CISC 2210 – Introduction to Discrete Structures

Midterm Exam 2
Sample Problems

• These problems are taken from past years exams.
• Some of the past year exam problems appear in the assignments.
1 \textbf{Counting}

1. \textbf{Definition:} For \( n \geq 2 \), a permutation on the numbers \( \{1, 2, \ldots, n\} \) is a \( \{1, n\} \)-permutation if 1 appears before \( n \) in the permutation.

\textbf{Examples:} For \( n = 6 \), the permutations (312564) and (416352) are \( \{1, n\} \)-permutations while the permutations (643521) and (612453) are not \( \{1, n\} \)-permutations.

(a) List all the \( \{1, n\} \)-permutations for \( n = 2 \).
(b) List all the \( \{1, n\} \)-permutations for \( n = 3 \).
(c) List all the \( \{1, n\} \)-permutations for \( n = 4 \).
(d) For \( n \geq 2 \), let \( P(n) \) be the number of \( \{1, n\} \)-permutations on the numbers \( \{1, 2, \ldots, n\} \). Find a closed form for \( P(n) \). Justify your answer.

2. \textbf{Definition:} For \( n \geq 2 \), a permutation on the numbers \( \{1, 2, \ldots, n\} \) is a \( \{1, 2\} \)-permutation if 2 appears immediately after 1 in the permutation.

\textbf{Examples:} For \( n = 6 \), the permutations (312564) and (453612) are \( \{1, 2\} \)-permutations while the permutations (643521) and (164253) are not \( \{1, 2\} \)-permutations.

(a) List all the \( \{1, 2\} \)-permutations for \( n = 2 \).
(b) List all the \( \{1, 2\} \)-permutations for \( n = 3 \).
(c) List all the \( \{1, 2\} \)-permutations for \( n = 4 \).
(d) For \( n \geq 2 \), let \( P(n) \) be the number of \( \{1, 2\} \)-permutations on the numbers \( \{1, 2, \ldots, n\} \). Find a closed form for \( P(n) \). Justify your answer.

3. \textbf{Definition:} For \( n \geq 2 \), a list with repetitions on the letters \( \{A, C, G, T\} \) is an \( \{AT\} \)-list if A and T each appears exactly once in the list and T appears immediately after A.

\textbf{Examples:} For \( n = 6 \), the lists (ATGCAG) and (GGCAT) are \( \{AT\} \)-lists while the lists (CCTAGG), (AGGTC), and (ATTGGG), are not \( \{AT\} \)-lists.

(a) List all the \( \{AT\} \)-lists for \( n = 2 \).
(b) List all the \( \{AT\} \)-lists for \( n = 3 \).
(c) List all the \( \{AT\} \)-lists for \( n = 4 \).
(d) For \( n \geq 2 \), let \( L(n) \) be the number of \( \{AT\} \)-lists of length \( n \). Find a closed form for \( L(n) \). Justify your answer.

4. \textbf{Definition:} For \( n \geq 2 \), a list with repetitions on the letters \( \{A, C, G, T\} \) is a \( \{CG\} \)-list if C and G each appears exactly once in the list and C appears immediately before G.

\textbf{Examples:} For \( n = 6 \), the lists (ATTTCG) and (CGTATA) are \( \{CG\} \)-lists while the lists (TTGCAA), (TACAGA), and (CCTGG), are not \( \{CG\} \)-lists.

(a) List all the \( \{CG\} \)-lists for \( n = 2 \).
(b) List all the \( \{CG\} \)-lists for \( n = 3 \).
(c) List all the \( \{CG\} \)-lists for \( n = 4 \).
(d) For \( n \geq 2 \), let \( L(n) \) be the number of \( \{CG\} \)-lists of length \( n \). Find a closed form for \( L(n) \). Justify your answer.
5. The goal is to count the number of length-\(k\) (\(1 \leq k \leq 10\)) passwords without repetitions on the ten digits 0, 1, \ldots, 9 given some restrictions. Note that in such passwords, each digit may appear only once but may not appear at all.

(a) How many length-2 passwords exist? Justify your answer.
How many length-2 passwords exist in which the last digit must be 0? Justify your answer.
How many length-2 passwords exist in which the first digit cannot be 0? Justify your answer.

(b) How many length-3 passwords exist? Justify your answer.
How many length-3 passwords exist in which the last digit must be 0? Justify your answer.
How many length-3 passwords exist in which the first digit cannot be 0? Justify your answer.

(c) For \(1 \leq k \leq 10\), as a function of \(k\), how many length-\(k\) passwords exist? Justify your answer.
For \(1 \leq k \leq 10\), as a function of \(k\), how many length-\(k\) passwords exist in which the last digit must be 0? Justify your answer.
For \(1 \leq k \leq 10\), as a function of \(k\), how many length-\(k\) passwords exist in which the first digit cannot be 0? Justify your answer.

2 Combinatorics

1. For \(n \geq 3\), the goal is to simplify the expression \(B(n)\) to contain only one binomial coefficient term instead of two binomial coefficient terms.

\[
B(n) \equiv \binom{n+1}{3} - \binom{n}{3}
\]

(a) What is the value of \(B(n)\) for \(n = 3\)?
(b) What is the value of \(B(n)\) for \(n = 4\)?
(c) What is the value of \(B(n)\) for \(n = 5\)?
(d) Simplify \(B(n)\) for any \(n \geq 3\). Prove that your simpler expression is identical to the original expression.

**Hint:** There is a “long way” that simplifies the expression using the explicit formula for \(\binom{n}{k}\).
But there is a short cut that is based on the recursive property of \(\binom{n}{k}\).

2. For \(n \geq 3\), the goal is to simplify the expression \(B(n)\) to contain only one binomial coefficient term instead of two binomial coefficient terms.

\[
B(n) \equiv \binom{n}{3} - \binom{n-1}{2}
\]

(a) What is the value of \(B(n)\) for \(n = 4\)?
(b) What is the value of \(B(n)\) for \(n = 5\)?
(c) What is the value of \(B(n)\) for \(n = 6\)?
(d) Simplify \(B(n)\) for any \(n \geq 4\). Prove that your simpler expression is identical to the original expression.

**Hint:** There is a “long way” that simplifies the expression using the explicit formula for \(\binom{n}{k}\).
But there is a short cut that is based on the recursive property of \(\binom{n}{k}\).
3 Induction

1. Prove the following identity by induction for \( n \geq 1 \).
\[
\sum_{i=1}^{n} (6i) = 3n(n + 1)
\]

2. Prove the following identity by induction for \( n \geq 1 \).
\[
\sum_{i=1}^{n} (i + 3) = \frac{n(n + 7)}{2}
\]

3. Prove the following identity by induction on \( n \geq 1 \) or by any other method.
\[
\sum_{i=1}^{n} (6i + 3) = 3n(n + 2)
\]

4. Prove the following identity by induction on \( n \geq 1 \) or by any other method.
\[
\sum_{i=1}^{n} (4i - 3) = n(2n - 1)
\]

5. Prove the following identity by induction on \( n \geq 1 \) or by any other method.
\[
\sum_{i=1}^{n} (i \cdot i!) = (n + 1)! - 1
\]

4 Recursion

1. Solve the recursive formula by finding the \textbf{exact} formula for \( T(n) \) as a function of \( n \). Assume that \( n \) is a positive integer.
\[
T(n) = \begin{cases} 
2 & \text{for } n = 1 \\
3 & \text{for } n = 2 \\
2T(n-1) - T(n-2) & \text{for } n > 2 
\end{cases}
\]

Explain how you found your solution and prove your answer. Guesses alone get only partial credit.

2. Solve the following recursions by finding the \textbf{exact} formula for \( S(n) \) as a function of \( n \). Assume that \( n \) is a positive integer.
\[
S(n) = \begin{cases} 
2 & \text{for } n = 1 \\
4 & \text{for } n = 2 \\
2S(n-1) - S(n-2) & \text{for } n > 2 
\end{cases}
\]

Explain how you found your solution and prove your answer. Guesses alone get only partial credit.

3. Solve the following recursions by finding the \textbf{exact} formula for \( T(n) \) as a function of \( n \). Assume that \( n \) is a positive integer.
\[
T(n) = \begin{cases} 
1 & \text{for } n = 1 \\
2 & \text{for } n = 2 \\
2T(n-2) - T(n-1) + 3 & \text{for } n > 2 
\end{cases}
\]

Explain how you found your solution and prove your answer. Guesses alone get only partial credit.
4. Solve the following recursions by finding the exact formula for $S(n)$ as a function of $n$. Assume that $n$ is a positive integer.

$$S(n) = \begin{cases} 
2 & \text{for } n = 1 \\
4 & \text{for } n = 2 \\
2S(n-2) - S(n-1) + 6 & \text{for } n > 2 
\end{cases}$$

Explain how you found your solution and prove your answer. Guesses alone get only partial credit.

5. A binary string is 1-even if the length of any sequence of consecutive 1’s in the string is even.

For example, (0110), (11011), (00000), and (0011110) are 1-even binary strings while (010), (11100), and (1100111) are not 1-even binary strings.

(a) List all the 1-even strings of length 2.
(b) List all the 1-even strings of length 3.
(c) List all the 1-even strings of length 4.
(d) For $k \geq 1$, as a function of $k$, how many 1-even binary strings of length $k$ exist? Justify your answer.

**Hint:** Think of Fibonacci numbers and their recursive definition. Then consider the two cases in which the first bit in the string is 1 or 0.

6. A binary string is 1-lonely if it does not contain consecutive 1s.

For example, (0010) and (01001) are 1-lonely binary strings while (11000), (01110), and (010110) are not 1-lonely binary strings.

(a) List all the 1-lonely binary strings of length 2.
(b) List all the 1-lonely binary strings of length 3.
(c) List all the 1-lonely binary strings of length 4.
(d) For $k \geq 1$, as a function of $k$, how many 1-lonely binary strings of length $k$ exist? Justify your answer.

**Hint:** Think of Fibonacci numbers and their recursive definition. Then consider the two cases in which the first bit in the string is 1 or 0.