1 Induction: 25 Credits

• Answer problem 1 if the last two digits of your ID are less or equal to 24.
• Answer problem 2 if the last two digits of your ID are greater or equal to 28 and less or equal to 55.
• Answer problem 3 if the last two digits of your ID are greater or equal to 56.

1. Prove the following identity by induction for $n \geq 1$

$$\sum_{i=1}^{n} (4i) = 2n(n+1)$$

2. Prove the following identity by induction

$$\sum_{i=1}^{n} (6i) = 3n(n+1)$$

3. Prove the following identity by induction

$$\sum_{i=1}^{n} (8i) = 4n(n+1)$$

Remark: The proof must be a proof by induction. Other proofs give no credit.
1. Solve the recursive formula by finding the \textbf{exact} formula for \( T(n) \) as a function of \( n \). Assume that \( n \) is a positive integer.

\[
T(n) = \begin{cases} 
2 & \text{for } n = 1 \\
3 & \text{for } n = 2 \\
2T(n - 1) - T(n - 2) & \text{for } n > 2 
\end{cases}
\]

Explain how you found your solution and prove your answer. Guesses alone get only partial credit.

2. Solve the following recursions by finding the \textbf{exact} formula for \( S(n) \) as a function of \( n \). Assume that \( n \) is a positive integer.

\[
S(n) = \begin{cases} 
2 & \text{for } n = 1 \\
4 & \text{for } n = 2 \\
2S(n - 1) - S(n - 2) & \text{for } n > 2 
\end{cases}
\]

Explain how you found your solution and prove your answer. Guesses alone get only partial credit.
In this problem

- $p = 13$ if the last digit of your ID is 1 or 7.
- $p = 17$ if the last digit of your ID is 0, 2, 3, or 9.
- $p = 19$ if the last digit of your ID is 4, 5, 6, or 8.

Justify your answers to the following three questions:

1. What is the inverse of 2 modulo $p$?
2. What is the inverse of 3 modulo $p$?
3. What is the inverse of 4 modulo $p$?

Remark and hint: You cannot use a calculator mainly because finding the answers does not require complicated math.
4 Counting: 25 credits

• Answer Problem 1 if the last digit of your ID is 1, 4, 5, or 6.
• Answer Problem 2 if the last digit of your ID is 0, 2, 3, 7, 8, or 9.

1. Definition: For $n \geq 2$, a permutation on the numbers \{1, 2, \ldots, n\} is a \{1, n\}-permutation if 1 appears before $n$ in the permutation.

Examples: For $n = 6$, the permutations (312564) and (416352) are \{1, n\}-permutations while the permutations (643521) and (612453) are not \{1, n\}-permutations.

(a) List all the \{1, n\}-permutations for $n = 2$.
(b) List all the \{1, n\}-permutations for $n = 3$.
(c) List all the \{1, n\}-permutations for $n = 4$.
(d) For $n \geq 2$, let $P(n)$ be the number of \{1, n\}-permutations on the numbers \{1, 2, \ldots, n\}. Find a closed form for $P(n)$. Justify your answer.

2. Definition: For $n \geq 2$, a permutation on the numbers \{1, 2, \ldots, n\} is a \{1, 2\}-permutation if 2 appears immediately after 1 in the permutation.

Examples: For $n = 6$, the permutations (312564) and (453612) are \{1, 2\}-permutations while the permutations (643521) and (164253) are not \{1, 2\}-permutations.

(a) List all the \{1, 2\}-permutations for $n = 2$.
(b) List all the \{1, 2\}-permutations for $n = 3$.
(c) List all the \{1, 2\}-permutations for $n = 4$.
(d) For $n \geq 2$, let $P(n)$ be the number of \{1, 2\}-permutations on the numbers \{1, 2, \ldots, n\}. Find a closed form for $P(n)$. Justify your answer.