1. A legal password of length 5 may contain any of the 26 letters \{A, B, \ldots, Z\} or any of the 10 digits \{0, 1, \ldots, 9\} where characters may appear more than once in the password.

(a) How many legal passwords are there if a password must start with a letter followed by 4 digits?

**Answer:** \(26 \times 10^4 = 260000\)

**Explanation:** There are 26 options for the first location in the password and 10 options for each one of the following 4 locations in the password. The selection of which character appears in each location is independent and therefore there are

\[
26 \times 10 \times 10 \times 10 \times 10 = 26 \times 10^4
\]

legal passwords that start with a letter followed by 4 digits.

(b) How many legal passwords are there if a password must contain exactly 1 letter and exactly 4 digits?

Note that unlike part (a), the single letter may appear in any place.

**Answer:** \(5 \times 26 \times 10^4 = 1300000\)

**Explanation:** First fix the location of the letter. There are 5 options. For each one of these options, the answer is identical to part (a) which is \(26 \times 10^4\). As a result, there are

\[
5 \times 26 \times 10^4
\]

legal passwords that contain exactly 1 letter and exactly 4 digits.

(c) How many legal passwords are there that contain only letters or only digits?

**Answer:** \(26^5 + 10^5 = 11981376\)

**Explanation:** There are \(26^5\) legal passwords that contain only letters and there are \(10^5\) legal passwords that contain only digits. As a result, there are

\[
26^5 + 10^5
\]

legal passwords that contain only letters or only digits.

(d) How many legal passwords are there that do not contain two adjacent letters and do not contain two adjacent digits?

**Answer:** \(26^3 \times 10^2 + 26^2 \times 10^3 = 2433600\)

**Explanation:** Denote by \(L\) any of the 26 letters and by \(D\) any of the 10 digits. There are only two types of passwords that do not contain two adjacent letters and do not contain two adjacent digits. The first type is \(LDLDL\) and the second type is \(DLDDL\). There are

\[
26^3 \times 10^2
\]

passwords of the first type because a password of this type contains exactly 3 letters and exactly 2 digits in specific locations. Similarly, there are

\[
26^2 \times 10^3
\]

passwords of the second type because a password of this type contains exactly 2 letters and exactly 3 digits in specific locations. All together, there are

\[
26^3 \times 10^2 + 26^2 \times 10^3
\]

legal passwords that do not contain two adjacent letters and do not contain two adjacent digits.
2. (a) Which is larger \( \binom{6}{3} \) or \( \binom{6}{2} \)?

Answer: \( \binom{6}{2} = 20 > \binom{6}{3} = 15 \)

Explanation: The following expressions are computed based on the formula \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \):

\[
\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = 6 \cdot 5 \cdot 4 \cdot \frac{2 \cdot 1}{3 \cdot 2} = \frac{6 \cdot 5}{3 \cdot 2} = \frac{120}{6} = 20
\]
\[
\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)4 \cdot 3 \cdot 2 \cdot 1} = \frac{6 \cdot 5}{2 \cdot 2} = \frac{30}{2} = 15
\]

(b) Which is larger \( \binom{2022}{2} \) or \( \binom{2022}{3} \)?

Answer: \( \binom{2022}{3} > \binom{2022}{2} \)

Explanation: The formula \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) implies that

\[
\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2) \cdots 2 \cdot 1}{(2 \cdot 1)(n-2)(n-3) \cdots 2 \cdot 1} = \frac{n(n-1)}{2}
\]
\[
\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)(n-3) \cdots 2 \cdot 1}{(3 \cdot 2 \cdot 1)(n-3)(n-2) \cdots 2 \cdot 1} = \frac{n(n-1)(n-2)}{6}
\]

Therefore

\[
\binom{2022}{3} = \frac{2022 \cdot 2021 \cdot 2020}{6} = \frac{2022 \cdot 2021}{2} \times \frac{2020}{3} = \left( \binom{2022}{2} \right) \times \frac{2020}{3} > \left( \binom{2022}{2} \right) (* \text{because } \frac{2020}{3} > 1 *)
\]

(c) Is there an integer \( n \geq 3 \) for which \( \binom{n}{3} < \binom{n}{2} \)?

Answer: Yes. \( \binom{4}{3} = 6 \) is larger than \( \binom{3}{3} = 4 \) and \( \binom{3}{2} = 3 \) is larger than \( \binom{3}{3} = 1 \).

Explanation:

\[
\binom{4}{3} = \frac{4 \cdot 3 \cdot 2}{6} = \frac{24}{6} = \frac{4}{2} = \frac{4 \cdot 3}{2} = \binom{4}{2}
\]
\[
\binom{3}{3} = \frac{3 \cdot 2 \cdot 1}{6} = \frac{6}{6} = \frac{3}{2} = \binom{3}{2}
\]

Remark: \( \binom{5}{3} = \binom{5}{2} = 10 \). The following is a proof that for \( n \geq 6 \) it always the case that \( \binom{n}{3} > \binom{n}{2} \):

\[
\binom{n}{3} = \frac{n(n-1)(n-2)}{6} = \frac{n(n-1)}{2} \times \frac{n-2}{3}
\]
\[
> \frac{n(n-1)}{2} (* \text{because } \frac{n-2}{3} > \frac{4}{3} > 1 \text{ for } n \geq 6 *)
\]

\[
= \binom{n}{2}
\]
3. Prove by induction that $3^{n-1} > 2^n$ for $n \geq 3$.

The inequality is false for $1 \leq n \leq 2$ and true for $3 \leq n \leq 4$:

- $3^0 = 1 < 2^1 = 2$
- $3^1 = 3 < 2^2 = 4$
- $3^2 = 9 > 2^3 = 8$
- $3^3 = 27 > 2^4 = 16$

Proof by induction:

- **Induction base.** $3^2 = 9 > 2^3 = 8$ for $n = 3$.
- **Induction hypothesis.** Assume that $3^{k-1} > 2^k$ for $k \geq 3$.
- **Inductive step.** Prove that $3^k > 2^{k+1}$ for $k \geq 3$.

\[
3^k = 3 \times 3^{k-1} \quad \text{(* algebra *)}
\]
\[
> 3 \times 2^k \quad \text{(* induction hypothesis *)}
\]
\[
> 2 \times 2^k \quad \text{(* because } 3 > 2 \text{ *)}
\]
\[
= 2^{k+1} \quad \text{(* algebra *)}
\]

Another proof: The claim follows because $3^2 = 9 > 2^3 = 8$ and $3^i \geq 2^i$ for $i \geq 0$.

\[
3^{n-1} = 3^2 \times 3^{n-3}
\]
\[
= 9 \times 3^{n-3}
\]
\[
> 8 \times 2^{n-3}
\]
\[
= 2^3 \times 2^{n-3}
\]
\[
= 2^n
\]
4. Solve the following double recursive formulas. Find the closed-forms for both $T(n)$ and $S(n)$ as a function of $n$. Assume that $n$ is a positive integer.

\[
T(n) = \begin{cases} 
3 & \text{for } n = 1 \\
S(n) + 1 & \text{for } n > 1 
\end{cases} 
\]
\[
S(n) = \begin{cases} 
2 & \text{for } n = 1 \\
T(n-1) + 1 & \text{for } n > 1 
\end{cases} 
\]

Answer: $T(n) = 2n + 1$ and $S(n) = 2n$

Proof: First transform the given double recursive formulas in which $T(n)$ is defined by $S(n)$ and $S(n)$ is defined by $T(n-1)$ to a standard single recursive formula in which $S(n)$ is defined by $S(n-1)$ for $n > 1$.

\[
S(n) = T(n-1) + 1 = (S(n-1) + 1) + 1 = S(n-1) + 2
\]

Next compute $S(n)$ with a bottom-up evaluation.

\[
\begin{align*}
S(1) &= 2 = 2 \cdot 1 \\
S(2) &= S(1) + 2 = 4 = 2 \cdot 2 \\
S(3) &= S(2) + 2 = 6 = 2 \cdot 3 \\
S(4) &= S(3) + 2 = 8 = 2 \cdot 4 \\
&\vdots \\
S(n) &= 2 \cdot n
\end{align*}
\]

Next prove by induction that $S(n) = 2n$ for $n \geq 1$.

- Induction base. $S(1) = 2 = 2 \cdot 1$.
- Induction hypothesis. $S(n-1) = 2(n-1)$ for $n \geq 2$.
- Inductive step. For $n \geq 2$,

\[
\begin{align*}
S(n) &= S(n-1) + 2 \quad \text{(* definition of } S(n) *) \\
&= 2(n-1) + 2 \quad \text{(* induction hypothesis *)} \\
&= 2n \quad \text{(* algebra *)}
\end{align*}
\]

Finally, express $T(n)$ as a function of $n$ given the closed form for $S(n)$.

\[
T(n) = S(n) + 1 = 2n + 1
\]

Remark: In a similar proof, it can be shown that $T(n) = T(n) + 2$ and since $T(1) = 3$ it follows that $T(n) = 2n + 1$ which in turn implies that $S(n) = 2n$. 