CISC 2210 – Introduction to Discrete Structures

Midterm Project

Mar 17 – Mar 24, 2020

Name: .................................................................

Id: .................................................................

<table>
<thead>
<tr>
<th>Problem</th>
<th>Maximum Points</th>
<th>Your Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>80</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>560</td>
<td></td>
</tr>
<tr>
<td>Grade</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Guidelines

• Submission rules:
  – The due date is at 12pm on Tuesday Mar 24, 2020.
  – Email the instructor one PDF file that contains all of your hand-written answers.
  – Alternatively, email an identical type-written copy of the project.

• Answering questions:
  – Answer all seven problems without help of any other person including your fellow students.
  – You may use notes, books, and any other resources.

• Grading rules:
  – You get 30% of the credit if you do not answer a problem or part of a problem.
  – Answers without explanations (except problem 3) will get zero credit.
  – There will be no partial credit for “almost” correct answers or for incomplete proofs.
  – You must write everything inside the given space.
  – Non readable answers will not be graded.

• Finalizing grades:
  – You must be able to orally explain your answers and solve similar problems.
  – Some students may be asked to explain their answers in a video conference meeting with the instructor.
  – Students who would fail to convince the instructor that they understand their submitted answers risk failing the whole project.
1. 36 students take the Discrete Structures class. They had 3 quizzes called $X$, $Y$, and $Z$.
   - Surprisingly, only 1 student did not get an $A$ on any quiz while the rest of the students got an $A$ (aced) on at least one quiz.
   - 21 students aced $X$, 19 students aced $Y$, and 17 students aced $Z$.
   - 9 students aced both $X$ and $Y$, 11 students aced both $X$ and $Z$, and 8 students aced both $Y$ and $Z$.

(a) How many students aced at least one quiz? _____
(b) How many students aced all three quizzes? ______
(c) How may students aced both $X$ and $Y$ but not $Z$? _____
(d) How may students aced both $X$ and $Z$ but not $Y$? _____
(e) How may students aced both $Y$ and $Z$ but not $X$? _____
(f) How many students aced only $X$? ______
(g) How many students aced only $Y$? ______
(h) How many students aced only $Z$? ______

Justify your answers.
2. Prove the following logic distributive law.

\[ x \land (y \lor z \lor w) \equiv (x \land y) \lor (x \land z) \lor (x \land w) \]
3. Let $S \subseteq U = \{2, 3, \ldots, 21\}$ be a non-empty set containing integers larger than 1 and smaller than 22. The following are three statements about $S$:

(A) $S$ does not contain prime numbers.
(B) $S$ contains at least two even numbers.
(C) $S$ contains at most two odd numbers.

In the following eight questions you are asked to find the smallest and the largest possible sets $S_{\text{min}}$ and $S_{\text{max}}$ for which some of the statements are TRUE and some of the statements are FALSE. $S_{\text{min}}$ should contain the minimum possible number of members from $U$ while $S_{\text{max}}$ should contain the maximum possible number of members from $U$.

(a) Find $S_{\text{min}}$ and $S_{\text{max}}$ for which all three statements are TRUE.

(b) Find $S_{\text{min}}$ and $S_{\text{max}}$ for which statements A and B are TRUE but statement C is FALSE.

(c) Find $S_{\text{min}}$ and $S_{\text{max}}$ for which statements A and C are TRUE but statement B is FALSE.

(d) Find $S_{\text{min}}$ and $S_{\text{max}}$ for which statements B and C are TRUE but statement A is FALSE.

(e) Find $S_{\text{min}}$ and $S_{\text{max}}$ for which statement A is TRUE but statements B and C are FALSE.

(f) Find $S_{\text{min}}$ and $S_{\text{max}}$ for which statement B is TRUE but statements A and C are FALSE.

(g) Find $S_{\text{min}}$ and $S_{\text{max}}$ for which statement C is TRUE but statements A and B are FALSE.

(h) Find $S_{\text{min}}$ and $S_{\text{max}}$ for which all three statements are FALSE.
4. The goal is to count the number of passwords with repetitions of length 7 on the digits \{0, 1, \ldots, 9\}. Note that unlike numbers, such passwords may start with a 0.

Answer the following eight questions. Explain how you came up with your answers. You do not need to evaluate expressions for large numbers like \( n^i \).

(a) How many passwords of length 7 are there?

(b) How many passwords of length 7 are there that contain only odd digits?

(c) How many passwords of length 7 are there that do not contain the digit 0?

(d) How many passwords of length 7 are there that contain the digit 0 at least once?
(e) How many passwords of length 7 are there that contain the digit 0 exactly once?

(f) How many passwords of length 7 are there that contain the digit 0 at most once?

(g) How many passwords of length 7 are there that contain the digit 0 exactly twice?

(h) How many passwords of length 7 are there that contain the digit 0 exactly once and the digit 1 exactly once?
5. Using the identity

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

simplify the following expression as a fraction with as few as possible terms

\[
\binom{m}{3} + \binom{m+1}{3} + \binom{m+2}{3}
\]

You must show your work.
6. Prove the following identity by induction on \( n \geq 1 \) or by any other method.

\[
\sum_{i=1}^{n} (6i + 3) = 3n(n + 2)
\]
7. Prove the following identity by induction on \( n \geq 1 \) or by any other method.

\[
\sum_{i=1}^{n} (i \cdot i!) = (n + 1)! - 1
\]