Discrete Structures: Midterm Exam II – Solutions

1. For $n \geq 3$, the goal is to simplify the expression $B(n)$ to contain only one binomial coefficient term instead of two binomial coefficient terms.

$$B(n) \equiv \binom{n + 1}{3} - \binom{n}{3}$$

(a) What is the value of $B(n)$ for $n = 3$?

$$\binom{4}{3} - \binom{3}{3} = \frac{4!}{3!(4 - 3)!} - \frac{3!}{3!(3 - 3)!} = \frac{4!}{3!} - \frac{3!}{3!} = \frac{4!}{3!1!} - \frac{3!}{3!0!} = \frac{24}{6} - \frac{6}{6} = 4 - 1 = 3$$

(b) What is the value of $B(n)$ for $n = 4$?

$$\binom{5}{3} - \binom{4}{3} = \frac{5!}{3!(5 - 3)!} - \frac{4!}{3!(4 - 3)!} = \frac{5!}{3!2!} - \frac{4!}{3!1!} = \frac{5!}{3!2!} - \frac{4!}{3!1!} = \frac{120}{24} - \frac{24}{6} = 10 - 4 = 6$$

(c) What is the value of $B(n)$ for $n = 5$?

$$\binom{6}{3} - \binom{5}{3} = \frac{6!}{3!(6 - 3)!} - \frac{5!}{3!(5 - 3)!} = \frac{6!}{3!3!} - \frac{5!}{3!2!} = \frac{6!}{3!3!} - \frac{5!}{3!2!} = \frac{720}{120} - \frac{120}{12} = 20 - 10 = 10$$
(d) Simplify $B(n)$ for any $n \geq 3$.

**Method 1:**

\[
\binom{n+1}{3} - \binom{n}{3} = \frac{(n+1)!}{3!(n-2)!} - \frac{n!}{3!(n-3)!}
= \frac{(n+1)n(n-1)}{6} - \frac{n(n-1)(n-2)}{6}
= \frac{(n+1)n(n-1) - n(n-1)(n-2)}{6}
= \frac{n(n-1)((n+1) - (n-2))}{6}
= \frac{n(n-1)3}{6}
= \frac{n(n-1)}{2}
= \binom{n}{2}
\]

**Method 2:** Recall the recursive identity for $\binom{n}{k}$:

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

By replacing $k$ with 3 and $n$ with $n+1$ (and therefore $n-1$ with $n$) the identity becomes

\[
\binom{n+1}{3} = \binom{n}{3} + \binom{n}{2}
\]

By rearranging terms, this identity is equivalent to the following identity that is the answer to the problem

\[
\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}
\]
2. The goal is to count the number of length-$k$ \((1 \leq k \leq 10)\) passwords without repetitions on the ten digits \(0, 1, \ldots, 9\) given some restrictions. Note that in such passwords, each digit may appear only once but may not appear at all.

(a) \(i\). How many length-2 passwords exist?

**Answer:** \(90 = 10 \times 9\).

**Proof:** There are 10 options for the first digit and only 9 options for the second digit that must be different from the first digit.

\(ii\). How many length-2 passwords exist in which the last digit must be 0?

**Answer:** 9.

**Proof:** The last digit is 0. Then there are only 9 options for the first digit that cannot be 0.

\(iii\). How many length-2 passwords exist in which the first digit cannot be 0?

**Answer 1:** \(81 = 9 \times 9\).

**Proof:** There are 9 options for the first digit that cannot be 0 and 9 options for the second digit that must be different from the first digit.

**Answer 2:** \(81 = 90 - 9\).

**Proof:** The first digit is either 0 or not 0. Therefore, the number of passwords in which the first digit cannot be 0 is the total number of passwords minus the number of passwords in which the first digit is 0 which is equal to the number of passwords in which the last digit is 0. Thus, the answer to (iii) is the answer to (i) minus the answer to (ii).

(b) \(i\). How many length-3 passwords exist?

**Answer:** \(720 = 10 \times 9 \times 8\).

**Proof:** There are 10 options for the first digit, 9 options for the second digit that must be different from the first digit, and only 8 options for the third digit that must be different from the first and the second digits.

\(ii\). How many length-3 passwords exist in which the last digit must be 0?

**Answer:** \(9 \times 8\).

**Proof:** The last digit is 0. Then there are 9 options for the first digit that cannot be 0 and only 8 options for the second digit that must be different from the first and the last digits.

\(iii\). How many length-3 passwords exist in which the first digit cannot be 0?

**Answer 1:** \(648 = 9 \times 9 \times 8\).

**Proof:** There are 9 options for the first digit that cannot be 0. Then there are 9 options for the second digit that must be different from the first digit and 8 options for the third digit that must be different from the first and the second digits.

**Answer 2:** \(648 = 720 - 72\).

**Proof:** The first digit is either 0 or not 0. Therefore, the number of passwords in which the first digit cannot be 0 is the total number of passwords minus the number of passwords in which the first digit is 0 which is equal to the number of passwords in which the last digit is 0. Thus, the answer to (iii) is the answer to (i) minus the answer to (ii).
(c) i. For $1 \leq k \leq 10$, as a function of $k$, how many length-$k$ passwords exist? 

**Answer:** 

\[
10 \times 9 \times \cdots \times (10 - k + 1) = \frac{10 \times 9 \times \cdots \times 1}{(10 - k) \times (10 - k - 1) \times \cdots \times 1} = \frac{10!}{(10 - k)!}
\]

**Proof:** There are 10 options for the first digit, there are 9 options for the second digit that must be different from the first digit, and so on until there are only $10 - k + 1$ options for the $k^{th}$ digit that must be different from the first $k - 1$ digits.

ii. For $1 \leq k \leq 10$, as a function of $k$, how many length-$k$ passwords exist in which the last digit must be 0? 

**Answer:** 

\[
9 \times 8 \times \cdots \times (10 - k + 1) = \frac{9 \times 8 \times \cdots \times 1}{(10 - k) \times (10 - k - 1) \times \cdots \times 1} = \frac{9!}{(10 - k)!}
\]

**Proof:** The last digit is 0. Then there are 9 options for the first digit that cannot be 0, there are 8 options for the second digit that must be different from the first and the last digits, and so on until there are only $10 - k + 1$ options for the $(k - 1)^{th}$ digit that must be different from the the first $k - 2$ digits and the last digit.

iii. For $1 \leq k \leq 10$, as a function of $k$, how many length-$k$ passwords exist in which the first digit cannot be 0? 

**Answer 1:** 

\[
9 \times 9 \times 8 \times \cdots \times (10 - k + 1) = \frac{9 \times 9 \times 8 \times \cdots \times 1}{(10 - k) \times (10 - k - 1) \times \cdots \times 1} = \frac{9 \times 9!}{(10 - k)!}
\]

**Proof:** There are 9 options for the first digit that cannot be 0. Then there are 9 options for the second digit that must be different from the first digit, 8 options for the third digit that must be different from the first and the second digits, and so on until there are only $10 - k + 1$ options for the $k^{th}$ digit that must be different from the the first $k - 1$ digits.

**Answer 2:** 

\[
\frac{10!}{(10 - k)!} - \frac{9!}{(10 - k)!} = \frac{10 \times 9!}{(10 - k)!} - \frac{1 \times 9!}{(10 - k)!} = \frac{(10 - 1) \times 9!}{(10 - k)!} = \frac{9 \times 9!}{(10 - k)!}
\]

**Proof:** The first digit is either 0 or not 0. Therefore, the number of passwords in which the first digit cannot be 0 is the total number of passwords minus the number of passwords in which the first digit is 0 which is equal to the number of passwords in which the last digit is 0. Thus, the answer to (iii) is the answer to (i) minus the answer to (ii).
3. Prove the following identity by induction on $n \geq 1$ or by any other method.

$$\sum_{i=1}^{n} (6i - 3) = 3n^2$$

Proof by induction:

- **Notations.**

  $L(n) = (6 \cdot 1 - 3) + (6 \cdot 2 - 3) + \cdots + (6n - 3)$

  $R(n) = 3n^2$

- **Induction base.** Prove that $L(1) = R(1)$:

  $L(1) = 6 \cdot 1 - 3 = 3$

  $R(1) = 3 \cdot 1^2 = 3$

- **Induction hypothesis.** Assume that $L(n - 1) = R(n - 1)$ for $n > 1$:

  $L(n - 1) = (6 \cdot 1 - 3) + (6 \cdot 2 - 3) + \cdots + (6(n - 1) - 3)$

  $R(n - 1) = 3(n - 1)^2$

  $= 3(n^2 - 2n + 1)$

  $= 3n^2 - 6n + 3$

- **Inductive step.** Prove that $L(n) = R(n)$ for $n > 1$:

  $L(n) = (6 \cdot 1 - 3) + (6 \cdot 2 - 3) + \cdots + (6(n - 1) - 3) + (6n - 3)$ \hspace{1cm} (* definition of $L(n)$ *)

  $= L(n - 1) + (6n - 3)$ \hspace{1cm} (* definition of $L(n - 1)$ *)

  $= R(n - 1) + (6n - 3)$ \hspace{1cm} (* induction hypothesis *)

  $= (3n^2 - 6n + 3) + (6n - 3)$ \hspace{1cm} (* evaluation of $R(n - 1)$ *)

  $= 3n^2$ \hspace{1cm} (* algebra *)

  $= R(n)$ \hspace{1cm} (* definition of $R(n)$ *)
Second proof: Recall that

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

This identity implies the following,

\[
\begin{align*}
L(n) &= \sum_{i=1}^{n} (6i - 3) \quad (* \text{definition of } L(n) \ast) \\
&= \sum_{i=1}^{n} (6i) + \sum_{i=1}^{n} (-3) \quad (* \text{algebra} \ast) \\
&= 6 \sum_{i=1}^{n} i - 3n \quad (* \text{algebra} \ast) \\
&= 6 \frac{n(n+1)}{2} - 3n \quad (* \text{applying above identity} \ast) \\
&= 3n(n + 1) - 3n \quad (* \text{algebra} \ast) \\
&= 3n^2 + 3n - 3n \quad (* \text{algebra} \ast) \\
&= 3n^2 \quad (* \text{algebra} \ast) \\
&= R(n) \quad (* \text{definition of } R(n) \ast)
\end{align*}
\]

Third proof: Recall that the sum of an arithmetic progression with \( n \) numbers is the sum of the first and the last numbers multiplied by \( n/2 \). The identity is about the sum of the following arithmetic progression with \( n \) numbers

\[ 3, 9, 15, \ldots, 6n - 3 \]

As a result the sum of this arithmetic progression is

\[
\begin{align*}
L(n) &= 3 + 9 + 15 + \cdots + (6n - 3) \quad (* \text{definition of } L(n) \ast) \\
&= \frac{n}{2} (3 + (6n - 3)) \quad (* \text{sum of arithmetic progression} \ast) \\
&= \frac{n(6n)}{2} \quad (* \text{algebra} \ast) \\
&= 3n^2 \quad (* \text{algebra} \ast) \\
&= R(n) \quad (* \text{definition of } R(n) \ast)
\end{align*}
\]
4. A binary string is 1-even if the length of any sequence of consecutive 1’s in the string is even. For example, (0110), (1101), (00000), and (0011110) are 1-even binary strings while (010), (1100), and (1100111) are not 1-even binary strings.

(a) There are two 1-even binary strings of length 2

(00) (11)

There are two non-1-even binary string of length 2

(01) (10)

(b) There are three 1-even binary strings of length 3

(000) (011) (110)

There are five non-1-even binary string of length 3

(001) (010) (100) (101) (111)

(c) There are five 1-even binary strings of length 4

(0000) (0011) (0110) (1100) (1111)

There are eleven non-1-even binary string of length 4

(0001) (0010) (0100) (0101) (0111) (1000) (1001) (1010) (1011) (1101) (1110)

(d) For \( k \geq 1 \), denote by \( G_k \) the number of 1-even binary strings of length \( k \). Trivially, (0) is a 1-even binary string and (1) is not. Combining this with the results from parts (a), (b), and (c), it follows that

\[
G_1 = 1 \quad G_2 = 2 \quad G_3 = 3 \quad G_4 = 5
\]

Recall that the first six Fibonacci numbers \( F_k \) for \( 0 \leq k \leq 5 \) are

\[
F_0 = 0 \quad F_1 = 1 \quad F_2 = 1 \quad F_3 = 2 \quad F_4 = 3 \quad F_5 = 5
\]

**Proposition:** For \( k \geq 1 \), there are \( G_k = F_{k+1} \) 1-even binary strings of length \( k \).

**Proof Sketch:** The claim is true for \( k = 1 \) and \( k = 2 \). It remains to show that \( G_k = G_{k-1} + G_{k-2} \) for \( k \geq 3 \). Consider the following two cases depending on the value of the first bit of a given 1-even binary string.

- The first bit is 0: In this case, the remaining \( k - 1 \) bits must be a 1-even binary string of length \( k - 1 \).
- The first bit is 1: In this case, the second bit must be 1 and the remaining \( k - 2 \) bits must be a 1-even binary string of length \( k - 2 \).

In the other direction, every 1-even binary string of length \( k - 1 \) becomes a 1-even binary string of length \( k \) by appending 0 at its beginning and every 1-even binary string of length \( k - 2 \) becomes a 1-even binary string of length \( k \) by appending 11 at its beginning. The above arguments can be used to rigorously prove the correctness of the recursive formula \( G_k = G_{k-1} + G_{k-2} \) and complete the proof.

**Example:** In part (c) the first three 1-even binary strings of length 4 are the three 1-even binary strings of length 3 with 0 appended at their beginning and the last two 1-even binary strings of length 4 are the two 1-even binary strings of length 2 with 10 appended at their beginning.