and \( (2, 1) \) are **good** permutations.

**n = 3:** Out of the six permutations, the three permutations \( (1, 2, 3), (2, 1, 3), \) and \( (1, 3, 2) \) are **good** permutations while the three permutations \( (3, 1, 2), (2, 3, 1), \) and \( (3, 2, 1) \) are **bad** permutations since either \( \pi(1) = 3 \) or \( \pi(3) = 1 \).

**n = 4:** Out of the 24 permutations, only the five permutations \( (1, 2, 3, 4), (2, 1, 3, 4), (1, 3, 2, 4), (1, 2, 4, 3), \) and \( (2, 1, 4, 3) \) are **good** permutations.

For \( n \geq 1 \), let \( G_n \) be the number of **good** permutations. The above shows that \( G_1 = 1 = F_2 \), \( G_2 = 2 = F_3 \), \( G_3 = 3 = F_4 \), and \( G_4 = 5 = F_5 \).

For \( n \geq 3 \), let \( \pi \) be a **good** permutation. Therefore, \( \pi(n) = n \) or \( \pi(n) = n - 1 \) because otherwise \( |\pi(n) - n| \geq 2 \).

- Assume \( \pi(n) = n \). Then \( \pi' = (\pi(1), \pi(2), \ldots, \pi(n-1)) \) must be a **good** permutation for the numbers \( \{1, 2, \ldots, n-1\} \). It follows that there are \( G_{n-1} \) **good** permutations \( \pi \) in which \( \pi(n) = n \).
- Assume \( \pi(n) = n - 1 \). Then \( \pi(n-1) = n \) because otherwise \( \pi(i) = n \) for some \( i \leq n - 2 \) which contradicts the **goodness** of \( \pi \). Moreover, \( \pi'' = (\pi(1), \pi(2), \ldots, \pi(n-2)) \) must be a **good** permutation for the numbers \( \{1, 2, \ldots, n-2\} \). It follows that there are \( G_{n-2} \) **good** permutations \( \pi \) in which \( \pi(n) = n - 1 \) and \( \pi(n - 1) = n \).

The above two cases imply that \( G_n = G_{n-1} + G_{n-2} \) for \( n \geq 3 \). Since \( G_1 = 1 = F_2 \) and \( G_2 = 2 = F_3 \), it follows that the number of **good** permutations for \( n \geq 1 \) is the Fibonacci number \( F_{n+1} \).

**Example:** For \( n = 5 \), there are \( F_6 = 8 \) **good** permutations. In \( F_5 = 5 \) of them \( \pi(5) = 5 \):

\[
(1, 2, 3, 4, 5) \quad (2, 1, 3, 4, 5) \quad (1, 3, 2, 4, 5) \quad (1, 2, 4, 3, 5) \quad (2, 1, 4, 3, 5)
\]

and in \( F_4 = 3 \) of them \( \pi(4) = 5 \) and \( \pi(5) = 4 \):

\[
(1, 2, 3, 5, 4) \quad (2, 1, 3, 5, 4) \quad (1, 3, 2, 5, 4)
\]