Structure, problem selection, and credit:

- You have 75 minutes to complete the exam.
- You will only get partial credit if you fail to justify or prove your answers. You will get 20% of the credit for any problem or part of it if you leave the allocated space for the answer empty. You will get zero credit for wrong answers.

**Honor code:** Students are expected to do this exam by themselves without any external help from other people, the Internet, books, or notes. Cheaters will be punished severely. At minimum, they will fail the exam, but they may fail the whole class. In addition, students who cheat risk disciplinary measures by Brooklyn College and CUNY.
1. For \( n \geq 3 \), the goal is to simplify the expression \( B(n) \) to contain only one binomial coefficient term instead of two binomial coefficient terms.

\[
B(n) \equiv \binom{n+1}{3} - \binom{n}{3}
\]

(a) What is the value of \( B(n) \) for \( n = 3 \)?

(b) What is the value of \( B(n) \) for \( n = 4 \)?

(c) What is the value of \( B(n) \) for \( n = 5 \)?

(d) Simplify \( B(n) \) for any \( n \geq 3 \). Prove that your simpler expression is identical to the original expression.

**Hint:** There is a “long way” that simplifies the expression using the explicit formula for \( \binom{n}{k} \). But there is a short cut that is based on the recursive property of \( \binom{n}{k} \).
2. The goal is to count the number of length-$k$ ($1 \leq k \leq 10$) passwords without repetitions on the ten digits 0, 1, . . . , 9 given some restrictions. Note that in such passwords, each digit may appear only once but may not appear at all.

(a) How many length-2 passwords exist? Justify your answer.

How many length-2 passwords exist in which the last digit must be 0? Justify your answer.

How many length-2 passwords exist in which the first digit cannot be 0? Justify your answer.

(b) How many length-3 passwords exist? Justify your answer.

How many length-3 passwords exist in which the last digit must be 0? Justify your answer.

How many length-3 passwords exist in which the first digit cannot be 0? Justify your answer.
(c) For $1 \leq k \leq 10$, as a function of $k$, how many length-$k$ passwords exist? Justify your answer.

For $1 \leq k \leq 10$, as a function of $k$, how many length-$k$ passwords exist in which the last digit must be 0? Justify your answer.

For $1 \leq k \leq 10$, as a function of $k$, how many length-$k$ passwords exist in which the first digit cannot be 0? Justify your answer.
3. Prove the following identity by induction on $n \geq 1$ or by any other method.

$$\sum_{i=1}^{n} (6i - 3) = 3n^2$$
4. A binary string is 1-even if the length of any sequence of consecutive 1’s in the string is even. For example (0110), (11011), (00000), and (0011110) are 1-even binary strings while (010), (11100), and (1100111) are not 1-even binary strings.

(a) List all the 1-even strings of length 2.

(b) List all the 1-even strings of length 3.

(c) List all the 1-even strings of length 4.

(d) For \( k \geq 1 \), as a function of \( k \), how many 1-even binary strings of length \( k \) exist? Justify your answer.

**Hint:** Think of Fibonacci numbers and their recursive definition. Then consider the two cases in which the first bit in the string is 1 or 0.